$\boxed{1}$ S has an arbitrage $\iff$ $\bar{S}$ has an arbitrage

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In the lectures, thanks to Lemma 5.8, we know that ξ is self-financing for S $\iff$ ξ is self-financing for $\bar{S}$ (i.e. $d\bar{V}^{\xi}_t = \xi\, d\bar{S}_t$)

Moreover since r is bounded we have that

$$\beta(t) = \exp\left(-\int_0^t r(s)\,ds\right) \geq C \quad P\text{-a.s. for a constant } C>0.$$

and this yields that ξ is admissible for S, i.e.

$$V^{\xi}_t \geq -K_{\xi} \quad P\text{-a.s. for all } t\in[0,T].$$

$$\Updownarrow$$

$$\bar{V}^{\xi}_t = V^{\xi}_t\,\beta(t) \geq -C K_{\xi} \quad P\text{-a.s. for all } t\in[0,T]$$

, that is, ξ is admissible for $\bar{S}$.

Similarly ξ is an arbitrage for S

$$\iff \left\{\begin{array}{l} \exists\, \xi \in \mathcal{A} \text{ s.t.} \\ V^{\xi}_0 = 0,\ V^{\xi}_T \geq 0,\ P\text{-a.s. and } P(V^{\xi}_T>0)>0 \end{array}\right\}$$

$$\iff \left\{\begin{array}{l} \exists\, \xi \in \mathcal{A} \text{ s.t.} \\ \bar{V}^{\xi}_0 = 0,\ \bar{V}^{\xi}_T = \beta(T)\,V^{\xi}_T \geq 0 \ P\text{-a.s. and} \\ P(\bar{V}^{\xi}_T>0) = P(\beta(T)V^{\xi}_T > 0) \underset{\beta(T)>0}{=} P(V^{\xi}_T>0)>0 \end{array}\right.$$

$\iff$ ξ is an arbitrage for $\bar{S}$.

$\boxed{2}$

S is already normalized $(S^0_t \equiv 1)$.

S has an arbitrage $\iff \exists\, \xi \in \mathcal{A}$ s.t. $V^\xi_T \geq V^\xi_0$ P-a.s. and $P(V^\xi_T > V^\xi_0) > 0$

Let ξ be as above. Then, define $\tilde{\xi} = (\tilde{\xi}_0(t), \dots, \tilde{\xi}_n(t))$ as follows. Take $\tilde{\xi}_i(t) := \xi_i(t)$ $i = 1, \dots, n$ and then choose $\tilde{\xi}_0(0)$ such that $V^{\tilde{\xi}}_0 = 0$ and $\tilde{\xi}_0(t)$ such that $\tilde{\xi}$ is self-financing.

Then,

$$V^{\tilde{\xi}}_t = \tilde{\xi}(t) \cdot S_t \overset{(1)}{=} \int_0^t \tilde{\xi}(s)\, dS_s \overset{(2)}{=} \int_0^t \xi(s)\, dS_s$$

$$\overset{(3)}{=} V^\xi_t - V^\xi_0$$

(1) $V^{\tilde{\xi}}_0 = 0$ and $\tilde{\xi}$ is self-financing.

(2) $S^0_t \equiv 1 \Rightarrow dS^0_t = 0$ and $\tilde{\xi}_i = \xi_i$ $i = 1, \dots, n$.

(3) ξ is self-financing

Finally, $\tilde{\xi} \in \mathcal{A}$ s.t. $V^{\tilde{\xi}}_T = V^\xi_T - V^\xi_0 \geq 0$ P-a.s.

$$P(V^{\tilde{\xi}}_T > 0) = P(V^\xi_T > V^\xi_0) > 0.$$

$\boxed{3}$ Let $\xi(t) = (\xi_0, \xi_1, \dots, \xi_n)$ be the constant portfolio. Prove that ξ is self-financing.

$$V_t^{\xi} = \xi \cdot S_t = \sum_{i=0}^{n} \xi_i \cdot S_t^i$$

Taking the Itô differential

$$dV_t^{\xi} = \sum_{i=0}^{n} d(\xi_i \cdot S_t^i) \overset{(1)}{=} \sum_{i=0}^{n} \xi_i \, dS_t^i = \xi \, dS_t$$

Product rule for semimartingales (1)

$$d(a \cdot Z_t) = a \, dZ_t + Z_t \underbrace{\cancel{da}}_{0} + \cancel{d[a, Z]_t}$$

$\boxed{4}$ S is normalized and Assumption 5.20 holds

$n = m$ and σ is invertible and with a bounded inverse $\|\sigma(t)^{-1}\| \leq C$.

