BLACK-76 FORMULA (NO-JUMP CASE)

MAT4770 & MAT9770 LECTURE NOTES

Abstract. We treat the case when forward dynamics does not have any jump components. In this situation we can derive explicit pricing formulas for the plain vanilla options being slight extensions of the Black-76 Formula.

Section 9.1.1: The Case of no Jumps - The Black-76 Formula

Consider a call option written on a forward contract, with exercise time $T > 0$ and strike price $K > 0$. The forward contract has maturity $\tau \geq T$, and we suppose the risk-neutral dynamics are given as follows

$$\frac{df(t,\tau)}{f(t,\tau)} = \sum_{k=1}^{p} \sigma_k(t,\tau)\, dW_k(t), \tag{0.1}$$

where W_k are p independent Brownian motions under the risk-neutral probability $\mathbb{Q}$. Recalling the forward price dynamics resulting from a geometric spot model derived in Proposition 4.8., we have

$$\sigma_k(t,\tau) = \sum_{i=1}^{m} \sigma_{ik}(t) \exp\left(-\int_t^{\tau} \alpha_i(u)\, du\right).$$

The speeds of mean reversion are described by the functions α_i, and the spot volatilities by σ_{ik}. The forward dynamics in (0.1) can also come from the direct modelling of the forward price curve as analysed in Chapter 6 (...).

The following Proposition states the price of a cal option and is known as the Black-76 Formula.

Proposition 1. *The price of a call option at time $t \leq T$, written on a forward with delivery at time τ, where the option has exercise time $T \leq \tau$ and strike price K, is*

$$C(t;T,K,\tau) = e^{-r(T-t)} \left\{ f(t,\tau)\, \Phi(d_1) - K\Phi(d_2) \right\}.$$

Here,

$$d_1 = d_2 + \sqrt{\sum_{k=1}^{p} \int_t^T \sigma_k^2(u,\tau)\, du}, \tag{0.2}$$

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$$d_2 = \frac{\ln\left(f\left(t,\tau\right)/K\right) - \frac{1}{2}\sum_{k=1}^{p}\int_t^T \sigma_k^2\left(u,\tau\right)du}{\sqrt{\sum_{k=1}^{p}\int_t^T \sigma_k^2\left(u,\tau\right)du}}, \tag{0.3}$$

and Φ is the cumulative standard normal probability distribution function.

Proof. consider the case $p = 1$. We have that

$$\ln f\left(T,\tau\right) \stackrel{d}{=} \ln f\left(t,\tau\right) - \frac{1}{2}\int_t^T \sigma^2\left(u,\tau\right)du + X\sqrt{\int_t^T \sigma^2\left(u,\tau\right)du},$$

where X is a standard normally distributed random variable. From general option theory, the price is defined as the present expected payoff, with expectation taken under the risk-neutral probability. Hence,

$$\begin{aligned}
&C\left(t;T,K,\tau\right)\\
&\quad= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left[\max\left(f\left(T,\tau\right)-K,0\right) \mid \mathcal{F}_t\right]\\
&\quad= e^{-r(T-t)}\mathbb{E}\left[\max\left(e^{\ln f(t,\tau)-\frac{1}{2}\int_t^T\sigma^2(u,\tau)du+X\sqrt{\int_t^T\sigma^2(u,\tau)du}}-K,0\right)\right]\\
&\quad= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left[\max\left(f\left(t,\tau\right)e^{-\frac{1}{2}\int_t^T\sigma^2(u,\tau)du+X\sqrt{\int_t^T\sigma^2(u,\tau)du}}-K,0\right)\right].
\end{aligned}$$

Note that we get a positive payoff from the option only when

$$X > \frac{\ln\left(\frac{K}{f(t,\tau)}\right) + \frac{1}{2}\int_t^T \sigma^2\left(u,\tau\right)du}{\sqrt{\int_t^T \sigma^2\left(u,\tau\right)du}} = -d_2.$$

