TRIAL EXAM SPRING 2018

MAT4770 & MAT9770

Excercise 1

a. Define a CAR(p)-process X, for $p \geq 1$. Consider a $p-$dimensional stochastic process $\vec{Z}(t) \in \mathbb{R}^p$s.t. has the following dynamics

$$d\vec{Z}(t) = A\vec{Z}(t)\,dt + \vec{e}_p \sigma dB(t), \tag{0.1}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_p & -\alpha_{p-1} & -\alpha_{p-2} & \cdots & -\alpha_1 \end{bmatrix} \in M_{p\times p}(\mathbb{R}), \qquad \alpha_1, \ldots, \alpha_p \text{are all positive numbers,}$$

$B(t)$ is a 1-dimensional Brownian motion in $\mathbb{R}$,

$$\vec{e}_p = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \in M_{p\times 1}(\mathbb{R}).$$

We define the CAR(p)-process as the stochastic process $\vec{X}(t)$ given by

$$\vec{X}(t) = \vec{e}_1^{\,T} \vec{Z}(t) = Z_1(t). \tag{0.2}$$

b. Let a commodity price follow a CAR(p) $-$process, that is $S(t) = X(t)$. Introduce a pricing measure $\mathbb{Q} \sim \mathbb{P}$ by using a constant market price of risk θ in Girsanov's Theorem. Derive the forward price $t \mapsto f(t,\tau)$, $t \leq \tau$ for a contract delivering at time $\tau > 0$. Analyse what happens with the forward price when $\tau \to \infty$ in view of properties of the CAR(p) $-$process. By the arbitrage free-argument, we know the forward price $f(t,\tau)$ is given by the following expression known as the arbitrage free forward pricing formula

$$f(t,\tau) = \mathbb{E}_{\mathbb{Q}}\left[S(\tau) \mid \mathcal{F}_t\right]. \tag{0.3}$$

Date: 10th April 2019.

In order to compute this we first need to derive the risk-free dynamics for S, through Girsanov's Theorem. Notice $S(\tau) = X(\tau)$.

Theorem 1. *(Girsanov's Theorem) Let $\theta \in \mathbb{R}$ and define forall $t \leq \tau$, $dW(t) = dB(t) - \theta dt$ and $\mathcal{E}(t) = \exp\left\{\theta B(t) - \frac{1}{2}\theta^2 t\right\}$.*

If $t \mapsto \mathcal{E}(t)$; $t \leq \tau$ is a martingale, then $\exists! \mathbb{Q} \sim \mathbb{P}$ such that $W(t)$ is a $\mathbb{Q}-$Brownian motion for $t \leq \tau$. Moreover

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \mathcal{E}(t), \quad \forall t \leq \tau.$$

Now we know from Equation (0.1), that is the $\mathbb{P}-$dynamics of $\vec{Z}(t)$ and Girsanov's Theorem (1) we can write

$$\begin{aligned} d\vec{Z}(t) &= A\vec{Z}(t)\,dt + \vec{e}_p\sigma\left(dW(t) + \theta dt\right), \\ &= A\vec{Z}(t)\,dt + \sigma\theta\vec{e}_p dt + \sigma\vec{e}_p dW(t). \end{aligned}$$

In order to find the solution to this equation we consider the process $e^{-At}\vec{Z}(t)$, then we know

$$\begin{aligned} d\left(e^{-At}\vec{Z}(t)\right) &= -Ae^{-At}\vec{Z}(t)\,dt + e^{-At}d\vec{Z}(t), \\ &= -Ae^{-At}\vec{Z}(t)\,dt + e^{-At}\left[A\vec{Z}(t)\,dt + \sigma\theta\vec{e}_p dt + \sigma\vec{e}_p dW(t)\right], \\ &= e^{-At}\vec{e}_p\sigma\theta dt + e^{-At}\vec{e}_p\sigma dW(t). \end{aligned}$$

Now integrating in $[t, \tau]$ for all $t \leq \tau$ we have

$$\begin{aligned} \int_t^\tau d\left(e^{-Au}\vec{Z}(u)\right) &= \int_t^\tau e^{-Au}\vec{e}_p\sigma\theta du + \int_t^\tau e^{-Au}\vec{e}_p\sigma dW(u), \\ e^{-A\tau}\vec{Z}(\tau) &= e^{-At}\vec{Z}(t) + \int_t^\tau e^{-Au}\vec{e}_p\sigma\theta du + \int_t^\tau e^{-Au}\vec{e}_p\sigma dW(u), \\ \vec{Z}(\tau) &= e^{A(\tau-t)}\vec{Z}(t) + \int_t^\tau e^{A(\tau-u)}\vec{e}_p\sigma\theta du + \int_t^\tau e^{A(\tau-u)}\vec{e}_p\sigma dW(u). \end{aligned}$$