Prove that

If all bounded claims are attainable $\Big\}$ $\Rightarrow$ Any lower bounded claim F s.t. $E_Q[F^2] < +\infty$ is attainable.

For any F lower bounded s.t. $E_Q[F^2] < +\infty$, we need to prove that $\exists\, \xi \in \mathcal{A}$ and $x \in \mathbb{R}$ such that

$$F = V_T^{\xi, x} := x + \int_0^T \xi(s)\, dS_s$$

and $\bar{V}_t^{\xi, x} = V_t^{\xi, x}$ is a Q-martingale.

We can choose bounded T-claims F_k such that

$$F_k \to F \text{ in } L^2(Q) \Rightarrow E_Q[F_k] \longrightarrow E_Q[F]$$

By assumption $\exists\ \xi^k = (\xi_0^k, \xi_1^k, \dots, \xi_n^k) \in \mathcal{A}$ an constants $V_k(0)$ such that

$$F_k = V_k(0) + \int_0^T \xi^k(s)\, dS_s = E_Q[F_k] + \int_0^T \hat{\xi}^k(s)\, \sigma_s\, dB_s^Q$$

where $\hat{\xi}^k = (\xi_1^k, \dots, \xi_n^k)$.

Then, for $n, m \in \mathbb{N}$

$$E_Q\Big[\int_0^T \|\hat{\xi}^n(t)\sigma_t - \hat{\xi}^m(t)\sigma_t\|^2 ds\Big]$$

Itô isometry:

$$= E_Q\left[\left(\int_0^T \hat{\xi}^n(t)\,\sigma_t\,dB_t^Q - \int_0^T \hat{\xi}^m(t)\,\sigma_t\,dB_t^Q\right)^2\right]$$

$$= E_Q\left[\left(F_n - F_m - \left(E_Q[F_n] - E_Q[F_m]\right)\right)^2\right]$$

$$\leq 2\,E_Q\left[(F_n - F_m)^2\right] + 2\left(E_Q[F_n] - E_Q[F_m]\right)^2 \to 0$$

because $F_n \xrightarrow[n\to\infty]{L^2} F$ and $E_Q[F_n] \xrightarrow[n\to\infty]{} E_Q[F]$ and in particular they are Cauchy sequences.

This implies that $\{\hat{\xi}^n(t)\,\sigma_t\}_{n\geq 1}$ is a Cauchy sequence in

$L^2([0,T]\times\Omega,\ \mathcal{B}([0,T])\otimes\mathcal{F}_T,\ \lambda\otimes Q)$ which is complete.

Hence, let ψ_t be the limit of $\hat{\xi}^n(t)\,\sigma_t$ in $L^2(\lambda\otimes Q)$ and we have

$$F = E_Q[F] + \int_0^T \psi_t\,dB_t^Q$$

Since σ^{-2} is bounded, we can write

$$\int_0^T \psi_t\,dB_t^Q = \int_0^T \psi_t\,\sigma_t^{-1}\sigma_t\,dB_t^Q$$

So $\hat{\xi}(t) := \psi_t\,\sigma_t^{-1}$ is the replicating portfolio for the risky asset,

and we choose ξ_0 such that $\xi = (\xi_0, \hat{\xi})$ is self-financing;

Since $F = V_T^{\xi, E_Q}$ is lower bounded $\xi \in \mathcal{A}$.

5) Here we will apply Theorem 5.16. with $r=0$.

a) $$dS^1_t = 3\,dt + dB^1_t + dB^2_t$$
$$dS^2_t = -dt + dB^1_t - dB^2_t$$

$\mu = (3,-1)^T$ and $\sigma = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

the system $\sigma\theta = \mu$ has the form

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \Rightarrow \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & -1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -2 & -4 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right) \quad \boxed{\theta_1 = 1,\ \theta_2 = 2.}$$

Since θ is constant, it satisfies Novikov's condition.

