Chapter 8

[1] a), b) and c) I ignore these exercises.
They only present notational complexity.

d) $\tilde{N} = \{\tilde{N}_t\}_{t\geq 0} = \{N_t - \lambda t\}_{t\geq 0}$ where N is a Poisson process with intensity $\lambda > 0$.
Let $\mathbb{F} = \mathbb{F}^N$ be the minimal augmented filtration generated by N.

- $\tilde{N}_t$ is $\mathcal{F}_t$-measurable.
- $E[|\tilde{N}_t|] \underset{\substack{\uparrow \\ N_t \geq 0}}{\leq} E[N_t] + \lambda t \underset{\substack{\uparrow \\ N_t \sim \text{Poiss}(\lambda t)}}{<} +\infty$
- $E[\tilde{N}_t \mid \mathcal{F}_s] = E[\tilde{N}_t - \tilde{N}_s + \tilde{N}_s \mid \mathcal{F}_s]$

(linearity of cond. expect. & $\tilde{N}_s$ is $\mathcal{F}_s$-meas.) $= E[\tilde{N}_t - \tilde{N}_s \mid \mathcal{F}_s] + \tilde{N}_s$

$= E[N_t - N_s \mid \mathcal{F}_s] - \lambda(t-s) + \tilde{N}_s$

($N_t - N_s$ is indep. of $\mathcal{F}_s$ because N is a Lévy process) $= E[N_t - N_s] - \lambda(t-s) + \tilde{N}_s$

($N_t \sim \text{Pois}(\lambda t)$, $N_s \sim \text{Pois}(\lambda s)$.) $= \lambda t - \lambda s - \lambda(t-s) + \tilde{N}_s = \tilde{N}_s$ ✓

e) Note that $\tilde{N}$ is a Lévy process because it is the sum of two indep. Lévy processes.

$\hat{N} = \{\hat{N}_t\}_{t\geq 0} = \{(\tilde{N}_t)^2 - \lambda t\}_{t\geq 0}$. N and $\mathbb{F}$ as in the previous exercise.

- $\hat{N}_t$ is $\mathcal{F}_t$-meas. because it is obtained by applying a Borel measurable function to $\tilde{N}_t$, which is $\mathcal{F}_t$-measurable.

- $E[\,|\hat{N}_t|\,] \le E[\,|\tilde{N}_t|^2\,] + \lambda t$,

and

$$E[\,|\tilde{N}_t|^2\,] \le 2\,E[\,|N_t|^2 + |\lambda t|^2\,]$$

$$\le 2\,(\,E[N_t^2] + |\lambda t|^2) < +\infty$$

↑ $N_t \sim \text{Poiss}(\lambda t)$

- Note that

$$\hat{N}_t = (\tilde{N}_t)^2 - \lambda t = (\tilde{N}_t - \tilde{N}_s + \tilde{N}_s)^2 - \lambda t$$

$$= (\tilde{N}_t - \tilde{N}_s)^2 + 2(\tilde{N}_t - \tilde{N}_s)\tilde{N}_s + (\tilde{N}_s)^2 - \lambda s - \lambda(t-s)$$

Taking cond. expect.

$$E[\hat{N}_t \mid \mathcal{F}_s] \overset{(1)}{=} E[(\tilde{N}_t - \tilde{N}_s)^2 \mid \mathcal{F}_s] + 2\tilde{N}_s\,E[\tilde{N}_t - \tilde{N}_s \mid \mathcal{F}_s] + \hat{N}_s - \lambda(t-s)$$

$$= E[(\tilde{N}_t - \tilde{N}_s)^2] + 2\tilde{N}_s\,E[\tilde{N}_t - \tilde{N}_s] + \hat{N}_s - \lambda(t-s)$$

$$= \lambda(t-s) + 0 + \hat{N}_s - \lambda(t-s) = \hat{N}_s \quad ✓$$

(1) $\tilde{N}_s$ is $\mathcal{F}_s$-meas.

(2) $\tilde{N}_t - \tilde{N}_s$ is indep. of $\mathcal{F}_s$ because $\tilde{N}$ is a Lévy process.

(3) $E[\tilde{N}_t] = E[\tilde{N}_s] = 0$ because $\tilde{N}$ is a martingale starting at 0.

