Exercises Chapter 6: Infinitely Divisible Distributions

1) X, Y independ. rand. variables with densities f_X and f_Y, respect. Show that $X+Y$ has density

$$f_{X+Y}(x) = \int_{\mathbb{R}^d} f_X(x-y) f_Y(y) dy. \quad x \in \mathbb{R}^d.$$

We assume that X, Y take values in $\mathbb{R}^d$. Consider the $\mathbb{R}^d \times \mathbb{R}^d$-valued rand. variable $(U, V) := (X+Y, Y) = g(X, Y)$.

Thanks to the independence of X and Y, we have that

$$f_{X,Y}(x,y) = f_X(x) f_Y(y).$$

Next note that the function $g: \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d \times \mathbb{R}^d$, $(x,y) \longmapsto (x+y, y) =: (u,v)$ is a bijection and $g^{-1}: \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d \times \mathbb{R}^d$, $(u, v) \longmapsto (u-v, v)$. Moreover,

$$J_{g^{-1}}(u,v) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \Rightarrow |\det J_{g^{-1}}(u,v)| = 1.$$

Hence,

$$f_{U,V}(u,v) = f_{X,Y}(g^{-1}(u,v)) |\det J_{g^{-1}}(u,v)| \mathbb{1}_{g(\mathbb{R}^d \times \mathbb{R}^d)}(u,v)$$
$$= f_X(u-v) f_Y(v) \mathbb{1}_{\mathbb{R}^d \times \mathbb{R}^d}(u,v)$$

We are interested in the marginal density of U, which is obtained by integrating $f_{U,V}$ with respect to v.

$$f_U(u) = \int_{\mathbb{R}^d} f_X(u-v) f_Y(v) dv$$

which is the desired result.

$\boxed{2}$ $X \sim N(0,1)$. Show $\varphi_X(u) = e^{-\frac{u^2}{2}}$.

As $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ is symmetric with respect to 0.

we have

$$\varphi_X(u) = E[e^{iuX}] = \int_{\mathbb{R}} (\cos(ux) + i\sin(ux)) f_X(x)\,dx$$

$$= \int_{\mathbb{R}} \cos(ux) f_X(x)\,dx$$

Taking the derivative with respect to u (under the integral sign)

we get

$$\begin{cases} \varphi'_X(u) = -\int_{\mathbb{R}} x\sin(ux) f_X(x)\,dx = \left\{ \begin{array}{l} w = \sin(ux),\ dw = u\cos(ux)dx \\ dv = -x f_X(x)dx,\ v = f_X(x) \end{array} \right\} \\ \quad = \underbrace{\cancel{[\sin(ux) f_X(x)]_{-\infty}^{+\infty}}}_{0} - \int_{-\infty}^{+\infty} u\cos(ux) f_X(x)\,dx \\ \quad = -u\,\varphi_X(u) \\ \varphi_X(0) = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{\varphi'_X(u)}{\varphi_X(u)} = -u \\ \varphi_X(0) = 1 \end{cases} \Rightarrow \log(\varphi_X(u)) = K - \frac{u^2}{2}$$

$$\Downarrow$$

$$\varphi_X(u) = e^K e^{-\frac{u^2}{2}} \underset{\uparrow}{=} e^{-\frac{u^2}{2}}$$

$\varphi_X(0) = 1 \Rightarrow K = 0$

$\boxed{3}$ $X \sim \text{Poiss}(\lambda)$. Show that $\varphi_X(u) = \exp(\lambda(e^{iu}-1))$.

Recall that the law of a Poisson distribution is supported on the non-negative integers $\mathbb{Z}_+$ and it is given by

$$P_X(k) = e^{-\lambda} \frac{\lambda^k}{k!} \qquad k = 0, 1, 2 \ldots$$

$$\varphi_X(u) = E[e^{iuX}] = \sum_{k \geq 0} e^{iuk} P_X(k)$$

$$= \sum_{k \geq 0} e^{iuk} e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k \geq 0} \frac{(e^{iu}\lambda)^k}{k!}$$

$$= e^{-\lambda} e^{e^{iu}\lambda} = e^{\lambda(e^{iu}-1)}$$

4) $X \sim \Gamma(n,\lambda)$, $n \in \mathbb{N}$, $\lambda > 0$, $f_X(x) = \dfrac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!}$, $x > 0$

show that

$$(P_X)^{1/n} = P_Y \quad \text{with} \quad Y \sim Exp(\lambda).$$

$P_X^{1/n}$ is such that $\left(P_X^{1/n}\right)^{*n} = P_X$.

Hence, we only need to show that $P_Y^{*n} = P_X$

When $n = 1$, the result is clear because $\Gamma(1,\lambda) \sim Exp(\lambda)$.