Hence $\nexists$ A.O.

b) $$dS^1_t = dt + dB^1_t + dB^2_t - dB^3_t \qquad \mu = (1,5)^T$$
$$dS^2_t = 5dt - dB^1_t + dB^2_t + dB^3_t \qquad \sigma = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}$$

We try to solve the system $\sigma\theta = \mu$ for θ.

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 2 & 0 & 6 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 3 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & 3 \end{array}\right) \Rightarrow \begin{cases} \theta_2 = 3 \\ \theta_1 = \theta_3 - 2 \\ \theta_3 \in \mathbb{R} \end{cases}$$

Infinite solutions, satisfy Novikov's

Hence, $\nexists$ A.O.

c) $dS^1_t = dt + dB^1_t + dB^2_t - dB^3_t$ $\mu = (1, 5)^T$

$dS^2_t = 5dt - dB^1_t - dB^2_t + dB^3_t$ $\sigma = \begin{pmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$

We try to solve $\sigma\theta = \mu$ for θ.

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 6 \end{array}\right) \Rightarrow \nexists \text{ solutions}$$

Clearly $\xi = (-S^1_0 - S^2_0, 1, 1)$ and $V^\xi_0 = 0$ is an arbitrage

$$V^\xi_T = \int_0^T 1\, dS^1_u + \int_0^T 1\, dS^2_u = 6T > 0.$$

d) $dS^1_t = dt + dB^1_t + dB^2_t - dB^3_t$ $\mu = (1, -3)^T$

$dS^2_t = -3dt - 3dB^1_t - 3dB^2_t + dB^3_t$ $\sigma = \begin{pmatrix} 1 & 1 & -1 \\ -3 & -3 & 1 \end{pmatrix}$

We try to solve $\sigma\theta = \mu$ for θ

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ -3 & -3 & 1 & -3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 0 & -2 & 0 \end{array}\right) \Rightarrow \begin{array}{l} \theta_3 = 0 \\ \theta_1 = 1 - \theta_2 \\ \theta_2 \in \mathbb{R} \end{array}$$

There are ∞ solutions satisfying Novikov's condition

$\Rightarrow \nexists$ A.O.

e) $dS^1_t = dt + dB^1_t + dB^2_t$ $\mu = (1, 2, 3)^T$

$dS^2_t = 2dt + dB^1_t - dB^2_t$ $\sigma = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}$

$dS^3_t = 3dt - dB^1_t + dB^2_t$

We try to solve $\sigma\theta = \mu$ for θ.

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 2 \\ -1 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 2 & 4 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 5 \end{array}\right)$$

$\Rightarrow \nexists$ solutions $\Rightarrow$

Clearly $\xi = (-S_0^2 - S_0^3, 0, 1, 1)$ and $V_0^\xi = 0$ is an A.O.

$$V_T^\xi = \int_0^T dS_t^2 + \int_0^T dS_t^3 = \int_0^T 5\,dt = 5T > 0$$

f) $dS_t^1 = dt + dB_t^1 + dB_t^2$ $\quad \mu = (1, 2, -2)^T$

$dS_t^2 = 2dt + dB_t^1 - dB_t^2$ $\quad \sigma = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}$

$dS_t^3 = -2dt - dB_t^1 + dB_t^2$

We try to solve $\sigma\theta = \mu$ for θ

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 2 \\ -1 & 1 & -2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 3/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} \theta_1 = 3/2 \\ \theta_2 = -1/2 \end{array}$$

Satisfy Novikov's cond.

$\Rightarrow \nexists$ A.O.

5.6 For the nonarbitrage markets of Exercise 5.5, find the ones that are complete. For those which are not complete find a T- claim which is not attainable

---

Here we will use Theo 5.26. $\operatorname{rank}(\sigma) = \dim(\operatorname{col}(\sigma)) = \dim(\operatorname{row}(\sigma))$

We will check that $\operatorname{rank}(\sigma) = m$.

a) $\sigma = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} \Rightarrow \operatorname{rank}(\sigma) = \dim(\operatorname{row}(\sigma)) = 2$

$m = 2$

Market is complete

b) $\sigma = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix}$, $m = 3$ but $\operatorname{rank}(\sigma) \leq \min(n, m) = 2$

the market is not complete.

c) Arbitrage existed.

d) $\sigma = \begin{pmatrix} 1 & 1 & -1 \\ -3 & -3 & 1 \end{pmatrix}$, $m = 3$ but $\operatorname{rank}(\sigma) \leq \min(n, m) = 2$

the market is not complete.

e) Arbitrage existed

f) $\sigma = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}$, $m = 2$ $\operatorname{rank}(\sigma) = \dim(\operatorname{col}(\sigma)) = 2$

the 2 column vectors are indep.