Trivially by the definition of CAR(p) $-$process we know that the risk-free dynamics for $\vec{X}(\tau) = e_1^T\vec{Z}(\tau)$,therefore we can write the forward using Equation (0.3) as follows.

$$\begin{aligned} f(t,\tau) &= \vec{e}_1^T\mathbb{E}_{\mathbb{Q}}\left[\vec{Z}(\tau) \mid \mathcal{F}_t\right] \\ &= \vec{e}_1^T e^{A(\tau-t)}\vec{Z}(t) + \sigma\theta\int_t^\tau \vec{e}_1^T e^{A(\tau-t)}\vec{e}_p ds. \end{aligned}$$

Now we do a change of variables as follows to reexpress the previous equation

$$f(t,\tau) = \vec{e}_1^T e^{A(\tau-t)}\vec{Z}(t) + \sigma\theta\int_0^{\tau-t} \vec{e}_1^T e^{As}\vec{e}_p ds.$$

Where $\int_0^{\tau-t} \vec{e}_1^T e^{As}\vec{e}_p ds = \vec{e}_1^T A^{-1}\left(e^{A(\tau-t)} - Id\right)\vec{e}_p$, as the expression is integrable given that A is always invertible. Note that matrix A has eigenvalues with negative real part $(\lambda_1, \ldots, \lambda_p)$ and eigenvectors $(\vec{v}_1, \ldots, \vec{v}_p)$ which form a basis.

$$\vec{Z}(t) = \sum_{i=1}^{p} z_i(t)\,\vec{v}_i.$$

Therefore we have

$$\begin{aligned} e^{A(\tau-t)}\vec{Z}(t) &= \sum_{i=1}^{p} z_i(t)\,e^{A(\tau-t)}\vec{v}_i \\ &= \sum_{i=1}^{p} z_i(t)\,e^{\lambda_i(\tau-t)}\vec{v}_i \\ &= \sum_{i=1}^{p} z_i(t)\,e^{Re(\lambda_i)(\tau-t)}e^{iIm(\lambda_i)(\tau-t)}\vec{v}_i. \end{aligned}$$

Since the real part is negative we know that $e^{Re(\lambda_i)(\tau-t)} \to 0$, when $\tau \to \infty$. Therefore we have

$$\lim_{\tau\to\infty} \vec{e}_1^T e^{A(\tau-t)}\vec{Z}(t) = 0.$$

Similarly

$$\vec{e}_1^T A^{-1}\left(e^{A(\tau-t)} - Id\right)\vec{e}_p \overset{\tau\to\infty}{\longrightarrow} -\vec{e}_1^T A^{-1}\vec{e}_p.$$

Therefore we have that

$$\lim_{\tau\to\infty} f(t,\tau) = -\sigma\theta\vec{e}_1^T A^{-1}\vec{e}_p,$$

this is in the long end we get a constant as $\tau \to \infty$.

c. Suppose the interest rate is zero, $r = 0$. Consider a forward contract delivering the accumulated value of S over the period $[\tau_1, \tau_2]$, with $\tau_2 > \tau_1 > 0$. Derive an expression for the price $F(t, \tau_1, \tau_2)$, $\quad t \le \tau_1$..

$$F(t,\tau_1,\tau_2) = \mathbb{E}_{\mathbb{Q}}\left[\int_{\tau_1}^{\tau_2} S(u)\,du \mid \mathcal{F}_t\right],$$

by section (b) in this exercise we have

$$\begin{aligned} F(t,\tau_1,\tau_2) &= \int_{\tau_1}^{\tau_2} f(t,u)\,du \\ &= \vec{e}_1^T \int_{\tau_1}^{\tau_2} e^{A(u-t)}du\vec{Z}(t) + \sigma\theta\vec{e}_1^T A^{-1}\int_{\tau_1}^{\tau_2}\left(e^{A(u-t)} - Id\right)du\vec{e}_p \\ &= \vec{e}_1^T A^{-1}\left(e^{A(\tau_2-t)} - e^{A(\tau_1-t)}\right)\vec{Z}(t) \\ &\quad + \sigma\theta\vec{e}_1^T A^{-1}\left(A^{-1}\left(e^{A(\tau_2-t)} - e^{A(\tau_1-t)}\right) - Id(\tau_2-\tau_1)\right)\vec{e}_p. \end{aligned}$$

d. Imagine that the temperature market trades a forward with "delivery" over the month of July (with forward price F_J), as well as four

weekly contracts "delivering" the first, second third and fourth week of July (with forward prices F_1, F_2, F_3, F_4). We assume that these four weeks cover exactly the month of July. By "delivery", we mean a stream of money matching the measured accumulated temperature over the period. Suppose that $F_J > F_1 + F_2 + F_3 + F_4$. Show how you can create an arbitrage position. Does your model in 1c allow for such an arbitrage? If $F_J > F_1 + F_2 + F_3 + F_4$ holds, then one can build the following arbitrage strategy.

