Exercises

1. Suppose we have

$$dX_t = \alpha dt + \sigma dB_t + \int_{|x|<1} h(x)\,\tilde{N}(dt,dx) + \int_{|x|\geq 1} k(x)\,N(dt,dx),$$
$$X_0 = x_0,$$

where α, σ are constants and h and k are given functions from $\mathbb{R} \to \mathbb{R}$.

   (a) Use Itô's formula to find dY_t when $Y_t = \exp(X_t)$.

   (b) How do we choose α, σ, h and k if we want Y_t to solve the SDE

$$dY_t = Y_{t-}\left(\beta dt + \theta dB_t + \lambda \int_{|x|<1} x\tilde{N}(dt,dx) + \rho \int_{|x|\geq 1} xN(dt,dx)\right),$$

   for given constants β, θ, λ and ρ.

2. Solve the following Lévy SDEs:

   (a) Let m, σ, λ be constants and

$$dX_t = (m - X_t)\,dt + \sigma dB_t + \lambda \int_{|x|<1} x\tilde{N}(dt,dx) + \rho \int_{|x|\geq 1} xN(dt,dx),$$
$$X_0 = x_0.$$

   (b) Let

$$dX_t = \alpha dt + \gamma X_{t-}\left(\int_{|x|<1} x\tilde{N}(dt,dx) + \int_{|x|\geq 1} xN(dt,dx)\right),$$
$$X_0 = x_0.$$

3. Let $h \in L^2(\mathbb{R})$ be deterministic and define

$$Y_t = \exp\left\{\int_0^t \int_{\mathbb{R}_0} h(s)\,x\tilde{N}(ds,dx) - \int_0^t \int_{\mathbb{R}_0} \left(e^{h(s)x} - 1 - h(s)\,x\right)\nu(dx)\,ds\right\}.$$

   Show that

$$dY_t = Y_{t-}\int_{\mathbb{R}_0} \left(e^{h(t)x} - 1\right)\tilde{N}(dt,dx).$$

4. Show that, under some conditions on $\gamma(s,x)$ (assumed to be deterministic) we have

$$\mathbb{E}\left[\exp\left(\int_0^t \int_{\mathbb{R}_0} \gamma(s,x)\,\tilde{N}(ds,dx)\right)\right] = \exp\left(\int_0^t \int_{\mathbb{R}_0} \left\{e^{\gamma(s,x)} - 1 - \gamma(s,x)\right\}\nu(dx)\,ds\right).$$

5. Let

$$dX_t^i = \int_{\mathbb{R}_0} \gamma_i(t,x)\,\tilde{N}(dt,dx), \qquad i = 1,2,$$

   be two one-dimensional Itô-Lévy processes. Prove that

$$X_t^1 X_t^2 = X_0^1 X_0^2 + \int_0^t X_{s-}^1 dX_s^2 + \int_0^t X_{s-}^2 dX_s^1 + \int_0^t \int_{\mathbb{R}_0} \gamma_1(s,x)\,\gamma_2(s,x)\,N(ds,dx).$$

6. Define, with suitable conditions on $\theta(s,x)$,

$$Z_t(\theta) = \exp\left(\int_0^t \int_{\mathbb{R}_0} \log(1-\theta(s,x))\, \tilde{N}(ds,dx) \right.$$
$$\left. + \int_0^t \int_{\mathbb{R}_0} \{\log(1-\theta(s,x)) + \theta(s,x)\}\, \nu(dx)\, ds \right).$$

Show that

$$dZ_t(\theta) = -Z_{t-}(\theta) \int_{\mathbb{R}_0} \theta(t,x)\, \tilde{N}(dt,dx).$$

7. Let $N = \{N_t\}_{t\geq 0}$ be a Poisson process with intensity parameter λ. Compute $\int_0^t N_{s-} dN_s$ and $\int_0^t N_s dN_s$.

8. Decide whether or not the following markets have arbitrages. If the market has an arbitrage find one.

   (a) Consider

$$\begin{aligned} dS_t^0 &= 0, \quad S_0^0 = 1, \\ dS_t^1 &= S_{t-}^1 \left(\alpha dt + \int_{\mathbb{R}_0} x \tilde{N}(dt,dx) \right), \quad S_0^1 > 0, \end{aligned}$$

   where ν is supported in $(-1,+\infty)$, $\alpha \in \mathbb{R}$ is a constant and $\int_{\mathbb{R}} |z|\, \nu(dz) > |\alpha|$.

   (b) Consider

$$\begin{aligned} dS_t^0 &= 0, \quad S_0^0 = 1, \\ dS_t^1 &= S_{t-}^1 \left(-dt - 1dB_t + 3 \int_{\mathbb{R}_0} x \tilde{N}(dt,dx) \right), \quad S_0^1 > 0, \\ dS_t^2 &= S_{t-}^2 \left(4dt + 2dB_t - 6 \int_{\mathbb{R}_0} x \tilde{N}(dt,dx) \right), \quad S_0^2 > 0. \end{aligned}$$