5.7 Find $z \in \mathbb{R}$ and φ such that

$$F = z + \int_0^T \varphi_t \, dB_t$$

for

a) $F = \sum_{i=1}^{n} (B_T^i)^2$, b) $F = \exp\left(\sum_{i=1}^{n} B_T^i\right)$

We will apply Theorem 5.34 with

$dY_t = dB_t$

$Y_0 = y$

(Note that we assume $P = Q \Rightarrow B = B^Q$.)

---

a) $h(x) = \sum_{i=1}^{n} x_i^2$

$$E^y[h(Y_{T-t})] = E[h(y + B_{T-t})]$$

$$= \sum_{i=1}^{n} E[(y_i + B_{T-t}^i)^2] = |y|^2 + 2\sum_{i=1}^{n} y_i \underbrace{\cancel{E[B_{T-t}^i]}}_{0}$$

$$+ \sum_{i=1}^{n} E[(B_{T-t}^i)^2]$$

$$= |y|^2 + n(T-t)$$

thus,

$$\frac{\partial}{y_i} E^y[h(Y_{T-t})] = 2y_i$$

and

$$B_T^2 = |y|^2 + nT + \sum_{i=1}^{n} \int_0^T 2(y_i + B_t^i) \, dB_t^i$$

b) $h(x) = \exp(x_1 + \dots + x_n)$

$$E^y[h(Y_{T-t})] = E[h(y + B_{T-t})]$$

$$= \exp\left(\sum_{i=1}^{n} y_i\right) E\left[\exp\left(\sum_{i=1}^{n} B_{T-t}^i\right)\right]$$

$$= \exp\left(\sum_{i=1}^{n} y_i\right) \left(E\left[\exp\left(B^1_{T-t} \right)\right]\right)^n$$

B^i_{T-t} i.i.d.

$$= \exp\left(\sum_{i=1}^{n} y_i\right) \exp\left(n \frac{(T-t)}{2} \right)$$

Hence

$$E^y[h(Y_t)] = \exp\left\{ \sum_{i=1}^{n} y_i + n\frac{T}{2} \right\}$$

$$\frac{\partial}{\partial y} E^y[h(Y_{T-t})] = \exp\left\{ \sum_{i=1}^{n} y_i + n\frac{(T-t)}{2} \right\}$$

and we can conclude.

$$\exp\left(\sum_{i=1}^{n} B^i_T\right) = \exp\left\{ \sum_{i=1}^{n} y_i + n\frac{T}{2} \right\}$$

$$+ \sum_{i=1}^{n} \exp\left(n\frac{(T-t)}{2} + \sum_{i=1}^{n} y_i + \sum_{i=1}^{n} B^i_t \right) dB^i_t$$

$\boxed{5.8}$ Let S be a geometric Brownian motion

$$dS_t = \mu S_t \, dt + \sigma S_t \, dB_t$$

$\mu, \sigma \neq 0$ constants, $S_0 = s > 0$

Find $z \in \mathbb{R}$ and ψ s.t.

$$S_T = z + \int_0^T \psi_t \, dB_t$$

We will use theorem 5.37 with

$Y_t = S_t$ and $h(x) = x$.

Let $X_t = \log(S_t)$, then using Itô's formula

$$dX_t = \frac{1}{S_t} dS_t - \frac{1}{2S_t^2} d[S]_t$$

$$= \mu \, dt + \sigma \, dB_t - \frac{1}{2S_t^2} \sigma^2 \, dt = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma \, dB_t$$

$$\Rightarrow S_t = \exp(X_t) = \exp\left(X_0 + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right)$$

then,

$$= s \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right)$$

$$E^s[h(S_{T-t})] = E\left[s \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)(T-t) + \sigma B_{T-t}\right)\right]$$

$$= s \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)(T-t)\right\} E\left[e^{\sigma B_{T-t}}\right]$$