The result follows by induction. Let $A = (0,a]$, $a > 0$.

$$P_Y^{*n}(A) = P_Y^{*(n-1)} * P_Y(A) \underset{\substack{\uparrow \\ \text{def of } * \text{ product}}}{=} \int_{\mathbb{R}\times\mathbb{R}} 1\!\!1_A(x+y)\, P_Y^{*(n-1)} \otimes P_Y(dx,y)$$

$$= \int_{\mathbb{R}\times\mathbb{R}} 1\!\!1_A(x+y)\, \frac{\lambda^{n-1} x^{n-2} e^{-\lambda x}}{(n-2)!} 1\!\!1_{(0,+\infty)}(x)\, \lambda e^{-\lambda y} 1\!\!1_{(0,+\infty)}(y)\, dx \otimes dy$$

$$= \int_D \frac{\lambda^n x^{n-2} e^{-\lambda(x+y)}}{(n-2)!}\, dx \otimes dy \qquad \left(\begin{aligned} D &= \{(x,y) \in \mathbb{R}^2 : 0 \le x+y \le a\} \\ &= \{0 \le x \le a,\ 0 \le y \le a-x\} \end{aligned} \right)$$

$$= \int_0^a \frac{\lambda^{n-1} x^{n-2} e^{-\lambda x}}{(n-2)!} \left(\int_0^{a-x} \lambda e^{-\lambda y}\, dy \right) dx$$

$$= \int_0^a \frac{\lambda^{n-1} x^{n-2} e^{-\lambda x}}{(n-2)!} \left(1 - e^{-\lambda(a-x)}\right) dx = (*)$$

To get the density of the law P_Y^{*n} we need to differentiate with respect to a.

$$\frac{d}{da}(*) = \frac{\lambda^{n-1} a^{n-2} e^{-\lambda a}}{(n-2)!} \underbrace{\cancel{\left(1 - e^{-\lambda(a-a)}\right)}}_{0} + \int_0^a \frac{\lambda^{n-1} x^{n-2}}{(n-2)!} e^{-\lambda x} \cdot \lambda e^{-\lambda(a-x)}\, dx$$

$$= \int_0^a \lambda^n \frac{x^{n-2}}{(n-2)!} e^{-\lambda a} \, dx = \lambda^n e^{-\lambda a} \int_0^a \frac{x^{n-2}}{(n-2)!} \, dx$$

$$= \lambda^n e^{-\lambda a} \frac{a^{n-1}}{(n-1)!}$$

which is the density of $X \sim \Gamma(n, \lambda)$

5)

μ is inf. div. $\iff$ For each $n \in \mathbb{N}$, $\exists (\mu)^{1/n} \in \mathcal{M}_1(\mathbb{R}^d)$

for which $\mathcal{E}_\mu(u) = \left(\mathcal{E}_{\mu^{1/n}}(u)\right)^n$ $u \in \mathbb{R}^d$.

μ is inf. divis. $\overset{\text{Def}}{\iff}$ For each $n \in \mathbb{N}$, $\exists \mu^{1/n} \in \mathcal{M}_1(\mathbb{R}^d)$ such that $(\mu^{1/n})^{*n} = \mu$.

Therefore we only need to prove that

$$\mathcal{E}_{(\mu^{1/n})^{*n}}(u) = \left(\mathcal{E}_{\mu^{1/n}}(u)\right)^n$$

Using Proposition 4.3. and induction it follows that

$$\forall n \in \mathbb{N}, \int_{\mathbb{R}^d} f(z)(\mu_1 * \cdots * \mu_n)(dz) = \int_{\underbrace{\mathbb{R}^d \times \cdots \times \mathbb{R}^d}_{n \text{ times}}} f(x_1 + \cdots + x_n)\, \mu_1 \otimes \cdots \otimes \mu_n(dx_1, \ldots, dx_n)$$

for $f \in \mathcal{B}_b(\mathbb{R}^d)$.

Applying this result to $e^{i\langle u, z\rangle}$ and $\mu_i = \mu^{1/n}$ $i = 1, \ldots, n$.

we get

$$\mathcal{E}_{(\mu^{1/n})^{*n}}(u) = \int_{\mathbb{R}^d} e^{i\langle u, z\rangle} (\mu^{1/n})^{*n}(dz)$$

$$= \int_{\mathbb{R}^d \times \cdots \times \mathbb{R}^d} e^{i\langle u, x_1 + \cdots + x_n\rangle} (\mu^{1/n})^{\otimes n}(dx_1, \ldots, dx_n)$$

$$\overset{\text{Fubini}}{=} \int_{\mathbb{R}^d} e^{i\langle u, x_1\rangle} \mu^{1/n}(dx_1) \cdots \int_{\mathbb{R}^d} e^{i\langle u, x_n\rangle} \mu^{1/n}(dx_n)$$

$$= \mathcal{E}_{\mu^{1/n}}(u) \cdots \mathcal{E}_{\mu^{1/n}}(u) = \left(\mathcal{E}_{\mu^{1/n}}(u)\right)^n.$$

6) Let X and Y be two indep. inf. divis. r.v.
Prove that $X+Y$ is also infin. divisible.