Then we can rewrite the previous expression as

$$\begin{aligned}
&C\left(t;T,K,\tau\right)\\
&\quad= e^{-r(T-t)}\int_{-d_2}^{\infty}\left(f\left(t,\tau\right)e^{-\frac{1}{2}\int_t^T\sigma^2(u,\tau)du+x\sqrt{\int_t^T\sigma^2(u,\tau)du}}-K\right)\phi\left(x\right)dx\\
&\quad= e^{-r(T-t)}\left(f\left(t,\tau\right)\int_{-d_2}^{\infty}e^{-\frac{1}{2}\int_t^T\sigma^2(u,\tau)du+x\sqrt{\int_t^T\sigma^2(u,\tau)du}}\phi\left(x\right)dx - K\int_{-d_2}^{\infty}\phi\left(x\right)dx\right),
\end{aligned}$$

where $\phi\left(x\right) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ is the density of the standard normally distributed random variable X. Note in the following lines that we will make use of the symmetry of the standard normal density function. Therefore we can rewrite the call prices as follows

$$\begin{aligned}
&C\left(t;T,K,\tau\right)\\
&\quad= \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\left(f\left(t,\tau\right)\int_{-d_2}^{\infty}e^{-\frac{1}{2}\int_t^T\sigma^2(u,\tau)du+x\sqrt{\int_t^T\sigma^2(u,\tau)du}}e^{-\frac{x^2}{2}}dx - K\int_{-d_2}^{\infty}e^{-\frac{x^2}{2}}dx\right)\\
&\quad= \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\left(f\left(t,\tau\right)\int_{-d_2}^{\infty}e^{-\frac{1}{2}\int_t^T\sigma^2(u,\tau)du+x\sqrt{\int_t^T\sigma^2(u,\tau)du}-\frac{x^2}{2}}dx\right) - Ke^{-r(T-t)}\Phi\left(d_2\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \left(f(t,\tau) \int_{-d_2}^{\infty} e^{-\frac{1}{2}\left(x^2 - 2x\sqrt{\int_t^T \sigma^2(u,\tau)du} + \int_t^T \sigma^2(u,\tau)du\right)} \right) - Ke^{-r(T-t)}\Phi(d_2) \\
&= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \left(f(t,\tau) \int_{-d_2}^{\infty} e^{-\frac{1}{2}\left(x - \sqrt{\int_t^T \sigma^2(u,\tau)du}\right)^2} dx \right) - Ke^{-r(T-t)}\Phi(d_2) \\
&= e^{-r(T-t)} f(t,\tau)\, \Phi\left(- \left(d_2 + \sqrt{\int_t^T \sigma^2(u,\tau)\, du} \right)\right) - Ke^{-r(T-t)}\Phi(d_2) \\
&= e^{-r(T-t)} \left(f(t,\tau)\, \Phi(d_1) - K\Phi(d_2)\right).
\end{aligned}
$$

□

We now turn our attention to the question of hedging the call option on the forward. From option theory, the delta hedging strategy is defined as follows

$$
\Delta(t;T,K,\tau) \triangleq \frac{\partial C(t;T,K,\tau)}{\partial f(t,\tau)}. \tag{0.4}
$$

The delta hedge gives the number of forwards one should have in the portfolio at all times up to exercise in a hedge of the call option. The strategy is derived in the following Proposition.

Proposition 2. *The delta hedge of the call option written on a forward with maturity at time τ, and where the option has exercise time $T \leq \tau$ and strike K, is given as*

$$
\Delta(t;T,K,\tau) = e^{-r(T-t)}\Phi(d_1),
$$

where Φ and d_1 are defined in the Prop. (1).

Proof. A differentiation leads to

$$
\Delta(t;T,K,\tau) = e^{-r(T-t)} \left\{ \Phi(d_1) + f(t,\tau)\,\Phi'(d_1)\frac{\partial d_1}{\partial f} - K\Phi'(d_2)\frac{\partial d_2}{\partial f} \right\},
$$

where d_2 is defined as in Prop. (1). We only need to prove that indeed

$$
f(t,\tau)\,\Phi'(d_1)\frac{\partial d_1}{\partial f} - K\Phi'(d_2)\frac{\partial d_2}{\partial f} = 0.
$$

This is easily done as shown in the following lines. Note first that $\frac{\partial d_1}{\partial f} = \frac{\partial d_2}{\partial f}$, therefore we have

$$
\begin{aligned}
0 &= f(t,\tau)\,\Phi'(d_1)\frac{\partial d_1}{\partial f} - K\Phi'(d_2)\frac{\partial d_2}{\partial f} \\
&\Leftrightarrow f(t,\tau)\,\Phi'(d_1) = K\Phi'(d_2) \\
&\Leftrightarrow \ln\frac{f(t,\tau)}{K} = \ln\frac{\Phi'(d_2)}{\Phi'(d_1)},
\end{aligned}
$$