$$= s \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)(T-t) + \frac{\sigma^2 (T-t)}{2}\right\}$$

$$= s \, e^{\mu (T-t)}$$

$\left(\begin{array}{l} Z \sim N(\mu, \sigma^2) \\ E[e^{\theta Z}] = \exp\left(\theta\mu + \frac{\sigma^2 \theta^2}{2}\right) \end{array} \right)$

$$\frac{\partial}{\partial s} E^s[h(S_{T-t})] = e^{\mu(T-t)}$$

then,

$$z = E^s[h(S_T)] = s\,e^{\mu T}$$

$$\psi_t = \frac{\partial}{\partial s} E^s[h(S_{T-t})]\Big|_{s=S_t} \sigma S_t$$

$$= e^{\mu(T-t)} \sigma S_t$$

Finally,

$$S_T = s\,e^{\mu T} + \int_0^T e^{\mu(T-t)} \sigma S_t \, dB_t$$

Note that

$$dS_T = s\mu e^{\mu T} dT + d\left(e^{\mu T} \int_0^T e^{-\mu t} \sigma S_t \, dB_t\right)$$

$$= s\mu e^{\mu T} dT + \mu e^{\mu T}\left(\int_0^T e^{-\mu t} \sigma S_t \, dB_t\right) dT + \cancel{e^{\mu T}}\, \cancel{e^{-\mu T}} \sigma S_T \, dB_T$$

$$= \mu\left(s e^{\mu T} + \int_0^T e^{\mu(T-t)} \sigma S_t \, dB_t\right) dT + \sigma S_T \, dB_T$$

$$= \mu S_T + \sigma S_T \, dB_T$$

5.9 $S = (S^0, S)$ $\quad dS^0_t = r S^0_t \, dt, \; S^0_0 = 1$

$r > 0$ constant

Find the price of the European T-claim $F = B_T$ and find the corresponding replicating portfolio in the following cases (μ and $\sigma \neq 0$ constants)

a) $dS_t = \mu S_t \, dt + \sigma S_t \, dB_t$

b) $dS_t = \sigma \, dB_t$

c) $dS_t = \mu S_t \, dt + \sigma \, dB_t$

$\left\{ S^0_t = e^{rt} \quad t \geq 0 \right.$ For all markets $\Big\}$

---

a) $dS_t = \mu S_t \, dt + \sigma S_t \, dB_t$

In this market consider $\theta \in L^0_{a,T}$ such that

$$\sigma S_t \cdot \theta = \mu S_t - r S_t \qquad \lambda \otimes P\text{-a.e.}$$

In this case

$$\theta = \frac{\mu - r}{\sigma} \quad \text{constant.}$$

By Theorem 5.16 we know that this market has no arbitrage opportunities.

$$Z_t = \exp\left\{ - \frac{\mu - r}{\sigma} B_t - \frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2 t \right\}$$

is a true martingale, $dQ = Z_T \, dP$ is a probab. measure and the process

$$B^Q_t = \frac{\mu - r}{\sigma} t + B_t \quad \text{is a } Q\text{- B.m.}$$

By Corollary 5.27 the market is complete.

By Theorem 5.31 the price of $F = B_T$ is given by $E_Q[e^{-rT} B_T]$.

$$E_Q[e^{-rT} B_T] = E_Q\left[e^{-rT}\left(B^Q_T - \frac{\mu - r}{\sigma} T\right)\right]$$

$$= e^{-rT} \underbrace{E_Q[B^Q_T]}_{0} - e^{-rT}\left(\frac{\mu - r}{\sigma}\right) T$$

B^Q is a Q-B.m.

The replicating portfolio will satisfy:

$$e^{-rt}\xi(t)\sigma S_t = \psi_t \Rightarrow \xi(t) = \frac{e^{rt}\psi_t}{\sigma S_t}$$

where ψ_t is such that

$$e^{-rT} B_T = E_Q[e^{-rT} B_T] + \int_0^T \psi_t \, dB^Q_t$$

We write $e^{-rT} B_T$ in term of B^Q_T

$$e^{-rT} B_T = e^{-rT} B^Q_T - \underbrace{e^{-rT}\frac{\mu - r}{\sigma} T}_{E_Q[e^{-rT} B_T]}$$

$$= E_Q[e^{-rT} B_T] + \int_0^T e^{-rT} \, dB^Q_T$$

$\Rightarrow \psi_t = e^{-rT}$ (deterministic and constant)

