HEATH-JARROW-MORTON (HJM) APPROACH

MAT4770 & MAT9770 LECTURE NOTES

ABSTRACT. HJM-approach to modelling forward prices, section 6.1-6.3. In fixed income markets, instead of modelling prices via spot-rate models, the dynamics of the forward rates are directly specified.

SECTION 6: MODELLING FORWARDS AND SWAPS USING HEATH-JARROW-MORTON APPROACH

Originally this was an approach to model forward rates in interest rate models. The approach was brought later to model forward prices.

Section 6.1.: The HJM modelling idea for forward contracts. Recall: We have looked at spot price models, introduced the transformation (Girsanov + Esscher) to go from $\mathbb{P}$ to $\mathbb{Q}$ dynamics and therefore computed the forward prices.

$$\text{Spot Price } + \mathbb{Q} - \text{dynamics} \Rightarrow \text{compute fwd price},$$
$$S(t) + \text{Girsanov/Esscher} \Rightarrow f(t,T) = \mathbb{E}_{\mathbb{Q}}\left[S(T) \mid \mathcal{F}_t\right].$$

Simple case: We assume the following spot model and the following $\mathbb{P}-$dynamics:

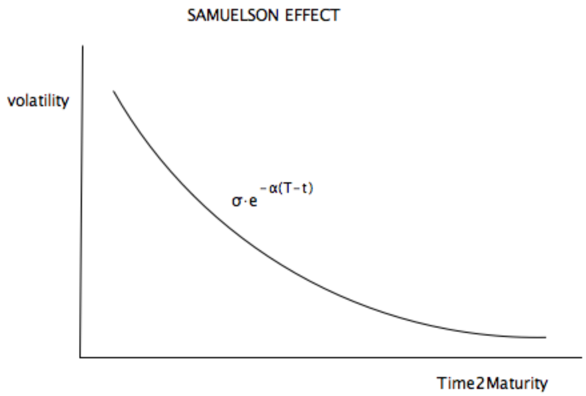
$$S(t) = \exp\{X(t)\}$$
$$dX(t) = (\mu - \alpha X(t))\,dt + \sigma dB_t.$$

Under the $\mathbb{Q}$ and by (Girsanov's transform) we know $dB^{\theta}(t) = dB(t) + \theta dt$. The dynamics of the forward rates can be, therefore, stated under the risk-neutral measure given by means of the arbitrage-free forward pricing formula $f(t,T) = \mathbb{E}_{\mathbb{Q}}\left[S(T) \mid \mathcal{F}_t\right]$

$$\frac{df(t,T)}{f(t,T)} = \sigma e^{-\alpha(T-t)} dB^{\theta}(t). \tag{0.1}$$

Remark 1. Remember to differentiate between the vol spot (σ), and the vol forward $\left(\sigma e^{-\alpha(T-t)}\right)$ in the previous expression. We are now considering a time dependent volatility which produces what is known as the Samuelson Effect. This implies that the closer to maturity we are, the higher the volatility will be and so the forward volatility will converge to the instanteneous volatility.

Date: 1st April 2019.



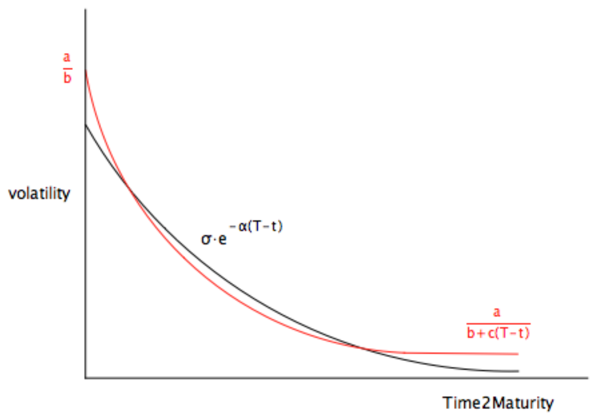
- Question: What is the price of a call option on a forward? (We will apply a modification of BS in order to make volatility time dependent.
- HJM Approach: Consists in modelling the forward price directly!
  - If we are option pricing we will use the $\mathbb{Q}-$dynamics.
  - If we are doing risk management we will use the $\mathbb{P}-$dynamics.
- HJM Constraints:
  - We do not need any spot price model.
  - No specification of the risk premium θ in the change of measure $\mathbb{Q}$. When doing real hedging, risk premium will include a hedging pressure (A farmer/producer will set a negative premium and a power consumer will input a positive premium).

Now we need to specify the model for the forward under $\mathbb{Q}$:

$$\frac{df\left(t,T\right)}{f\left(t,T\right)}=\sigma\left(t,T\right)dW\left(t\right), \tag{0.2}$$

where W is a $\mathbb{Q}-$Brownian motion.