9. Consider the following market

$$\begin{aligned} dS_t^0 &= 0, \quad S_0^0 = 1, \\ dS_t^1 &= S_{t-}^1 \left(\mu_1 dt + \gamma_1^1 d\eta_t^1 + \gamma_2^1 d\eta_t^2 \right), \quad S_0^1 > 0, \\ dS_t^2 &= S_{t-}^2 \left(\mu_2 dt + \gamma_1^2 d\eta_t^1 + \gamma_2^2 d\eta_t^2 \right), \quad S_0^2 > 0, \end{aligned}$$

   where μ_i and $\gamma_j^i, i,j = 1,2,$ are constants and η^1 and η^2 are independent Lévy martingales of the form

$$d\eta_t^i = \int_{\mathbb{R}_0} x \tilde{N}^i(dt,dx), \qquad i = 1,2.$$

   Assume that the matrix $\gamma = \left(\gamma_j^i\right)_{1\leq i,j\leq 2}$ is invertible, with inverse $\gamma^{-1} = \lambda = \left(\lambda_j^i\right)_{1\leq i,j\leq 2}$ and assume that

$$\int_{\mathbb{R}_0} |x|\, \nu^i(dx) > \left|\lambda_1^i \mu_1 + \lambda_2^i \mu_2\right|, \qquad i = 1,2.$$

   Show that this market is arbitrage free.

1 $$dY_t = \alpha\,dt + \sigma\,dB_t + \int_{|x|<1} h(x)\,\tilde{N}(dt,dx) + \int_{|x|\geq 1} k(x)\,N(dt,dx)$$

where α, σ are constants, h, k functions

a) Use Itô's formula to find dY_t when $Y_t = \exp(X_t)$

b) How do we choose α, σ, h and k if we want to solve the SDE.

$$dY_t = Y_{t-}\left(\beta\,dt + \theta\,dB_t + \lambda\int_{|x|<1} x\,\tilde{N}(dt,dx) + \rho\int_{|x|\geq 1} x\,N(dt,dx)\right)$$

for given constants β, θ, λ and ρ.

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a) Using Theorem 11.5 (Itô's formula 2) we get

$f(x) = f'(x) = f''(x) = e^x$

$$dY_t = d(\exp(X_t)) = \exp(X_{t-})\{\alpha\,dt + \sigma\,dB_t\} + \frac{1}{2}\exp(X_{t-})\sigma^2\,dt$$

$$+ \int_{|x|\geq 1}\{\exp(X_{t-} + k(x)) - \exp(X_{t-})\}\,N(dt,dx)$$

$$+ \int_{|x|<1}\{\exp(X_{t-} + h(x)) - \exp(X_{t-})\}\,\tilde{N}(dt,dx)$$

$$+ \int_{|x|<1}\{\exp(X_{t-} + h(x)) - \exp(X_{t-}) - h(x)\exp(X_{t-})\}\,\nu(dx)\,dt$$

$$= Y_{t-}\left\{\alpha + \frac{1}{2}\sigma^2 + \int_{|x|<1}\{e^{h(x)} - 1 - h(x)\}\,\nu(dx)\right\}dt$$

$$+ \quad Y_{t-}\, \sigma\, dB_t + Y_{t-} \int_{|x|\geq 1} \{e^{k(x)} - 1\}\, N(dt,dx)$$

$$+ \quad Y_{t-} \int_{|x|<1} \{e^{h(x)} - 1\}\, \tilde{N}(dt,dx)$$

b) Clearly, we match the terms and get

$$\beta = \alpha + \frac{1}{2}\sigma^2 + \int_{|x|<1} \{e^{h(x)} - 1 - h(x)\}\, \nu(dx)$$

$$\theta = \sigma$$

$$\lambda x = e^{h(x)} - 1 \quad \Rightarrow \quad h(x) = \log(1 + \lambda x)$$

$$\varrho x = e^{k(x)} - 1 \quad \Rightarrow \quad k(x) = \log(1 + \varrho x)$$

In order for these to make sense we need

$1 + \lambda x > 0$ ν-a.e. and $1 + \varrho x > 0$ ν-a.e.

$\Rightarrow$ ν must be supported in $(\max(-1/\lambda, -1/\varrho), +\infty)$

2) Solve

a) m, σ, λ constants and

$$dX_t = (m - X_t)\,dt + \sigma\, dB_t + \lambda \int_{|x|<1} x\, \hat{N}(dt, dx) + \ell \int_{|x|\geq 1} x\, N(dt, dx)$$

$$X_0 = x_0$$

b) Let α, γ be constants and

$$dY_t = \alpha\, dt + \gamma X_{t-} \left(\int_{|x|<1} x\, \hat{N}(dt, dx) + \int_{|x|\geq 1} x\, N(dt, dx) \right)$$

$$X_0 = x_0$$

Recall the integration by parts formula (or product rule)

$$d(X_t^1 X_t^2) = X_{s-}^1\, dX_s^2 + X_{s-}^2\, dX_s^1 + d[X^1, X^2]_s$$

To solve the SDEs we use the previous formula and the variation of constants techniques from O.D.E.s.

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a) We compute

$$d(e^t \cdot X_t) = e^t\, dX_t + X_{t-}\, e^t\, dt + [e^\cdot, X]_t = (*)$$

But $[e^\cdot, X] \equiv 0$ because e^t has continuous path of finite variation. (Section 9.3 in the notes)

$$(*) = e^t m\, dt - \cancel{X_t e^t dt} + e^t \sigma\, dB_t + \lambda e^t \int_{|x|<1} x\, \hat{N}(dt, dx)$$
$$+ \ell \int_{|x|\geq 1} x\, N(dt, dx) + \cancel{X_t e^t dt}$$

Since the r.h.s. of the previous equation does NOT depend explicitly on X we can integrate

$$e^t X_t = X_0 + \int_0^t m e^s \, ds + \int_0^t \sigma e^s \, dB_s + \int_0^t \int_{|x|<1} \lambda e^s x \, \hat{N}(ds, dx) + \int_0^t \int_{|x| \geq 1} \ell e^s x \, N(ds, dx)$$