Then the replic. portfolio is

$$\xi(t) = \frac{e^{-r(T-t)}}{\sigma S_t}$$ and $\xi_0(t)$ such that

$\xi = (\xi_0(t), \xi(t))_{t \in [0,T]}$ is self-financing.

the price will be $p(F) = E_Q[F] = E_Q[e^{-rT} B_T] = -e^{-rT}\left(\frac{\mu - r}{\sigma}\right) T$

Let

$$A(t) = \int_0^t \xi(u)\, dS_u - \xi(t) S_t$$

$$= \int_0^t \frac{e^{-r(T-u)}}{\sigma S_u}\, dS_u - \frac{e^{-r(T-t)}}{\sigma}$$

$$dA(t) = \frac{e^{-r(T-t)}}{\sigma} \frac{dS_t}{S_t} - \frac{r}{\sigma} e^{-r(T-t)}$$

$$= \frac{\mu - r}{\sigma} e^{-r(T-t)} dt + e^{-r(T-t)} dB_t$$

$$= e^{-r(T-t)} dB_t^Q$$

Then,

$$\xi_0(t) = \xi_0(0) + \int_0^t e^{-ru}\, dA(u) = \xi_0(0) + \int_0^t e^{-ru} e^{-r(T-u)}\, dB_u^Q$$

$$= \xi_0(0) + e^{-rT} B_t^Q$$

$$\xi_0(0) + \xi(0) S_0 = V_0^{\xi} = p(F) = -e^{-rT} \frac{\mu - r}{\sigma} T$$

$$\Rightarrow \xi_0(0) = \frac{-e^{-rT}}{\sigma S_0} S_0 - e^{-rT} \frac{\mu - r}{\sigma} T = -e^{-rT}\left(\frac{1}{\sigma} + \frac{\mu - r}{\sigma} T\right)$$

Hence,

$$\xi_0(t) = -e^{-rT}\left(\frac{1}{\sigma} + \frac{\mu - r}{\sigma} T\right) + e^{-rT} B_t^Q$$

$$V_T^{\xi} = \xi_0(T) e^{rT} + \xi(T) S_T = -\left(\frac{1}{\sigma} + \frac{\mu - r}{\sigma} T\right) + B_T^Q + \frac{1}{\sigma} = B_T$$

b) $dS_t = \sigma dB_t \Rightarrow \boxed{S_t = S_0 + \sigma B_t}$

Consider $\theta \in L^0_{a,T}$ s.t.

$$\sigma \theta = -r S_t \Rightarrow \theta_t = -\frac{r}{\sigma} S_t = -\frac{r}{\sigma}\{S_0 + \sigma B_t\}$$

We need to check that

$$Z_t = \exp\left\{-\int_0^t \theta_s \, dB_s - \frac{1}{2}\int_0^t |\theta_s|^2 ds\right\}$$

is a true martingale. We try Novikov.

$$(*) \quad E\left[\exp\left(\frac{1}{2}\int_0^T \frac{r^2}{\sigma^2}(S_0 + \sigma B_t)^2 dt\right)\right] < +\infty \, ?$$

Not straightforward.

We have a simple version of Fernique's theorem that says that if $X \sim N(0,\sigma^2)$ then

$$E[\exp(\beta X^2)] < +\infty \Leftrightarrow \beta < \frac{1}{2\sigma^2}$$

Although this result cannot be applied directly, it does hint at that $(*)$ is not finite.

Corollary 5.14 in Karatzas shows we only need to prove

$$E\left[\exp\left(\frac{1}{2}\int_{t_n}^{t_{n+1}} |\theta_t|^2 dt\right)\right] < +\infty, \; \forall n \geq 1$$

with $\{t_n\}_{n\geq 0}$ real numbers such that $0 = t_0 < t_1 < \dots < t_n \uparrow \infty$

This is a kind of "local Girsanov" and I think it would work in this case.

For the moment we assume that Z_t is a martingale.