- $E[(\tilde{N}_t - \tilde{N}_s)^2] = E[(N_t - N_s)^2 - 2\lambda(t-s)(N_t - N_s) + \lambda^2(t-s)^2]$

$$= E[(N_t - N_s)^2] - 2\lambda(t-s)E[N_t - N_s] + \lambda^2(t-s)^2$$

$\left(\begin{array}{l} Var[Z] = E[Z^2] - (E[Z])^2 \\ N_t - N_s \sim N_{t-s} \end{array} \right)$ $= Var[N_{t-s}] + (E[N_{t-s}])^2 - 2\lambda(t-s)\,E[N_{t-s}] + \lambda^2(t-s)$

$\left(\begin{array}{l} E[N_{t-s}] = \lambda(t-s) \\ Var[N_{t-s}] = \lambda(t-s) \end{array} \right)$ $= \lambda(t-s) + \lambda^2(t-s)^2 - 2\lambda^2(t-s)^2 + \lambda^2(t-s) = \lambda(t-s)$

2 B one dim. stand. B.m. $\mathbb{F} = \mathbb{F}^B$.

$$S_t := \inf\{s>0 : B_s = \frac{t}{\sqrt{2}}\} \qquad t \geq 0.$$

For each $t \geq 0$, S_t is a stopping time. (with respect to $\mathbb{F}$).

For each $\theta \in \mathbb{R}$. $M_t(\theta) = \exp(\theta B_t - \frac{1}{2}\theta^2 t)$ is an $\mathbb{F}$-martingale.

By Doob's optional stopping theorem, for $t \geq 0$, $n \in \mathbb{N}$, we have that $E[M_T(\theta) \mid \mathcal{F}_S] = M_S(\theta)$ for all bounded stopping times $S \leq T$, P.a.s. In particular, $E[M_T(\theta)] = E[M_S(\theta)]$

Take $S=0$ and $T = n \wedge S_t$ ← bounded., then

$$1 = E[M_0(\theta)] = E[M_{n\wedge S_t}] = E[\exp(\theta B_{n\wedge S_t} - \frac{1}{2}\theta^2(n\wedge S_t))]$$

For each $n \in \mathbb{N}$, $t \geq 0$ consider the set

$$A_{n,t} = \{\omega \in \Omega : S(t) \leq n\}$$

Then

$$E[\exp(\theta B_{n\wedge S_t} - \frac{1}{2}\theta^2(n\wedge S_t))] = E[\exp(\theta B_{n\wedge S_t} - \frac{1}{2}\theta^2(n\wedge S_t))(\mathbb{1}_{A_{n,t}} + \mathbb{1}_{A^c_{n,t}})]$$

$$= E[\exp(\theta B_{S_t} - \frac{1}{2}\theta^2 S_t)\mathbb{1}_{A_{n,t}}] + e^{-\frac{1}{2}\theta^2 n} E[e^{\theta B_n}\mathbb{1}_{A^c_{n,t}}]$$

But, for $\omega \in A^c_{n,t}$, $S_t(\omega) > n \Rightarrow B_n < t/\sqrt{2}$ and, therefore,

$$e^{-\frac{1}{2}\theta^2 n} E[e^{\theta B_n}\mathbb{1}_{A^c_{n,t}}] < \exp(-\frac{1}{2}\theta^2 n + \frac{t\theta}{\sqrt{2}}) \xrightarrow[n\to\infty]{} 0$$

Note that when $n \to \infty$ $A_{n,t} \uparrow \Omega$ and

By the monotone convergence theorem

$$1 = E[\exp(\theta B_{S_t} - \frac{1}{2}\theta^2 S_t)] = e^{\frac{\theta t}{\sqrt{2}}} E[e^{-\frac{1}{2}\theta^2 S_t}]$$

$$B_{S_t} = \frac{t}{\sqrt{2}}$$

$$\Rightarrow E[e^{-\frac{1}{2}\theta^2 S_t}] = e^{-\frac{\theta t}{\sqrt{2}}}$$

Making the substitution $\theta = \sqrt{2u}$ we get

$$E[\exp(-u S_t)] = e^{-t u^{1/2}}$$

which is the Laplace exponent of the Lévy subordinator.

(α-stable subordinator with $\alpha = 1/2$)