(1) $\sigma\left(t,T\right)=\sigma e^{-\alpha(T-t)}$.
(2) $\sigma\left(t,T\right)=\frac{a}{b+c\cdot(T-t)}$; $a,b,c>0$, known as ELVIZ (Electricity Visualization).



Section 6.2.: HJM modelling of forwards. Now, solving (0.2) we can write by using Itô formula on $\ln\left(f\left(t,T\right)\right)$, the following

$$f\left(t,T\right)=f\left(0,T\right)\exp\left(-\frac{1}{2}\int_0^t\sigma^2\left(s,T\right)ds+\int_0^t\sigma\left(s,T\right)dW\left(s\right)\right).$$

Remember, this is the solution to the Geometric Brownian Motion. The $\mathbb{Q}-$dynamics in the jump case would be

$$df\left(t,T\right)=f\left(t,T\right)\sigma\left(t,T\right)dW\left(t\right)+f\left(t,T\right)\eta\left(t,T\right)dI\left(t\right).$$

Note that with these dynamics we may obtain negative forward prices. By Itô formula again, we can preserve the positivity in $f\left(t,T\right)$, and the explicit dynamics become

$$f\left(t,T\right)=f\left(0,T\right)\exp\left(\int_0^t\alpha\left(s,T\right)ds+\int_0^t\sigma\left(s,T\right)dW\left(s\right)+\int_0^t\eta\left(s,T\right)dI\left(s\right)\right),\quad 0\le t\le T.$$

Note that, to ensure the process to be a martingale, we will need to have an α, which will depend on σ,η. In this case I is a CPP independent of the Brownian motion W. We proceed now to specify the formal set of conditions for this framework.

- **First set of conditions:**

for all $0\le t\le T$, α,σ,η are measurable, deterministic functions such that

$$\begin{aligned}
\alpha\left(\cdot,T\right):&[0,t]\longrightarrow\mathbb{R}; &&\int_0^t\left|\alpha\left(s,T\right)\right|ds<+\infty\\
\sigma\left(\cdot,T\right):&[0,t]\longrightarrow\mathbb{R}; &&\int_0^t\left|\sigma^2\left(s,T\right)\right|ds<+\infty\\
\eta\left(\cdot,T\right):&[0,t]\longrightarrow\mathbb{R}; &&\sum_{s_i<t}\left|\eta\left(s_i,T\right)\right|\left|\Delta I\left(s_i\right)\right|<+\infty\text{a.s.}
\end{aligned}$$

- **Second set of conditions:**

$$\mathbb{E}_{\mathbb{Q}}\left[f\left(t,T\right)\right]<\infty,\qquad \forall t\leq T.$$

This is left as an exercise for the students to derive!

- **Third set of conditions:** to ensure

$$t\mapsto f\left(t,T\right)\ \mathbb{Q}-\text{martingale}.$$

We set $T\geq t\geq s\geq 0$

$$\mathbb{E}_{\mathbb{Q}}\left[f\left(t,T\right)\mid\mathcal{F}_s\right]=f\left(0,T\right)\exp\left(\int_0^t\alpha\left(u,T\right)du\right)\cdot\mathbb{E}_{\mathbb{Q}}\left[\exp\left(\int_0^t\sigma\left(u,T\right)dW\left(u\right)+\int_0^t\eta\left(u,T\right)dI\left(u\right)\right)\mid\mathcal{F}_s\right].$$

Due to the independence of the CPP and the Brownian motion we can rewrite the previous expression as

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[f\left(t,T\right)\mid\mathcal{F}_s\right]&=f\left(0,T\right)\exp\left(\int_0^t\alpha\left(u,T\right)du\right)\exp\left(\int_0^s\sigma\left(u,T\right)dW\left(u\right)+\int_0^s\eta\left(u,T\right)dI\left(u\right)\right)\\&\quad\times\mathbb{E}_{\mathbb{Q}}\left[\exp\left(\int_s^t\sigma\left(u,T\right)dW\left(u\right)\right)\right]\mathbb{E}_{\mathbb{Q}}\left[\exp\left(\int_s^t\eta\left(u,T\right)dI\left(u\right)\right)\right]\\&=f\left(s,T\right)\exp\left(\int_s^t\alpha\left(u,T\right)du\right)\exp\left(\frac{1}{2}\int_s^t\sigma^2\left(u,T\right)du\right)\exp\left(\int_s^t\varphi\left(\eta\left(u,T\right)\right)du\right),\end{aligned}$$

where $\varphi\left(x\right)=\ln\mathbb{E}_{\mathbb{Q}}\left[e^{xI(1)}\right]$ is the log MGF of I. Therefore we have that the forward is a $\mathbb{Q}-$martingale iff

$$\int_s^t\alpha\left(u,T\right)du=-\frac{1}{2}\int_s^t\sigma^2\left(u,T\right)du-\int_s^t\varphi\left(\eta\left(u,T\right)\right)du,$$

forall $0\leq s\leq t\leq T$. This is a simplified version of proposition 6.1 in the book. As we have seen α is not a free parameter, it actually depends on σ,η.