$\Rightarrow$

$$X_t = \underbrace{e^{-t} X_0 + m e^{-t}(e^t - 1)}_{= m + (X_0 - m)e^{-t}} + \int_0^t \sigma e^{-(t-s)} \, dB_s + \int_0^t \int_{|x|<1} \lambda e^{-(t-s)} x \, \hat{N}(ds, dx) + \int_0^t \int_{|x| \geq 1} \ell e^{-(t-s)} x \, N(ds, dx)$$

b) Consider the process $G = \{G_t\}_{t \geq 0}$ given by

$$G_t = \exp\left(\int_0^t \int_{|x|<1} \theta(z) \, \hat{N}(ds, dz) + \int_0^t \int_{|x| \geq 1} \theta(z) \, N(ds, dz) - t \int_{|x|<1} \{ e^{\theta(z)} - 1 - \theta(z) \} \, \nu(dz) \right)$$

For $\theta(z)$ a deterministic function (to choose). By Itô's formula

$$dG_t = G_{t-} \Big\{ \int_{|x|<1} (e^{\theta(x)} - 1) \, \hat{N}(dt, dx) + \int_{|x| \geq 1} (e^{\theta x} - 1) \, N(dt, dx)$$

$$+ \cancel{\left(\int_{|x|<1} \left(e^{\theta(x)} - 1 - \theta(x) \right) \nu(dx) \right) dt}$$

$$- \cancel{\left(\int_{|x|<1} \left(e^{\theta(x)} - 1 - \theta(x) \right) \nu(dx) \right) dt}$$

$$= G_{t-} \left\{ \int_{|x|<1} \left(e^{\theta(x)} - 1 \right) \tilde{N}(dt, dx) + \int_{|x|\geq 1} \left(e^{\theta(x)} - 1 \right) N(dt, dx) \right\}$$

Next we consider the process

$$\tilde{X}_t = X_0 G_t + \alpha G_t \int_0^t G_s^{-1} ds$$

We will show that this process solves the SDE (by choosing an appropriate θ)

$$d\tilde{X}_t = X_0 \, dG_t + \alpha G_t \cdot G_t^{-1} dt + \alpha \left(\int_0^t G_s^{-1} ds \right) dG_t$$

$$+ \underbrace{d\left[\alpha G_\cdot , \int_0^\cdot G_s^{-1} ds \right]_t}_{= 0}$$

continuous and of finite variation.

$$= \alpha \, dt + \left(X_0 + \alpha \left(\int_0^t G_s^{-1} ds \right) \right) dG_t$$

$$= \alpha \, dt + \left(X_0 + \alpha \left(\int_0^t G_s^{-1} ds \right) \right) G_{t-} \, \cdot$$

$$\cdot \left(\int_{|x|<1} \left(e^{\theta(x)} - 1 \right) \tilde{N}(dt, dx) + \int_{|x|\geq 1} \left(e^{\theta(x)} - 1 \right) N(dt, dx) \right)$$

$$= \alpha \, dt + \tilde{X}_{t-} \left(\int_{|x|<1} \left(e^{\theta(x)} - 1 \right) \tilde{N}(dt, dx) + \int_{|x|\geq 1} \left(e^{\theta(x)} - 1 \right) N(dt, dx) \right)$$

Hence $\tilde{X}_t$ solves our equation if we choose θ

such that

$$e^{\theta(x)} - 1 = \gamma x \iff \theta(x) = \log(1 + \gamma x)$$

and Y is supported in $\left(-\frac{1}{\gamma}, +\infty\right)$.

3 $h \in L^2(\mathbb{R})$ be deterministic and define

$$Y_t = \exp\left(\int_0^t \int_{\mathbb{R}_0} h(s) x \tilde{N}(ds, dx) - \int_0^t \int_{\mathbb{R}_0} (e^{h(s)x} - 1 - h(s)x) \nu(dx) ds\right)$$

show that

$$dY_t = Y_{t-} \int_{\mathbb{R}_0} (e^{h(t)x} - 1) \tilde{N}(dt, dx)$$

Itô's formula $Y_t = \exp(X_t)$ with

$$dX_t = \int_{\mathbb{R}_0} h(t) x \tilde{N}(dt, dx) - \int_{\mathbb{R}_0} (e^{h(t)x} - 1 - h(t)x) \nu(dx) dt$$

$$= \int_{|x|<1} h(t) x \tilde{N}(dt, dx) + \int_{|x|\geq 1} h(t) x N(dt, dx)$$

$$- \int_{\mathbb{R}_0} (e^{h(t)x} - 1 - h(t) x \mathbb{1}_{\hat{B}_1(0)}(x)) \nu(dx) dt$$

$e^x = f(x) = f'(x) = f''(x)$.

$$dY_t = d(\exp(X_t)) = -\exp(X_{t-}) \int_{\mathbb{R}_0} (e^{h(t)x} - 1 - h(s)x \mathbb{1}_{\hat{B}_1(0)}(x)) \nu(dx) dt$$

$$+ \int_{|x|<1} (e^{X_{t-} + h(t)x} - e^{X_{t-}} - h(t) x e^{X_{t-}}) \nu(dx) dt$$

$$+ \int_{|x|<1} (e^{X_{t-} + h(t)x} - e^{X_{t-}}) \tilde{N}(dt, dx)$$

$$+ \int_{|x|\geq 1} (e^{X_{t-} + h(t)x} - e^{X_{t-}}) N(dt, dx)$$

$$= Y_{t-} \Big\{ -\int_{|x|\geq 1} (e^{h(t)x} - 1)\, \nu(dx)\, dt$$

$$+ \int_{|x|<1} (e^{h(t)x} - 1)\, \hat{N}(dt, dx)$$

$$+ \int_{|x|\geq 1} (e^{h(t)x} - 1)\, N(dt, dx) \Big\}$$

$$= Y_{t-} \int_{\mathbb{R}_0} (e^{h(t)x} - 1)\, \hat{N}(dt, dx)$$

Hidden assumption $\int_{|x|\geq 1} x^2\, \nu(dx) < +\infty$.