By Corollary 4.5 with $f = \mathbb{1}_A$, $A \in \mathcal{B}(\mathbb{R}^d)$ we can conclude that

$$P_{X+Y} = P_X * P_Y .$$

As X and Y are inf. divis., for each $n \in \mathbb{N}$ there exists $P_X^{1/n}, P_Y^{1/n} \in M_1(\mathbb{R}^d)$ such that $P_X = (P_X^{1/n})^{*n}$ and $P_Y = (P_Y^{1/n})^{*n}$.

Therefore,

$$P_{X+Y} = P_X * P_Y = (P_X^{1/n})^{*n} * (P_Y^{1/n})^{*n} = (P_X^{1/n} * P_Y^{1/n})^{*n}$$

By Prop. 4.3 (2) and (3) Commutativity and Associativity of the conv. product.

By Prop. 4.2 we know that $P_X^{1/n} * P_Y^{1/n} \in M_1(\mathbb{R}^d)$.

Hence, we can conclude by using Prop. 4.7, which shows the equivalence between Z being inf. divis. and P_Z having a convolution nth root that is itself a random variable.

In our case $Z = X+Y$ and $P_{X+Y}^{1/n} = P_X^{1/n} * P_Y^{1/n}$.

$\boxed{7}$ ν a Lévy measure on $\mathbb{R}^d$. Then, for all $\varepsilon > 0$

$\nu(\{y \in \mathbb{R}^d : |y|^2 > \varepsilon\}) < +\infty$ and conclude

that ν is σ-finite

Since ν is a Lévy measure we know that

$$\int_{\mathbb{R}^d} (1 \wedge |y|^2)\, \nu(dy) < +\infty \iff (1) \int_{|y|^2<1} |y|^2 \nu(dy) < +\infty \text{ and}$$

$$(2) \int_{|y|^2 \geq 1} \nu(dy) < +\infty.$$

If $\varepsilon \geq 1$ then (2) implies the result.

If $\varepsilon < 1$ then

$$\nu(\{y \in \mathbb{R}^d : |y|^2 > \varepsilon\}) = \int_{|y|^2 > \varepsilon} \nu(dy) = \int_{\varepsilon < |y|^2 < 1} \nu(dy) + \int_{|y|^2 \geq 1} \nu(dy)$$

The second integral is finite by (2).

$$\int_{\varepsilon < |y|^2 < 1} \nu(dy) = \frac{1}{\varepsilon} \int_{\varepsilon < |y|^2 < 1} \varepsilon\, \nu(dy) < \frac{1}{\varepsilon} \int_{\varepsilon < |y|^2 < 1} |y|^2 \nu(dy)$$

$$\underset{\substack{\uparrow \\ |y|^2 \geq 0 \\ \text{and } \{\varepsilon < |y|^2 < 1\} \subset \{|y|^2 < 1\}}}{<} \frac{1}{\varepsilon} \int_{|y|^2 < 1} |y|^2 \nu(dy) \underset{\substack{\uparrow \\ \text{By } (1).}}{<} +\infty$$

$$\mathbb{R}^d_0 = \biguplus_{n \geq 1} A_n, \quad \text{where} \quad A_1 = \{|y|^2 > 1\}, \quad A_n = \left\{\frac{1}{n} < |y|^2 \leq \frac{1}{n-1}\right\}, \; n \geq 2$$

Clearly $\nu(A_1) < +\infty$ and

$$\nu(A_n) \underset{\substack{\uparrow \\ A_n \subset \{|y|^2 > 1/n\}}}{\leq} \nu(|y|^2 > 1/n) < +\infty.$$

8 Show that the Lévy symbol η of an inf. divis. distribution μ is continuous at every $u \in \mathbb{R}^d$.

$$\eta(u) = i\langle b, u\rangle - \frac{1}{2}\langle u, Au\rangle + \int_{\mathbb{R}^d_0} \left(e^{i\langle u,y\rangle} - 1 - i\langle u,y\rangle \mathbb{1}_{B_1(0)}(y)\right)\nu(dy)$$

As $\langle \cdot,\cdot\rangle$ is continuous we only need to prove that

$$\psi(u) = \int_{\mathbb{R}^d_0} \underbrace{\left(e^{i\langle u,y\rangle} - 1 - i\langle u,y\rangle \mathbb{1}_{B_1(0)}(y)\right)}_{\xi(u,y)} \nu(dy)$$

is continuous with respect to u, that is, for any $u_n \xrightarrow[n\to\infty]{} u$

$$\psi(u) = \lim_{n\to\infty} \psi(u_n)$$

As $\xi(u,y)$ is continuous with respect to u, the result follows from dominated convergence. In order to apply dominated convergence we need to bound $|\xi(u_n,y)|$ by any $\tilde{\xi}(u) \in L^1(\nu)$