remember that $\Phi'(d_i) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_i^2}{2}}$,therefore $\ln \frac{\Phi'(d_2)}{\Phi'(d_1)} = \ln \frac{e^{-\frac{d_2^2}{2}}}{e^{-\frac{d_1^2}{2}}} = \ln e^{-\frac{d_2^2}{2}} - \ln e^{-\frac{d_1^2}{2}} = \frac{1}{2}\left(d_1^2 - d_2^2\right)$ .Now it is only left to prove that

$$\ln \frac{f(t,\tau)}{K} = \frac{1}{2}\left(d_1^2 - d_2^2\right).$$

To show that the following lines serve the purpose

$$\begin{aligned}
\frac{1}{2}\left(d_1^2 - d_2^2\right) &= \frac{1}{2}\left(d_1 + d_2\right)\left(d_1 - d_2\right) \\
\text{by eq. (0.2)} &= \frac{1}{2}\left(d_2 + \sqrt{\int_t^T \sigma^2(u,\tau)\,du} + d_2\right)\left(\sqrt{\int_t^T \sigma^2(u,\tau)\,du}\right) \\
&= \frac{1}{2}\left(2d_2 + \sqrt{\int_t^T \sigma^2(u,\tau)\,du}\right)\left(\sqrt{\int_t^T \sigma^2(u,\tau)\,du}\right) \\
\text{by eq. (0.3)} &= \left(\ln\left(f(t,\tau)/K\right) - \frac{1}{2}\int_t^T \sigma^2(u,\tau)\,du\right) + \frac{1}{2}\int_t^T \sigma^2(u,\tau)\,du \\
&= \ln\left(f(t,\tau)/K\right).
\end{aligned}$$

□

Remark. The following bulletpoints.

- We recognise the price and hedging strategy are analogous to the call option in the classical Black-Scholes contexts.
- The only difference is that the forward dynamics is a martingale in the risk-neutral setting (Remember HJM-approach) whereas in the Black-Scholes framework it is the discounted asset price (Spot price) which is a martingale. This leads to some minor modifications of the price and hedge in the case of forward options.

We will have a quick look at an example where the forward price dynamics comes from a Schwartz model with constant volatility and speed of mean reversion, i.e., assuming $p = m = 1$, we have

$$\sigma(u,\tau) = \sigma e^{-\alpha(\tau - u)}.$$

Thus the aggregated volatility to be inserted into the Black-76 Formula becomes

$$\int_t^T \sigma^2(u,\tau)\,du = \frac{\sigma^2}{2\alpha}\left(e^{-2\alpha(\tau-T)} - e^{-2\alpha(\tau-t)}\right).$$

It is obvious that the aggregated volatility increases with the exercise time and decreases with the maturity of the forward. Hence if the maturity of the forward is far into the future, the aggregated volatility will be relatively low if exercise of the option is close. The aggregated volatility is decreasing with an increasing speed of mean reversion α.

Now we will consider a call option written on a swap contract. Suppose that the delivery period is $[\tau_1, \tau_2]$, and consider the forward dynamics as considered in subsection 6.4 given by

$$\frac{dF(t,\tau_1,\tau_2)}{F(t,\tau_1,\tau_2)} = \sum_{k=1}^{p} \Sigma_k(t,\tau_1,\tau_2)\, dW_k(t).$$

Following the case of options on forwards, the following result is reached and the proof is left as an exercise for the reader. (Hint: closely follow the proof from Prop. (1))

Proposition 3. *Suppose a call option written on a swap contract with delivery period $[\tau_1, \tau_2]$, has exercise time $T \leq \tau_1$ and strike K. The option price at time t is then given as*

$$C(t;T,K,\tau_1,\tau_2) = e^{-r(T-t)} \left\{F(t,\tau_1,\tau_2)\,\Phi(d_1) - K\Phi(d_2)\right\},$$

where

$$d_1 = d_2 + \sqrt{\sum_{k=1}^{p} \int_t^T \Sigma_k^2(s,\tau_1,\tau_2)\, ds},$$

$$d_2 \frac{\ln\left(F(t,\tau_1,\tau_2)/K\right) - \frac{1}{2}\sum_{k=1}^{p}\int_t^T \Sigma_k^2(s,\tau_1,\tau_2)\, ds}{\sqrt{\sum_{k=1}^{p}\int_t^T \Sigma_k^2(s,\tau_1,\tau_2)\, ds}}.$$

The delta hedge of the option is given by

$$\Delta(t;T,K,\tau_1,\tau_2) = e^{-r(T-t)}\Phi(d_1).$$

Where Φ is the cummulative standard normal probability distribution function.