The Lévy process associated to N has finite second moment.

(5) Under some conditions on $\gamma(s,x)$ deterministic we have

$$E\left[\exp\left(\int_0^t \int_{\mathbb{R}_0} \gamma(s,x)\,\hat{N}(ds,dx)\right)\right] = \exp\left(\int_0^t \int_{\mathbb{R}_0} \left(e^{\gamma(s,x)} - 1 - \gamma(s,x)\right)\nu(dx)\,ds\right)$$

Identify an exponential martingale. You can use Proposition 10.5 (linking ordinary exponentials and stochastic exponentials) or using Theorem 12.10 combined with Novikov's condition for Itô-Lévy processes.

For instance, using Theo. 12.10. Assuming that

$$E\left[\int_0^t \int_{|x|\geq 1} |e^{\gamma(s,x)} - 1|\,\nu(dx)\,ds\right] \underset{\uparrow}{=} \int_0^t \int_{|x|\geq 1} |e^{\gamma(s,x)} - 1|\,\nu(dx)\,ds < \infty$$

γ is determin.

$$Y_t = \int_0^t \int_{\mathbb{R}_0} \gamma(s,x)\,\hat{N}(ds,dx) = -\int_0^t \int_{|x|\geq 1} \gamma(s,x)\,\nu(dx)\,ds + \int_0^t \int_{|x|<1} \gamma(s,x)\,\hat{N}(ds,dx) + \int_0^t \int_{|x|\geq 1} \gamma(s,x)\,N(ds,dx)$$

Then $e^{Y_t + \int_0^t G_s ds}$ is a local martingale $\Leftrightarrow$

$$0 = G_s - \int_{|x|\geq 1} \gamma(s,x)\,\nu(dx)\,ds + \int_{|x|<1} \left(e^{\gamma(s,x)} - 1 - \gamma(s,x)\right)\nu(dx) + \int_{|x|\geq 1} \left(e^{\gamma(s,x)} - 1\right)\nu(dx)$$

$$\Leftrightarrow -\int_{\mathbb{R}_0} \left(e^{\gamma(s,x)} - 1 - \gamma(s,x)\right)\nu(dx) = G_s$$

Hence

$$Y_t := \exp\left(\int_0^t \int_{\mathbb{R}_0} \gamma(s,x) \tilde{N}(ds,dx) - \int_0^t \int_{\mathbb{R}_0} (e^{\gamma(s,x)} - 1 - \gamma(s,x)) \nu(dx) \right) \text{ is}$$

a local martingale.

We need to find conditions on $\gamma(s,x)$ so that we have that Y is a martingale. Note that, then, we can claim that

$$1 = Y_0 = E[Y_t] = E[\exp(\int_0^t \int_{\mathbb{R}_0} \gamma(s,x) \tilde{N}(ds,dx))] \exp(-\int_0^t (e^{\gamma(s,x)} - 1 - \gamma(s,x)) \nu(dx))$$

$\uparrow$ because γ is deterministic

and we get the result.

Define $\lambda(s,x)$ such that $\gamma(s,x) = \log(1 - \lambda(s,x))$.

$(\lambda(s,x) = 1 - e^{\gamma(s,x)})$

Then,

$$Y_t = \exp\left(\int_0^t \int_{\mathbb{R}_0} \log(1 - \lambda(s,x)) \tilde{N}(ds,dx) + \int_0^t \int_{\mathbb{R}_0} \{\log(1 - \lambda(s,x)) + \lambda(s,x)\} \nu(dx) ds \right)$$

Now we can apply Theorem 12.23 (Novikov!)

$$E\left[\exp\left(\int_0^T \int_{\mathbb{R}_0} \{(1 - \lambda(s,x)) \log(1 - \lambda(s,x)) + \lambda(s,x)\} \nu(dx) ds\right)\right] < +\infty$$

$$\Updownarrow \quad \lambda(s,x) \text{ is deterministic}$$

$$\int_0^T \int_{\mathbb{R}_0} \{(1 - \lambda(s,x)) \log(1 - \lambda(s,x)) + \lambda(s,x)\} \nu(dx) ds < +\infty$$

$$\Updownarrow$$

$$\int_0^T \int_{\mathbb{R}_0} \{e^{\gamma(s,x)} \gamma(s,x) + 1 - e^{\gamma(s,x)}\} \nu(dx) < +\infty$$

5) $d X^i_t = \int_{\mathbb{R}_0} \gamma_i(t,x)\, \tilde{N}(dt,dx) \qquad i=1,2$

be two one-dimensional Itô-Lévy processes.

