Exercises

Chapter 7. Lévy Processes

[1] X a Lévy process (γ, A, ν).

a) $-X = \{-X_t\}_{t\in\mathbb{R}}$ is a L.P. with charact. $(-\gamma, A, \tilde{\nu})$, $\tilde{\nu}(B) = \nu(-B)$, $B \in \mathcal{B}(\mathbb{R}^d)$

b) $\{X_t + ct\}_{t\geq 0}$ is a L.P. find its characteristics.

$\boxed{\text{Check def. 5.1.}}$

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a) (1) $-X_0 = -1 \cdot 0 = 0$; P.a.s.

(2) Let $0 \leq t_0 < t_1 < \ldots < t_m \in \mathbb{R}_+$, $m \in \mathbb{N}$. $\{u_j\}_{j=1,\ldots,m} \in \mathbb{R}^d$

$$E\left[e^{i\sum_{j=1}^m \langle u_j, -X_{t_j} + X_{t_{j-1}}\rangle}\right] = E\left[e^{i\sum_{j=1}^m \langle -u_j, X_{t_j} - X_{t_{j-1}}\rangle}\right]$$

$$\underset{\substack{\uparrow \\ X \text{ has} \\ \text{ind. incren.}}}{=} \prod_{j=1}^m E\left[e^{i\langle -u_j, X_{t_j} - X_{t_{j-1}}\rangle}\right] = \prod_{j=1}^m E\left[e^{i\langle u_j, -X_{t_j} + X_{t_{j-1}}\rangle}\right]$$

(3) For any $s < t$, $\mathcal{L}(-X_t + X_s) = \mathcal{L}(-X_{t-s})$?

$u \in \mathbb{R}^d$, $E\left[e^{i\langle u, -X_t + X_s\rangle}\right] = E\left[e^{i\langle -u, X_t - X_s\rangle}\right]$

$$\underset{\substack{\uparrow \\ X \text{ has stat. incren.}}}{=} E\left[e^{i\langle -u, X_{t-s}\rangle}\right]$$

$$= E\left[e^{i\langle u, -X_{t-s}\rangle}\right] \checkmark$$

(4) For any $a > 0$,

$$\lim_{h\to 0} P(|-X_{t+h} + X_t| > a) = \lim_{h\to 0} P(|X_{t+h} - X_t| > a) \underset{\substack{\uparrow \\ X \text{ is} \\ \text{cont. in} \\ \text{probab.}}}{=} 0$$

The charact. of $-X$ ~~can~~ be extracted from the characteristic function of $-X_1$

$$E\left[e^{i\langle u, -X_1\rangle}\right] = E\left[e^{i\langle -u, X_1\rangle}\right] =$$

$$= \exp\Big(i\langle -u, \gamma\rangle - \frac{1}{2}\langle -u, A(-u)\rangle$$

$$+ \int_{\mathbb{R}^d_0} \{ e^{i\langle -u, y\rangle} - 1 - i\langle -u, y\rangle \mathbb{1}_{B_1(0)} \} \, \nu(dy) \Big)$$

$$= \exp\Big(i\langle u, -\gamma\rangle - \frac{1}{2}\langle u, Au\rangle$$

$$+ \int_{\mathbb{R}^d_0} \{ e^{i\langle u, -y\rangle} - 1 - i\langle u, -y\rangle \mathbb{1}_{B_1(0)}(y) \} \, \nu(dy) \Big) \qquad (*)$$

Image - measure Th.

$$\int_E f\circ T \, d\mu = \int_F f \, d\mu_T \quad , \text{ where } T: E \to F$$

$T: \mathbb{R}^d_0 \to \mathbb{R}^d_0$, $f(x) = e^{i\langle u, x\rangle} - 1 - i\langle u, x\rangle \mathbb{1}_{B_1(0)}(x)$

$y \mapsto x = -y$

↑ Note that

$\mathbb{1}_{B_1(0)}(y) = \mathbb{1}_{B_1(0)}(-y)$

$\mu = \nu$ and $\mu_T = \tilde{\nu}$.

Hence, we get

$$(*) = \exp\Big(i\langle u, -\gamma\rangle - \frac{1}{2}\langle u, Au\rangle + \int_{\mathbb{R}^d_0} \{ e^{i\langle u, x\rangle} - 1 - i\langle u, x\rangle \mathbb{1}_{B_1(0)}(x) \} \, \tilde{\nu}(dx) \Big)$$

$\Rightarrow -X$ has charact. $(-\gamma, A, \tilde{\nu})$.