Then $dQ = Z_T \, dP$ is a prob. measure and by Girsanov's theorem

$$B^Q_t = -\int_0^t \frac{r}{\sigma}(S_0 + \sigma B_s)\,ds + B_t = -\int_0^t \frac{r}{\sigma} S_s\,ds + B_t \text{ is a } Q\text{- B.m.}$$

and by Corollary 5.27, since $n = m = 1$ and σ is invertible, we can conclude the the market is complete.

By Theorem 5.32 the price of $F = B_T$ is given by $p(F) = E_Q[e^{-rT} B_T] = e^{-rT} E_Q[B_T]$

Moreover note that

$$dS_t = \sigma\, dB_t = \sigma \left\{ \frac{r}{\sigma} S_t\,dt + dB^Q_t \right\}$$

$$= r S_t\,dt + \sigma\, dB^Q_t$$

$$\Rightarrow S_t = S_0 e^{rt} + \sigma \int_0^t e^{r(t-u)}\,dB^Q_u$$

Therefore,

$$B_T = \frac{r}{\sigma}\int_0^T S_t\,dt + B^Q_T$$

$$= \frac{r}{\sigma}\int_0^T S_0 e^{rt}\,dt + r\int_0^T\int_0^t e^{r(t-u)}\,dB^Q_u\,dt + B^Q_T$$

$$\int_0^T\int_0^t r e^{r(t-u)}\,dB^Q_u\,dt \overset{\text{stoch. Fubini}}{=} \int_0^T\int_u^T r e^{r(t-u)}\,dt\,dB^Q_u$$

$$= \int_0^T \left(e^{r(T-u)} - 1\right) dB^Q_u$$

$$= \frac{S_0}{\sigma}\left(e^{rT}-1\right) + \int_0^T e^{r(T-u)}\, dB^Q_u - \cancel{B^Q_T} + \cancel{B^Q_T}$$

and

$$e^{-rT}B_T = \frac{S_0}{\sigma}\left(1-e^{-rT}\right) + \int_0^T \overbrace{e^{-ru}}^{=:\psi_u}\, dB^Q_u$$

the replicating strategy ξ will satisfy

$$e^{-rt}\xi(t)\sigma = \psi_t \Rightarrow \xi(t) = \frac{1}{\sigma}$$

And the price will be

$$p(F) = E_Q[F] = E_Q[e^{-rT}B_T] = \frac{S_0}{\sigma}\left(1-e^{-rT}\right)$$

Moreover

$$\xi_0(t) = E_Q[e^{-rT}B_T] + e^{-rt}A_t + \int_0^t r A_s e^{(r)}\, ds = (*)$$

where $A_t = \int_0^t \xi(u)\, dS_u - \xi(t)S_t = \frac{1}{\sigma}(S_t - S_0) - \frac{1}{\sigma}S_t$

$$= -\frac{S_0}{\sigma}$$

$$(*) = \frac{S_0}{\sigma}\left(1-e^{-rT}\right) + e^{-rt}\frac{S_0}{\sigma} + \frac{S_0}{\sigma}\left(e^{-rt}-1\right) = -\frac{S_0}{\sigma}e^{-rT}$$

$$V_0^\xi = \xi_0(0)S_0^0 + \xi_1(0)S_0 = -\frac{S_0}{\sigma}e^{-rT} + \frac{1}{\sigma}S_0$$

$$= \frac{S_0}{\sigma}\left(1-e^{-rT}\right) \checkmark$$

$$V_T^\xi = \xi_0(T)S_T^0 + \xi_1(T)S_T = -\frac{S_0}{\sigma} + \frac{1}{\sigma}S_T$$

$$= \frac{1}{\cancel{\sigma}}\left\{-\cancel{S_0} + \cancel{S_0} + \cancel{\sigma}B_T\right\} = B_T$$

c) $dS_t = \mu S_t dt + \sigma dB_t$

In this market consider $\theta \in L^0_{a,T}$ such that

$$\sigma\theta = \mu S_t - r S_t \qquad \lambda\otimes P\text{-a.e.}$$

$$\Rightarrow \quad \theta_t = \frac{(\mu - r) S_t}{\sigma}$$

Suppose that

$$Z_t := \exp\left(-\int_0^t \frac{(\mu-r)}{\sigma} S_t\, dt - \frac{1}{2}\int_0^t \frac{(\mu-r)^2}{\sigma^2} S_t^2\, dt\right.$$

is a martingale. Then under Q given by $dQ = Z_T\, dP$ we have that

$$B_t^Q = \int_0^t \frac{\mu - r}{\sigma} S_u\, du + B_t \quad \text{is a } Q\text{-B.m.}$$

By Theo. 5.16 and Corollary 5.27 we know that this market has no arbitrages and is complete..