Prove that

$$X^1_t X^2_t = X^1_0 X^2_0 + \int_0^t X^1_{s-}\, dX^2_s + \int_0^t X^2_{s-}\, dX^1_s$$

$$+ \int_0^t \int_{\mathbb{R}_0} \gamma_1(s,x)\, \gamma_2(s,x)\, N(ds,dx)$$

(Alternative Itô's formula)

By Theorem 12.11. we just have to show that

$$[X^1, X^2]_t = \int_0^t \int_{\mathbb{R}_0} \gamma_1(s,x)\gamma_2(s,x)\, N(ds,dx).$$

The processes X^1 and X^2 can be rewritten as

$$X^i_t = -\int_0^t \int_{|x|\geq 1} \gamma_i(s,x)\, \nu(dx)\, ds + \int_0^t \int_{|x|<1} \gamma_i(s,x)\, \tilde{N}(ds,dx)$$

$$+ \int_0^t \int_{|x|\geq 1} \gamma_i(s,x)\, N(ds,dx)$$

And using the formula for the quadratic variation of Itô-Lévy processes we get

$$[X^1, X^2]_t = \int_0^t \int_{|x|<1} \gamma_1(s,x)\gamma_2(s,x)\, N(ds,dx) + \int_0^t \int_{|x|\geq 1} \gamma_1(s,x)\gamma_2(s,x)\, N(ds,dx)$$

$$= \int_0^t \int_{\mathbb{R}_0} \gamma_1(s,x)\, \gamma_2(s,x)\, N(ds,dx)$$

6 Define, with suitable conditions on $\theta(s,*)$,

$$Z_t(\theta) = \exp\Big(\int_0^t \int_{\mathbb{R}_0} \log(1-\theta(s,x))\, \hat{N}(ds,dx) + \int_0^t \int_{\mathbb{R}_0} \{\log(1-\theta(s,x)) + \theta(s,x)\}\, \nu(dx)\, ds \Big)$$

Show that

$$dZ_t(\theta) = -Z_{t-}(\theta) \int_{\mathbb{R}_0} \theta(t,x)\, \hat{N}(dt,dx)$$

$$\begin{cases} \theta(s,x) < 1 & P \otimes \lambda \otimes \nu \text{ -a.e.} \\ \log(1-\theta(s,x)) \in \mathcal{P}_2([0,T]) & \end{cases}$$

$Z_t(\theta) = \exp(Y_t)$, where Y_t can be written as

$$Y_t = \int_0^t \int_{|x|<1} \log(1-\theta(s,x))\, \hat{N}(ds,dx) + \int_0^t \int_{|x|\geq 1} \log(1-\theta(s,x))\, N(ds,dx)$$

$$+ \int_0^t \int_{\mathbb{R}_0} \{\log(1-\theta(s,x))\, \mathbb{1}_{\{|x|<1\}} + \theta(s,x)\}\, \nu(dx)\, ds$$

By Itô's formula with $f(x) = e^x = f'(x)$

$$dZ_t(\theta) = \exp(Y_{t-}) \int_{\mathbb{R}_0} \{\log(1-\theta(t,x))\, \mathbb{1}_{\{|x|<1\}} + \theta(t,x)\}\, \nu(dx)\, dt$$

$$+ \int_{|x|<1} \{\exp(Y_{t-} + \log(1-\theta(t,x))) - \exp(Y_{t-})\}\, \hat{N}(dt,dx)$$

$$+ \int_{|x|\geq 1} \{ \exp(Y_{t-} + \log(1-\theta(t,x))) - \exp(Y_{t-}) \} N(dt,dx)$$

$$+ \int_{|x|<1} \{ \exp(Y_{t-} + \log(1-\theta(t,x))) - \exp(Y_{t-}) - \exp(Y_{t-}) \log(1-\theta(t,x)) \} \nu(dx)\,dt$$

$$= \exp(Y_{t-}) \Big[\int_{\mathbb{R}_0} \{ \cancel{\log(1-\theta(t,x))}\, 1_{\{|x|<1\}} + \theta(t,x) \} \nu(dx)$$

$$+ \int_{|x|<1} \{ \cancel{1} - \theta(t,x) - \cancel{1} \} \hat{N}(dt,dx)$$

$$+ \int_{|x|\geq 1} \{ 1 - \theta(t,x) - 1 \} N(dt,dx)$$

$$+ \int_{|x|<1} \{ \cancel{1} - \theta(t,x) - \cancel{1} - \cancel{\log(1-\theta(t,x))} \} \nu(dx)\,dt \Big]$$

$$= - Z_{t-}(\theta) \int_{\mathbb{R}_0} \theta(t,x) \hat{N}(dt,dx)$$

If you set $\lambda(t,x) = -\theta(t,x)$ then $\left(\begin{array}{l} \lambda(s,x) > -1 \\ \log(1+\lambda(s,x)) \in \mathcal{P}_2([0,T]) \end{array} \right)$

$$\tilde{Z}_t(\lambda) := \exp\Big(\int_0^t \int_{\mathbb{R}_0} \log(1+\lambda(s,x)) \hat{N}(ds,dx) + \int_0^t \int_{\mathbb{R}_0} \{ \log(1+\lambda(s,x)) - \lambda(s,x) \} \nu(dx)\,ds \Big)$$

Solves the S.D.E.