$\boxed{2}$ If X and Y are stoch. cont. processes then so is their sum

We have that for all $a > 0$

$$\lim_{h \to 0} P(|X_{t+h} - X_t| > a) = 0 \text{ and } \lim_{h \to 0} P(|Y_{t+h} - Y_t| > a) = 0$$

We can write

$$P(|X_{t+h} + Y_{t+h} - X_t - Y_t| > a)$$

$$\leq P\left(|X_{t+h} - X_t| > \frac{a}{2}\right) + P(|Y_{t+h} - Y_t| > a/2)$$

Taking $\overline{\lim}$ on the previous inequality we obtain

$$\overline{\lim_{h \to 0}} P(|X_{t+h} + Y_{t+h} - X_t - Y_t| > a) = 0$$

but as probabilities are non-negative we conclude that the limit exists and it is 0, which yields the stochastic continuity of $X + Y$.

b) $X_t + ct$ is a L.p.

(1) $X_0 + c \cdot 0 = 0$, P-a.s.

(2) Let $0 = t_0 < t_1 < \dots < t_n \in \mathbb{R}_+$, $n \in \mathbb{N}$, $\{u_j\}_{j=1,\dots,n} \subseteq \mathbb{R}^d$.

$$E[e^{i\sum_{j=1}^n \langle u_j, X_{t_j} - X_{t_{j-1}} + c(t_j - t_{j-1})\rangle}]$$

$$= e^{i\sum_{j=1}^n \langle u_j, c(t_j - t_{j-1})\rangle} E[e^{i\sum_{j=1}^n \langle u_j, X_{t_j} - X_{t_{j-1}}\rangle}]$$

$$= \prod_{j=1}^n e^{i\langle u_j, c(t_j - t_{j-1})\rangle} E[e^{i\langle u_j, X_{t_j} - X_{t_{j-1}}\rangle}]$$

$$= \prod_{j=1}^n E[e^{i\langle u_j, X_{t_j} - X_{t_{j-1}} + c(t_j - t_{j-1})\rangle}]$$ ✓

(3) Let $s \le t$.

$$E[e^{i\langle u, X_t + ct - X_s - cs\rangle}] = E[e^{i\langle u, X_t - X_s\rangle}] e^{i\langle u, c(t-s)\rangle}$$

$$= E[e^{i\langle u, X_{t-s}\rangle}] e^{i\langle u, c(t-s)\rangle} = E[e^{i\langle u, X_{t-s} + c(t-s)\rangle}]$$

(4) $$P(|X_{t+h} + c(t+h) - X_t - ct| > a)$$

$$= P(|X_{t+h} - X_t + ch| > a) \le P(|X_{t+h} - X_t| + |ch| > a)$$

$$= P(|X_{t+h} - X_t| > a - |ch|)$$

$$\underset{|h| < \frac{a}{2c}}{\le} P(|X_{t+h} - X_t| > \tfrac{a}{2})$$

Taking $\overline{\lim}$ in the previous inequality we get that the limit exists and it is zero.

3 The sum of two indep. Lévy processes X and Y is again a Lévy process

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(1) $X_0 + Y_0 = 0$, P-a.s ✓

(2) Let $0 \leq t_0 < t_1 < \dots < t_m \in \mathbb{R}_+$, $m \in \mathbb{N}$, $\{u_j\}_{j=1,\dots,m} \subseteq \mathbb{R}^d$

$$E[e^{i\sum_{j=1}^{m}\langle u_j, X_{t_j}+Y_{t_j}-X_{t_{j-1}}-Y_{t_{j-1}}\rangle}] = E[e^{i(\sum_{j=1}^{m}\langle u_j, X_{t_j}-X_{t_{j-1}}\rangle + \sum_{j=1}^{m}\langle u_j, Y_{t_j}-Y_{t_{j-1}}\rangle)}]$$

X and Y are ind. $= E[e^{i\sum_{j=1}^{m}\langle u_j, X_{t_j}-X_{t_{j-1}}\rangle}]\, E[e^{i\sum_{j=1}^{m}\langle u_j, Y_{t_j}-Y_{t_{j-1}}\rangle}]$

X and Y have indep incr. $= \prod_{j=1}^{m} E[e^{i\langle u_j, Y_{t_j}-Y_{t_{j-1}}\rangle}] \prod_{j=1}^{m} E[e^{i\langle u_j, X_{t_j}-Y_{t_{j-1}}\rangle}]$

X and Y are indep. $= \prod_{j=1}^{m} E[e^{i\langle u_j, X_{t_j}+Y_{t_j}-X_{t_{j-1}}-Y_{t_{j-1}}\rangle}]$ ✓

(3) $0 \leq s \leq t$, $u \in \mathbb{R}^d$

$$E[e^{i\langle u, X_t+Y_t-X_s-Y_s\rangle}] = E[e^{i\langle u, X_t-X_s\rangle} e^{i\langle u, Y_t-Y_s\rangle}]$$

X and Y are indep. $= E[e^{i\langle u, X_t-X_s\rangle}]\, E[e^{i\langle u, Y_t-Y_s\rangle}]$

X and X have stat. increm. $= E[e^{i\langle u, X_{t-s}\rangle}]\, E[e^{i\langle u, Y_{t-s}\rangle}]$

X and X are indep. $= E[e^{i\langle u, Y_{t-s}+Y_{t-s}\rangle}]$ ✓

(4) It follows from exercise 2 in this list and the hypothesis that X and Y are stochastically continuous.