- Today:
  - Sell July forward contract F_J at 0$
  - Buy F_1, F_2, F_3, F_4 contracts at 0$
  - Total cost of investment: 0$.
- July:
  - Deliver July forward contract (income of F_J\$)
  - Receive the weekly forward contracts and use them to deliver the July contract (outcome of $-(F_1 + F_2 + F_3 + F_4)$\$)
  - Total cost of investment: $F_J - (F_1 + F_2 + F_3 + F_4) > 0$. We have built an arbitrage strategy.

Our model in 1c does not allow arbitrage as we can split the integral $\int_{\tau_1}^{\tau_2} f(t, u)\, du$ into 4 weeks.

Exercise 2

Let $I(t)$ be a compound Poisson process with jump intensity $\lambda = 1$ and jump size distribution being exponential, with mean jumps equal to 1.

a. On $0 \leq t \leq T < \infty$, define a probability $\mathbb{Q} \sim \mathbb{P}$ by the Esscher transform of I. What is the logarithm of the moment generating function of I with respect to $\mathbb{Q}$. Knowing that the jumps are still exponentially distributed under $\mathbb{Q}$, find the jump intensity and the mean jump size under $\mathbb{Q}$. Discuss, in terms of the market price of risk (i.e., the parameter in the Esscher transform), what happens with the mean jump size. Consider a CPP given by $I(t) = \sum_{k=1}^{N(t)} J_k$, $\lambda = 1$, such that $J_k \sim \exp\{\lambda\}$. By the Esscher transform (always need to state conditions so that the Esscher transform is well defined) we have

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left\{\tilde{\theta} I(t) - \varphi\left(\tilde{\theta}\right) t\right\}; \qquad \varphi\left(\tilde{\theta}\right) = \log \mathbb{E}\left[e^{\tilde{\theta} I(1)}\right],$$

only defined for $\tilde{\theta}$'s where $\mathbb{E}\left[e^{\tilde{\theta} I(1)}\right] < +\infty$.

$$\mathbb{E}\left[e^{\tilde{\theta} I(1)}\right] = \mathbb{E}\left[e^{\tilde{\theta} \sum_{k=1}^{N(1)} J_k}\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[\mathbb{E}\left[e^{\tilde{\theta}\sum_{k=1}^{n} J_k} \mid N(1) = n\right]\right] \\
&= e^{-\lambda}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathbb{E}\left[e^{\tilde{\theta}J}\right]^n \\
&= \exp\left\{-\lambda + \lambda\mathbb{E}\left[e^{\tilde{\theta}J}\right]\right\}.
\end{aligned}
$$

Now $\varphi\left(\tilde{\theta}\right) = \lambda\left(\mathbb{E}\left[e^{\tilde{\theta}J}\right] - 1\right)$ as long as $\mathbb{E}\left[e^{\tilde{\theta}J}\right] < +\infty$, but $\mathbb{E}\left[e^{\tilde{\theta}J}\right] = \int_0^\infty e^{\tilde{\theta}z}e^{-z}dz = \frac{1}{1-\tilde{\theta}}$; $\tilde{\theta} < 1$. Thus when $\tilde{\theta} < 1$ we have the Esscher transform is well defined.

We will now find the log-MGF of I w.r.t. $\mathbb{Q}$.

$$
\begin{aligned}
\varphi_{\mathbb{Q}}(x) &= \log \mathbb{E}_{\mathbb{Q}}\left[e^{xI(1)}\right] \\
&= \log \mathbb{E}\left[e^{xI(1)}e^{\tilde{\theta}I(1)-\varphi\left(\tilde{\theta}\right)}\right] \\
&= \left(\log \mathbb{E}\left[e^{\left(x+\tilde{\theta}\right)I(1)}\right]\right) - \varphi\left(\tilde{\theta}\right) \\
&= \varphi\left(x+\tilde{\theta}\right) - \varphi\left(\tilde{\theta}\right), \quad \text{when } x+\tilde{\theta} < 1 \\
&= \lambda\left(\frac{x+\tilde{\theta}}{1-\left(x+\tilde{\theta}\right)} - \frac{\tilde{\theta}}{1-\tilde{\theta}}\right) \\
&= \lambda\frac{x}{\left(1+\tilde{\theta}\right)\left(1-\tilde{\theta}-x\right)}.
\end{aligned}
$$

If $J \sim \text{Exp}\{\xi\}$, under $\mathbb{Q}$, then

$$
\begin{aligned}
\varphi_{\mathbb{Q}}(x) &= \lambda_{\mathbb{Q}}\frac{x}{\xi - x}; \qquad x < \xi \\
&= \lambda\frac{x}{\left(1-\tilde{\theta}\right)\left(1-\tilde{\theta}-x\right)}.
\end{aligned}
$$