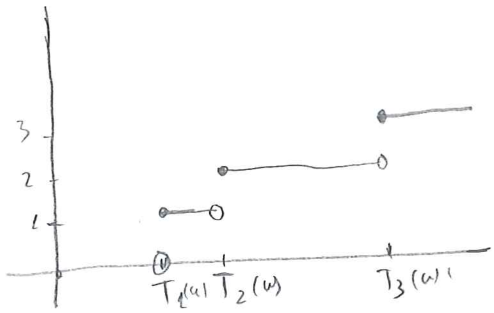
$$dY_t = Y_{t-} \int_{\mathbb{R}_0} \lambda(t,x) \hat{N}(dt,dx)$$

that is

$$\tilde{Z}_t(\lambda) = \mathcal{E}_t\Big(\int_0^{\cdot} \int_{\mathbb{R}_0} \lambda(t,x) \hat{N}(dt,dx) \Big)$$

17 Let $N = \{N_t\}_{t \geq 0}$ be a Poisson process with intensity parameter λ.

Compute $\int_0^t N_{s-}\, dN_s$ and $\int_0^t N_s\, dN_s$.



Let $T_0 \equiv 0$ and $\{T_n\}_{n \geq 1}$ the sequence of jump times of N.

Note that,

For $n \geq 0$, $N_t = n$ if $t \in [T_n, T_{n+1})$ and is càdlàg. But,

$N_{t-} = n$ if $t \in (T_n, T_{n+1}]$ for $n \geq 0$.

and hence it is càglàd (left continuous with right limits). Since the filtration is the one generated by N, we have left continuous + adapted $\Rightarrow$ predictable. The integral is pathwise.

$$\int_0^t N_{s-}\, dN_s = \sum_{k \geq 1} (k-1)\left(N_{t \wedge T_k} - N_{t \wedge T_{k-1}}\right)$$

$$\left(\text{if } t \in [T_{n-1}, T_n)\right) = \sum_{k=1}^{n-1} (k-1) \underbrace{\left(N_{T_k} - N_{T_{k-1}}\right)}_{=1} = \sum_{k=1}^{n-1} (k-1)$$

$$= \frac{(n-1)(n-2)}{2}$$

But note that if $t \in [T_{n-1}, T_n)$, $N_t = n-1$

and we can conclude

$$\int_0^t N_{s-}\, dN_s = \frac{N_t(N_t-1)}{2} \qquad t \geq 0.$$

On the other hand, we have

$$\int_0^t N_s\, dN_s = \sum_{k\geq 1} k\left(N_{t\wedge T_k} - N_{t\wedge T_{k-1}}\right)$$

$$(\text{if } t\in[T_{n-1}, T_n)) = \sum_{k=1}^{n-1} k\,\underbrace{(N_{T_k} - N_{T_{k-1}})}_{=1} = \sum_{k=1}^{n-1} k = \frac{(n-1)n}{2}$$

and we get that

$$\int_0^t N_s\, dN_s = \frac{N_t(N_t+1)}{2}.$$

Another approach to compute $\int_0^t N_{s-}\, dN_s$ is to use the Lévy-Itô decomposition of N (which is a Lévy process)

$$N_t = \int_{|x|\geq 1} x\, N(t, dx) \qquad \text{where } \nu(dx) = \delta_1(x)$$

Using Itô's formula with $f(x) = x^2$, $f'(x) = 2x$, we get

$$N_t^2 = \int_0^t \int_{|x|\geq 1} \{(N_{s-}+x)^2 - N_{s-}^2\}\, N(ds, dx)$$

$$= \int_0^t \int_{|x|\geq 1} \{ \cancel{N_{s-}^2} + 2 N_{s-} x + x^2 - \cancel{N_{s-}^2} \} N(ds, dx)$$

$$= 2 \int_0^t N_{s-} \underbrace{\int_{|x|\geq 1} x \, N(ds, dx)}_{d N_s} + \int_0^t \int_{|x|\geq 1} x^2 \, N(ds, dx)$$

$$= 2 \int_0^t N_{s-} \, dN_s + \sum_{0 \leq u \leq t} \Delta N_u \, \mathbb{1}_{\{1\}}(\Delta N_u)$$

$$\text{(finite variation)} = 2 \int_0^t N_{s-} \, dN_s + N_t$$

Hence, we have

$$N_t^2 = 2 \int_0^t N_{s-} \, dN_s + N_t \iff \int_0^t N_{s-} \, dN_s = \frac{N_t^2 - N_t}{2} = \frac{N_t (N_t - 1)}{2}$$

$\boxed{8}$ Decide whether or not the following markets have arbitrages. If the market has an arbitrage find one.

a) Consider
$$dS_t^0 = 0, \quad S_0^0 = 1.$$
$$dS_t^1 = S_{t-}^1 \left(\alpha\, dt + \int_{\mathbb{R}_0} x\, \tilde{N}(dt, dx)\right), \quad S_0^1 > 0$$

where ν is supported in $(-1, +\infty)$, $\alpha \in \mathbb{R}$ and $\int_{\mathbb{R}} |x|\nu(dx) > |\alpha|$ and is absolutely continuous with respect to λ.

b) Consider

$$d S_t^0 = 0, \quad S_0^0 = 1$$
$$d S_t^1 = S_{t-}^1 \left(-dt - 1\, dB_t + 3 \int_{\mathbb{R}_0} x\, \tilde{N}(dt, dx)\right) \quad S_0^1 > 0$$
$$d S_t^2 = S_{t-}^2 \left(4\, dt + 2\, dB_t - 6 \int_{\mathbb{R}_0} x\, \tilde{N}(dt, dx)\right), \quad S_0^2 > 0$$

---

a) We apply Theorem 13.14. We need to find $\theta_1 < 1$ such that

$$S_{t-} \int_{\mathbb{R}_0} x\, \theta(t, x)\, \nu(dx) = S_{t-}\, \alpha$$
$$\Updownarrow$$
$$\int_{\mathbb{R}_0} x\, \theta(x)\, \nu(dx) = \alpha \qquad \left(\text{We can choose } \theta \text{ to not depend on } t\right)$$