4) Compulsory assignment.

5) Prove that

a) $\mu^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{+\infty} (1 - e^{-\mu x}) \frac{dx}{x^{1+\alpha}}$, $\mu \geq 0$, $0 < \alpha < 1$

$$\int_0^{+\infty} (1-e^{-\mu x})\, x^{-1-\alpha}\, dx = -\int_0^{+\infty} \left(\int_0^x \mu e^{-\mu y} dy \right) x^{-1-\alpha} dx$$

$$= \left\{ \begin{matrix} 0 \le x < +\infty \\ 0 \le y \le x \end{matrix} \right\} \Leftrightarrow \left\{ \begin{matrix} y \le x < +\infty \\ 0 \le y < +\infty \end{matrix} \right\}$$

$$\overset{\text{Fubini}}{=} - \int_0^{+\infty} \Big(\underbrace{\int_y^{+\infty} x^{-1-\alpha} dx}_{= \frac{y^{-\alpha}}{\alpha}} \Big) \mu e^{-\mu y} dy$$

$$= \frac{\mu}{\alpha} \int_0^{+\infty} e^{-\mu y} y^{-\alpha} dy$$

$$= \left\{ \begin{matrix} x := \mu y \\ dx = \mu dy \end{matrix} \right\} = \frac{\mu^\alpha}{\alpha} \int_0^{+\infty} e^{-x} x^{-\alpha} dx$$

$$= \frac{\mu^\alpha}{\alpha} \Gamma(\alpha - 1) .$$

and the result follows.

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5) b) Let $a, b > 0$, $\mu \geq 0$. $t \geq 0$

$$\int_0^{+\infty} e^{-\mu x} \frac{b^{at}}{\Gamma(at)} x^{at-1} e^{-bx} dx = \left(1 + \frac{\mu}{b}\right)^{-at}.$$

$$\int_0^{+\infty} \frac{b^{at}}{\Gamma(at)} x^{at-1} e^{-(\mu+b)x} dx = \frac{b^{at}}{(\mu+b)^{at}} \underbrace{\int_0^{+\infty} \frac{(\mu+b)^{at}}{\Gamma(at)} x^{at-1} e^{-(\mu+b)x} dx}_{=1} = \frac{\cancel{b^{at}}}{\cancel{b^{at}}} \frac{1}{(\frac{\mu}{b}+1)^{at}} \checkmark$$

$\boxed{6}$ Mock exam

$\boxed{7}$ Let $\mu_t(dx) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx \quad t > 0$

$\mu_0(dx) = \delta_0$

Show that μ_t is weakly continuous convolution semigroup.

· Convolution semigroup

μ_s and μ_t are prob. meas. on $\mathbb{R}$. $\overset{\text{Prop. 4.2}}{\Rightarrow}$ $\mu_s * \mu_t$ is a prob. meas on $\mathbb{R}$

The charact. function of $\mu_s * \mu_t$ is given by

$$\varphi_{\mu_s * \mu_t}(u) = \int_{\mathbb{R}} e^{iux} \mu_s * \mu_t(dx)$$

$$(\text{Prop. 4.3 1)}) \quad = \int_{\mathbb{R}\times\mathbb{R}} e^{iu(x+y)} \mu_s \otimes \mu_t(dx,dy)$$

$$(\text{Fubini}) \quad = \int_{\mathbb{R}} e^{iux} \mu_s(dx) \int_{\mathbb{R}} e^{iuy} \mu_t(dy)$$

$$\left(\varphi_{N(0,\sigma^2)}(u) = e^{-\frac{\sigma^2 u^2}{2}}\right) \quad = e^{-\frac{su^2}{2}} e^{-\frac{tu^2}{2}}$$

$$= e^{-\frac{(s+t)u^2}{2}} = \varphi_{\mu_{s+t}}(u)$$

$$\Rightarrow \quad \mu_s * \mu_t = \mu_{s+t}$$

Weakly continuous at 0.

$f \in C_b$, $\int_{\mathbb{R}} f(x) \mu_t(dx) = \int_{\mathbb{R}} f(y\sqrt{t}) \mu_1(dy) \quad \left(\begin{array}{l} y = \frac{x}{\sqrt{t}} \\ dy = \frac{dx}{\sqrt{t}} \end{array}\right)$

$\lim_{t\to 0} f(y\sqrt{t}) = f(0)$ and $f(y\sqrt{t}) \le \|f\|_\infty \in L^1(\mu_1(dy))$

By dominated convergence $\lim_{t\to 0} \int_{\mathbb{R}} f(y\sqrt{t}) \mu_1(dy) = \int_{\mathbb{R}} f(0) \mu_1(dy) = f(0)$ ✓