The dynamics of S can be written as

$$dS_t = \mu S_t dt + \sigma dB_t$$

$$= r S_t dt + \sigma dB_t^Q$$

$$d(e^{-\mu t} S_t) = e^{-\mu t} dS_t - \mu e^{-\mu t} S_t dt$$

$$= \sigma e^{-\mu t} dB_t$$

$$\Rightarrow e^{-\mu t} S_t = S_0 + \int_0^t \sigma e^{-\mu u} dB_u$$

$$\Rightarrow \quad S_t = S_0 e^{\mu t} + \int_0^t \sigma e^{\mu(t-u)} dB_u$$

Similarly, we get

$$S_t = S_0 e^{rt} + \int_0^t \sigma e^{r(t-u)} dB_u^Q$$

We cannot write $F = B_T = H(S_T)$ for some H, so Corollary 5.35 or Theorem 5.34 cannot be applied. We need to find the mart. representation of $e^{-rT} B_T$ with respect to B^Q:

$$\boxed{\lambda := \frac{\mu - r}{\sigma}}$$

$$B_T = \int_0^T \frac{\mu - r}{\sigma} S_t \, dt + B_T^Q$$

$$= \int_0^T \lambda \left\{ S_0 e^{rt} + \int_0^t \sigma e^{r(t-u)} dB_u^Q \right\} dt + B_T^Q$$

$$= \int_0^T \lambda S_0 e^{rt} dt + \frac{\lambda\sigma}{r} \int_0^T \int_0^t r e^{r(t-u)} dB_u^Q \, dt + \int_0^T dB_T^Q$$

$$* \int_0^T \int_0^t r e^{r(t-u)} dB_u^Q \, dt \overset{\text{stoch. Fubini}}{=} \int_0^T \int_u^T r e^{r(t-u)} dt \, dB_u^Q$$

$$= \int_0^T \left(e^{r(T-u)} - 1\right) dB_u^Q$$

$$= \frac{\lambda}{r} S_0 \left(e^{rT} - 1\right) + \frac{\lambda\sigma}{r} \int_0^T \left\{ \left(e^{r(T-u)} - 1\right) + \frac{r}{\lambda\sigma} \right\} dB_u^Q$$

therefore

$$e^{-rT} B_T = \frac{\lambda}{r} S_0 (1 - e^{-rT}) + \frac{\lambda\sigma}{r} e^{-rT} \int_0^T \left\{ (e^{r(T-u)} - 1) + \frac{r}{\lambda\sigma} \right\} dB_u^Q$$

and $\psi_u = e^{-rT} \frac{\lambda\sigma}{r} \left\{ (e^{r(T-u)} - 1) + \frac{r}{\lambda\sigma} \right\}$

the replicating strategy ξ will satisfy

$$e^{-rt} \xi(t) \sigma = \psi_t \Rightarrow \xi(t) = e^{-r(T-t)} \frac{\lambda}{r} \left\{ (e^{r(T-t)} - 1) + \frac{r}{\lambda\sigma} \right\}$$

$$= e^{-r(T-t)} \frac{\lambda}{r} \left\{ e^{r(T-t)} + \frac{r - \lambda\sigma}{\lambda\sigma} \right\} =$$

$$= \frac{1}{\sigma} \left\{ \frac{\lambda\sigma}{r} + \frac{r - \lambda\sigma}{r} e^{-r(T-t)} \right\} = \frac{1}{\sigma} \left\{ \frac{\mu - r}{r} + \frac{2r - \mu}{r} e^{-r(T-t)} \right\}$$

$$= \frac{1}{\sigma} \left\{ -1 + \frac{\mu}{r} + \left(2 - \frac{\mu}{r}\right) e^{-r(T-t)} \right\}$$

and the price will be

$$p(F) = E_Q[e^{-rT} B_T] = \frac{\mu - r}{\sigma r} S_0 (1 - e^{-rT})$$

We choose ξ_0 such that (ξ, ξ_0) is self-financing.