If $\int_{\mathbb{R}} |x|\, \nu(dx) > |\alpha|$ we can choose $A \subset \mathbb{R}_0$ and bounded below such that you still have $|\alpha| \leq \int_A |x| \, \nu(dx) =: K < +\infty$

and then choose

$$\theta(x) = \frac{\alpha}{K} \mathbb{1}_A(x) \, \text{sign}(x)$$

Note that $|\theta(x)| \leq \frac{|\alpha|}{K} < 1$ and

$$\int_{\mathbb{R}_0} x \, \theta(x) \, \nu(dx) = \frac{\alpha}{K} \int_{\mathbb{R}_0} x \, \mathbb{1}_A(x) \, \text{sign}(x) \, \nu(dx)$$

$$= \frac{\alpha}{K} \int_A |x| \, \nu(dx) = \alpha$$

2) Novikov's condition is

$$E\left[\exp\left(\int_0^T \int_{\mathbb{R}_0} \{(1-\theta(x)) \log(1-\theta(x)) + \theta(x)\} \, \nu(dx) \, dt\right)\right] < +\infty$$

$\Updownarrow$ θ is deterministic and does not depend on t

$$\int_{\mathbb{R}_0} \{(1-\theta(x)) \log(1-\theta(x)) + \theta(x)\} \, \nu(dx) < +\infty$$

$\Updownarrow$ the integrand is 0 if $x \notin A$.

$$\int_A \{(1-\theta(x)) \log(1-\theta(x)) + \theta(x)\} \, \nu(dx) < +\infty$$

which holds because $\theta(x)$ can be chosen such that is bounded away (from below) from 1.

Then we can use θ in Theorem 13.14 and conclude that there are no arbitrages in this market.

b) In this case the equation 13.2.2 in theorem 13.14 reads

$$\begin{pmatrix} -1 \, S^1_{t-} \\ 2 \, S^2_{t-} \end{pmatrix} \theta_0(t) + \begin{pmatrix} 3 \, S^1_{t-} \int_{\mathbb{R}_0} x \, \theta_1(t,x) \nu(dx) \\ -6 \, S^2_{t-} \int_{\mathbb{R}_0} x \, \theta_1(t,x) \nu(dx) \end{pmatrix} = \begin{pmatrix} (-1-0) S^1_{t-} \\ (4-0) S^2_{t-} \end{pmatrix}$$

where $\theta_0(t), \theta_1(t,x) \in \mathbb{R}$

Define $\hat{\theta}_1(t) := \int_{\mathbb{R}_0} x \, \theta_1(t,x) \nu(dx)$ and simplify the previous system of equations.

$$-1 \, \theta_0(t) + 3 \, \hat{\theta}_1(t) = -1$$
$$2 \, \theta_0(t) - 6 \, \hat{\theta}_1(t) = 4$$

which is inconsistent (adding to the second equation two times the first equation you get $0 = 2$).

By Theorem 13.4 (last addition) we know that there exists an arbitrage. In this case it is easy to find.

In this market the portfolios are predictable processes

$$\xi = (\xi_0(t), \xi_1(t), \xi_2(t))_{0 \le t \le T} \in \mathbb{R}^3$$

By Lemma 13.7, $\xi_1(t), \xi_2(t)$ we can choose

$\xi_0(t)$ such that ξ is self-financing and we have freedom to choose X_0^ξ.

Take $X_0^\xi \equiv 0$ and $\xi_1(t) = \frac{2}{S^1_{t-}}$, $\xi_2(t) = \frac{1}{S^2_{t-}}$

and $\xi_0(t)$ such that ξ is self-financing.

$$X_t^\xi = \underbrace{X_0^\xi}_{\shortparallel \atop 0} + \int_0^t \xi_0(s)\, \underbrace{dS_s^0}_{\shortparallel \atop 0} + \int_0^t \xi_1(s)\, dS_s^1 + \int_0^t \xi_2(s)\, dS_s^2$$

$$= \int_0^t \xi_1(s)\, dS_s^1 + \int_0^t \xi_2(s)\, dS_s^2$$

$$= \int_0^t \frac{2}{S^1_{u-}} S^1_{u-} \left(- du - dB_u + 3 \int_{\mathbb{R}_0} x\, \tilde{N}(du, dx) \right)$$

$$+ \int_0^t \frac{1}{S^2_{u-}} S^2_{u-} \left(4\, du + 2\, dB_u - 6 \int_{\mathbb{R}_0} x\, \tilde{N}(du, dx) \right)$$

$$= \int_0^t 2\, du > 0$$

The value of this portfolio is lower-bounded $\Rightarrow$ It is admissible.

Clearly it is an arbitrage since

$$X_0^\xi = 0 \quad \text{and} \quad X_T^\xi = 2T > 0 \quad \text{a.s.}$$

9) Consider

$$dS^0_t = 0, \quad S^0_0 = 1$$

$$dS^1_t = S^1_{t-}\left(\mu_1 dt + \gamma^1_1 d\eta^1_t + \gamma^1_2 d\eta^2_t\right), \quad S^1_0 > 0$$

$$dS^2_t = S^2_{t-}\left(\mu_2 dt + \gamma^2_1 d\eta^1_t + \gamma^2_2 d\eta^2_t\right) \quad S^2_0 > 0$$

μ_j, γ^i_j constants and η^1 and η^2 are independent Lévy martingales of the form

$$d\eta^i_t = \int_{\mathbb{R}_0} x \, \tilde{N}^i(dt, dx), \quad i = 1, 2$$

Assume the matrix γ is invertible with inverse $\gamma^{-1} = \lambda = (\lambda^i_j)_{1 \le i,j \le 2}$ and assume that

$$\int_{\mathbb{R}_0} |x| \, \nu^i(dx) > |\lambda^i_1 \mu_1 + \lambda^i_2 \mu_2|, \quad i = 1, 2.$$

Show that this market is arbitrage free.