Section 6.3.: The HJM modelling idea for forward contracts. Denote by $F\left(t,\tau_1,\tau_2\right)$ the price at time t for a forward contract such that $t\leq\tau_1\leq\tau_2$

$$\frac{dF\left(t,\tau_1,\tau_2\right)}{F\left(t,\tau_1,\tau_2\right)}=\sigma\left(t,\tau_1,\tau_2\right)dW\left(t\right),\text{GMB},\tag{0.3}$$

where σ is deterministic. The solution to the previous SDE is as follows

$$F\left(t,\tau_1,\tau_2\right)=F\left(0,\tau_1,\tau_2\right)\exp\left\{-\frac{1}{2}\int_0^t\sigma^2\left(s,\tau_1,\tau_2\right)ds+\int_0^t\sigma\left(s,\tau_1,\tau_2\right)dW\left(s\right)\right\}.$$

Therefore the map $t\mapsto F\left(t,\tau_1,\tau_2\right)$ is a $\mathbb{Q}-$martingale. Therefore there is no arbitrage.

Example. Consider a forward contract covering the 2nd quarter (Q2), this is F_{Q2} delivering during April, May, June. On the other hand consider monthly forward contracts F_A April, F_M May, F_JJune delivering monthly the specified months.

In the Q2 forward contract you pay in total $90F_{Q2}$ (90 days 3 months).

In the monthly contracts you pay in total $30F_A + 30F_M + 30F_J$.

Consider the interest rate $r = 0$, i.e. discounting factor equals one (money value in april equals money value in june) and assume that $F_{Q2} > \frac{1}{30}(F_A + F_M + F_J)$, then we can build an arbitrage strategy as follows:

- Today:
  - Sell Q2 forward F_{Q2} at 0\$
  - Buy F_A, F_M, F_J forward contracts at 0\$
  - Total cost of investment: 0\$.
- April-June:
  - Receive $90F_{Q2}$\$ to deliver the electricity.
  - Deliver/Pay $-30(F_A + F_M + F_J)$\$ to receive the electricity each month.
  - Total cost of investment: $90F_{Q2} - 30(F_A + F_M + F_J) > 0$! (ARBITRAGE).

No arbitrage happens iff $F_{Q2} = \frac{1}{30}(F_A + F_M + F_J)$.

Recall that depending on when the settlement is done (usually at the end of the period, τ_2, known as “in arrears”), from the spot price formula we have

$$F(t, \tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} \hat{w}(u, \tau_1, \tau_2) f(t, u)\, du,$$

where $\hat{w}$ is the weight function that connected forwards with swaps in (4.3) in the Book of the course and is defined by

$$\hat{w}(u, \tau_1, \tau_2) = \frac{w(u)}{\int_{\tau_1}^{\tau_2} w(u)\, du}, \qquad w(u) = \begin{cases} 1 \\ e^{-ru} \end{cases}$$

We start by considering a partition $\{\tau_1, \tau_2, \ldots, \tau_N\}$, the no-arbitrage condition is given by

$$F(t, \tau_1, \tau_N) = \sum_{i=1}^{N-1} w_i F(t, \tau_i, \tau_{i+1}),$$

where

$$w_i = \frac{\int_{\tau_i}^{\tau_{i+1}} w(u)\, du}{\int_{\tau_1}^{\tau_N} w(u)\, du}.$$

Originally we modelled $t \mapsto f(t, T)$, for all $t \leq \tau < +\infty$. Now we want to model $t \mapsto F(t, \tau_1, \tau_2)$, for all $t \leq \tau_1 \leq \tau_2 \leq +\infty$, i.e. for any τ_1, τ_2. So what about no-arbitrage condition when modelling any forward?