In the proof of Theorem 4.14 we have that

$$|\xi(u_n,y)| \le \frac{1}{2}|u_n|^2|y|^2 \mathbb{1}_{B_1(0)}(y) + 2\,\mathbb{1}_{B_1^c(0)}(y)$$

$$\le \frac{1}{2}\sup_n |u_n|^2 |y|^2 \mathbb{1}_{B_1(0)}(y) + 2\,\mathbb{1}_{B_1^c(0)}(y) \in L^1(\nu)$$

Note that $\sup_n |u_n|^2 < +\infty$, because $u_n \xrightarrow[n\to\infty]{} u$.

9 Show that the Lévy symbol η of an inf. divis. distribution μ satisfies

$$|\eta(u)| \le C(1+|u|^2) \qquad u \in \mathbb{R}^d, \; C>0.$$

We have

$$\eta(u) = i\langle \gamma, u\rangle - \frac{1}{2}\langle u, Au\rangle + \int_{\mathbb{R}^d_0} \left(e^{i\langle u,y\rangle} - 1 - i\langle u, y\rangle \mathbb{1}_{B_1(0)}(y)\right)\nu(dy)$$

By the triangular inequality and Cauchy-Schwarz

$$\left|i\langle \gamma, u\rangle - \frac{1}{2}\langle u, Au\rangle\right| \le |\langle \gamma, u\rangle| + \frac{1}{2}|\langle u, Au\rangle|$$

$$\le |\gamma|\,|u| + \frac{1}{2}|u|\,|Au|$$

- $|u| \le 1 + |u|^2$
- $|Au| \le \|A\|_2 |u|$

$$\le |\gamma| + \left(\frac{1}{2}\|A\|_2 + 1\right)|u|^2$$

In Theo. 4.14 we showed that

$$\left|e^{i\langle u,y\rangle} - 1 - i\langle u, y\rangle \mathbb{1}_{B_1(0)}(y)\right| \le \frac{1}{2}|u|^2|y|^2 \mathbb{1}_{B_1(0)}(y) + 2\,\mathbb{1}_{B_1^c(0)}(y)$$

and, therefore,

$$\left|\int_{\mathbb{R}^d_0}\left(e^{i\langle u,y\rangle} - 1 - i\langle u, y\rangle \mathbb{1}_{B_1(0)}(y)\right)\nu(dy)\right| \le \frac{1}{2}\left(\int_{B_1(0)} |y|^2\nu(dy)\right)|u|^2 + 2\int_{B_1^c(0)}\nu(dy)$$

The result follows with

$$C = \max\left(|\gamma| + 2\int_{B_1^c(0)}\nu(dy),\; \frac{1}{2}\|A\|_2 + 1 + \frac{1}{2}\int_{B_1(0)}|y|^2\nu(dy)\right)$$

$\boxed{10}$ If μ is I.D then, for every $t \in \mathbb{R}_+$, μ^t is well defined and I.D

We have a distribution $\mu^{1/n}$ for any $n \in \mathbb{N}$, because μ is I.D.
It is I.D., since $\varphi_{\mu^{1/n}}(z) = \varphi_\mu(z)^{1/n} = \left(\varphi_\mu(z)^{1/nk}\right)^k$ for any $k \in \mathbb{N}$.

$$= \left(\varphi_{\mu^{1/nk}}(z)\right)^k$$

Hence for any $m, n \in \mathbb{N}$, $\mu^{m/n}$ is also I.D.
because

$$\mu^{m/n} = \mu^{1/n} \overset{m \text{ times}}{* \cdots *} \mu^{1/n}$$

and the convolution product of I.D. distributions is I.D. (See Exercise 6).

For any real number t, choose rational numbers $r_n \xrightarrow[n\to\infty]{} t$.

Then,

$$\varphi_{\mu^{r_n}}(z) = \varphi_\mu(z)^{r_n} \xrightarrow[n\to\infty]{} \varphi_\mu(z)^t \qquad \forall z \in \mathbb{R}^d.$$

$\varphi_\mu(z)^t$ is well defined but does not need to be the characteristic function of a distribution.
Since $\varphi_\mu(z)^t$ is a continuous function (due to the construction of the distinguished logarithm and $\varphi_\mu(z) \neq 0$.), it is continuous at 0. and, by Levy's continuity theorem Theor 2.28. we know that $\varphi_\mu(z)^t$ is indeed a charact. function. By Cramer-Wold's theorem $\mu^{r_n} \xrightarrow[n\to\infty]{h} \mu^t$ and μ^t is I.D. by Prop. 6.20.