We apply Theorem 13.14. In this case equation (11.2.2) in Theorem 13.14 is

$$(1) \; \cancel{S^1_{t-}}\left(\gamma^1_1 \int_{\mathbb{R}_0} x \, \theta^1_1(t,x)\nu^1(dx) + \gamma^1_2 \int_{\mathbb{R}_0} x \, \theta^2_1(t,x)\nu^2(dx)\right) = (\mu_1 - 0)\cancel{S^1_{t-}}$$

$$(2) \; \cancel{S^2_{t-}}\left(\gamma^2_1 \int_{\mathbb{R}_0} x \, \theta^1_1(t,x)\nu^1(dx) + \gamma^2_2 \int_{\mathbb{R}_0} x \, \theta^2_1(t,x)\nu^2(dx)\right) = (\mu_2 - 0)\cancel{S^2_{t-}}$$

a... . We can take θ_1 indep. of t, i.e., $\theta_1 = (\theta_1^1(x), \theta_1^2(x))$.

Call $\hat{\theta}_1^i = \int_{\mathbb{R}_0} x\, \theta_1^i(x)\, \nu^i(dx) \quad i=1,2$.

Then the previous system can be written as

$$\underset{\mathbb{R}^{2\times 2}}{\gamma}\ \underset{\mathbb{R}^{2\times 1}}{\hat{\theta}_1} = \underset{\mathbb{R}^{2\times 1}}{\mu} \iff \hat{\theta}_1 = \gamma^{-1}\mu = \lambda\mu$$

Then the previous system is equivalent to

$$(3)\quad \int_{\mathbb{R}_0} x\, \theta_1^1(x)\, \nu^1(dx) = \lambda_1^1 \mu_1 + \lambda_2^1 \mu_2$$

$$(4)\quad \int_{\mathbb{R}_0} x\, \theta_1^2(x)\, \nu^2(dx) = \lambda_1^2 \mu_1 + \lambda_2^2 \mu_2$$

By assumption

$$\int_{\mathbb{R}_0} |x|\, \nu^i(dx) > |\lambda_1^i \mu_1 + \lambda_2^i \mu_2| \quad i=1,2$$

(assuming $\nu^i \ll \lambda$ (Lebesgue measure)), by absolute continuity of the integrals we can find $A_i \subset \mathbb{R}_0$ with

$$|\lambda_1^i \mu_1 + \lambda_2^i \mu_2| < \int_{A_i} |x|\, \nu^i(dx) = K_i < +\infty$$

and, then, the functions

$$\theta_1^i(x) = \text{sign}(x) \, \frac{\lambda_1^i \mu_1 + \lambda_2^i \mu_2}{K_i} \, \mathbb{1}_{A_i}(x) \quad i=1,2.$$

solve (3) and (4). For $i=1,2$.

$$\int_{\mathbb{R}_0} x \, \theta_1^i(x) \, \nu^i(dx) = \int_{A_i} x \, \text{sign}(x) \, \frac{(\lambda_1^i \mu_1 + \lambda_2^i \mu_2)}{K_i} \, \nu^i(dx)$$

$$= (\lambda_1^i \mu_1 + \lambda_2^i \mu_2) \underbrace{\frac{\int_{A_i} |x| \, \nu^i(dx)}{K_i}}_{= 1}$$

By construction

$$|\theta_1^i(x)| = \left| \text{sign}(x) \, \frac{\lambda_1^i \mu_1 + \lambda_2^i \mu_2}{K_i} \, \mathbb{1}_{A_i}(x) \right|$$

$$\leq \frac{|\lambda_1^i \mu_1 + \lambda_2^i \mu_2|}{K_i} < 1.$$

Then, if we define

$$Z_t = \exp\left\{ \sum_{i=1}^{2} \int_0^t \int_{\mathbb{R}_0} \log(1-\theta_1^i(x)) \, \tilde{N}^i(ds,dx) \right.$$

$$\left. + \int_0^t \int_{\mathbb{R}_0} \{\log(1-\theta_1^i(x)) + \theta_1^i(x)\} \, \nu^i(dx) \right\}$$

~~then~~ and we put $dQ = Z_T \, dP$, we get that Q is a ELMM for (S^1, S^2) and there is no arbitrage.

see Theorem 13.14.

To check that Z_T defines a probability measure, we need to check that Z is a martingale.

By Theorem 12.23 (Novikov's) it is sufficient to check

$$E\left[\exp\left(\sum_{i=1}^{2}\int_0^T\int_{\mathbb{R}_0}\{(1-\theta_i^i(x))\log(1-\theta_i^i(x))+\theta_i^i(x)\}\nu^i(dx)\right)\right] < +\infty$$

Since θ is deterministic and does not depend on t, the last condition is equivalent to

$$\int_{\mathbb{R}_0}\{(1-\theta_i^i(x))\log(1-\theta_i^i(x))+\theta_i^i(x)\}\nu^i(dx) < +\infty$$

which holds because we can choose $\theta_i^i(x) \leq 1-\delta$ for some $\delta > 0$.