We start by fixing a delivery period τ_S a start date and τ_E an end date and continue by making a subdivision of the interval $[\tau_S, \tau_E]$ using an Euler discretization of the following form:

$$\tau_k = \tau_S + (k-1)\Delta; \qquad k = \{1, \ldots, N\},$$

such that $\tau_N = \tau_E$, with $\Delta = \frac{\tau_E - \tau_S}{N-1}$. Then the no-arbitrage condition is written as

$$F(t, \tau_S, \tau_E) = \sum_{k=1}^{N-1} \frac{\frac{1}{\Delta}\int_{\tau_k}^{\tau_{k+1}} w(u)\, du}{\int_{\tau_S}^{\tau_E} w(u)\, du} F(t, \tau_k, \tau_{k+1})\, \Delta.$$

If we now let $\Delta \searrow 0 \Rightarrow N \nearrow +\infty$ and therefore

$$\sum_{k=1}^{N-1} \frac{\frac{1}{\Delta}\int_{\tau_k}^{\tau_{k+1}} w(u)\, du}{\int_{\tau_S}^{\tau_E} w(u)\, du} \longrightarrow \int_{\tau_S}^{\tau_E} \frac{w(u)}{\int_{\tau_S}^{\tau_E} w(v)\, dv} du,$$

and therefore we have

$$F(t, \tau_S, \tau_E) = \sum_{k=1}^{N-1} \frac{\frac{1}{\Delta}\int_{\tau_k}^{\tau_{k+1}} w(u)\, du}{\int_{\tau_S}^{\tau_E} w(u)\, du} F(t, \tau_k, \tau_{k+1}) \xrightarrow{\Delta \searrow 0} \int_{\tau_s}^{\tau_E} \frac{w(u)}{\int_{\tau_S}^{\tau_E} w(v)\, dv} F(t, u, u)\, du,$$

where $F(t, u, u) = f(t, u)$. Hence the no-arbitrage condition is stated as (in continuous τ_1, τ_2)

$$F(t, \tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} \hat{w}(u, \tau_1, \tau_2)\, F(t, u)\, du.$$

Question now is: "does the model (0.3) satisfy the no-arbitrage condition?" Not in general unless we specify certain properties of σ such as

$$\sigma(t, \tau_1, \tau_2) = \sigma(t, \tau_1).$$

Proof. $w(u) = 1 \Rightarrow (\tau_2 - \tau_1) F(t, \tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} F(t, u, u)\, du$,

$$F(t, \tau_1, \tau_2) + (\tau_2 - \tau_1) \frac{\partial F}{\partial \tau_2}(t, \tau_1, \tau_2) = F(t, \tau_2, \tau_2)\, (*),$$

therefore

$$F(t, \tau_1, \tau_2) = F(0, \tau_1, \tau_2) \exp\left\{ -\frac{1}{2}\int_0^t \sigma^2(s, \tau_1, \tau_2)\, ds + \int_0^t \sigma(s, \tau_1, \tau_2)\, dW(s) \right\}$$

$$\frac{\partial F(t)}{\partial \tau_2} = \frac{\partial F(0)}{\partial \tau_2} \exp\{\cdots\} + F(0) \exp\{\cdots\}$$
$$\times \left(-\int_0^t \sigma(s, \tau_1, \tau_2) \frac{\partial \sigma}{\partial \tau_2}(s, \tau_1, \tau_2)\, ds + \int_0^t \frac{\partial \sigma}{\partial \tau_2}(s, \tau_1, \tau_2)\, dW(s) \right)$$

$$\frac{\partial F}{\partial \tau_2}(t) = \frac{\frac{\partial F(0)}{\partial \tau_2}}{F(0)} F(t) + F(t) \left(-\int_0^t \sigma(s) \frac{\partial \sigma}{\partial \tau_2}(s)\, ds + \int_0^t \frac{\partial \sigma}{\partial \tau_2}(s)\, dW(s) \right).$$

Now the L.H.S. of $(*)$ is given by

$$(*) = F(t, \tau_1, \tau_2) \left(1 + \frac{\frac{\partial F(0, \tau_1, \tau_2)}{\partial \tau_2}}{F(0, \tau_1, \tau_2)} (\tau_2 - \tau_1) \right.$$
$$\left. + (\tau_2 - \tau_1) \left(-\int_0^t \sigma(s, \tau_1, \tau_2) \frac{\partial \sigma}{\partial \tau_2}(s, \tau_1, \tau_2)\, ds + \int_0^t \frac{\partial \sigma}{\partial \tau_2}(s, \tau_1, \tau_2)\, dW(s) \right) \right),$$

where $\int_0^t \frac{\partial \sigma}{\partial \tau_2}(s, \tau_1, \tau_2)\, dW(s)$ is a Gaussian random variable with zero mean. The R.H.S. of $(*)$ is given by

$$(*) = F(t, \tau_1, \tau_2) > 0.$$

So if $\frac{\partial \sigma}{\partial \tau_2} \neq 0 \Rightarrow \mathbb{P}(\text{LHS of } (*) < 0) > 0.$ $\square$