Let $\xi = 1-\tilde{\theta}$ and $\lambda_{\mathbb{Q}} = \frac{\lambda}{1-\theta}$. If $\tilde{\theta} \in (0,1) \Rightarrow$ increased mean jump size we get more frequent jumps. If $\tilde{\theta} < 0 \Rightarrow$ we get smaller jumps on average and less frequent.

b. Define a spot price S with dynamics $S(t) = \exp(X(t) + Y(t))$, where Y has dynamics under $\mathbb{P}$ given by

$$dY(t) = -bY(t)\,dt - dI(t),$$

and

$$dX(t) = -\alpha X(t)\,dt + dB(t).$$

Find the characteristic function of $X(t) + Y(t)$, and find its limit when $t \to \infty$. Notice the negative jumps in Y.

$$
\begin{cases}
X(t) &= e^{-\alpha t}X(0) + \int_0^t e^{-\alpha(t-s)}dB(s), \\
Y(t) &= e^{-bt}Y(0) - \int_0^t e^{-b(t-s)}dI(s).
\end{cases}
$$

Characteristic function of

$$\begin{aligned}X(t)+Y(t) &= \mathbb{E}\left[e^{ix(X(t)+Y(t))}\right] \\ &= \exp\left\{ixe^{-\alpha t}X(0)+ixe^{-bt}Y(0)\right\} \\ &\quad\times \mathbb{E}\left[\exp\left\{ix\int_0^t e^{-\alpha(t-s)}dB(s)\right\}\right](*) \\ &\quad\times \mathbb{E}\left[\exp\left\{ix\int_0^t e^{-b(t-s)}dI(s)\right\}\right](**),\end{aligned}$$

where

$$\int_0^t e^{-\alpha(t-s)}dB(s)\sim N\left(0,\int_0^t e^{-2\alpha(t-s)}ds\right),$$

and

$$\begin{aligned}(*) &= \exp\left\{-\frac{x^2}{4\alpha}\left(1-e^{-2\alpha t}\right)\right\}\overset{t\nearrow\infty}{\longrightarrow} e^{-\frac{x^2}{4\alpha}}, \\ (**)\text{ by (a)} &= \exp\left\{\int_0^t \varphi\left(-ixe^{-b(t-s)}\right)ds\right\} \\ &= \exp\left\{\lambda\int_0^t \frac{-ixe^{-bs}}{1+ixe^{-bs}}ds\right\} \\ &= \exp\left\{\frac{\lambda}{b}\ln\left(1+ixe^{-bs}\right)|_{s=0}^t\right\} \\ &= \exp\left\{\frac{\lambda}{b}\ln\left(\frac{1+ixe^{-bx}}{1+ix}\right)\right\}\overset{t\nearrow\infty}{\longrightarrow} e^{\frac{\lambda}{b}\ln\left(\frac{1}{1+ix}\right)}.\end{aligned}$$

Therefore

$$\mathbb{E}\left[e^{ix(X(t)+Y(t))}\right]\overset{t\nearrow\infty}{\longrightarrow}\exp\left\{-\frac{x^2}{4\alpha}\right\}\frac{1}{(1+ix)^{\lambda/b}},$$

where $\exp\left\{-\frac{x^2}{4\alpha}\right\}$ is the normal distribution and $\frac{1}{(1+ix)^{\lambda/b}}$ is the Gamma distribution. So the product of the two distributions is somehow defining the characteristic function of the sum of the two random variables.

c. Derive the forward price $t\mapsto f(t,\tau)$ for $t\le\tau$, for a forward contract delivering at time $\tau>0$. The market price of risk (the parameter in the Esscher transform) is set equal to $\tilde{\theta}=1$, while the market price of risk associated with X is $\hat{\theta}=0$ (you do no change of probability for X).

$$\begin{aligned}f(t,\tau) &= \mathbb{E}_{\mathbb{Q}}\left[e^{X(\tau)+Y(\tau)}\mid\mathcal{F}_t\right] \\ &= \exp\left\{e^{-\alpha(\tau-t)}X(t)+e^{-b(\tau-t)}Y(t)\right\} \\ &\quad\times\mathbb{E}_{\mathbb{Q}}\left[e^{\int_t^\tau e^{-\alpha(\tau-s)}dB(s)}\right](*) \\ &\quad\times\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^\tau e^{-b(\tau-s)}dI(s)}\right](**),\end{aligned}$$

where

$$(*) = \exp\left\{\frac{1}{2}\int_t^\tau e^{-2\alpha(\tau-s)}ds\right\},$$

$$(**) = \exp\left\{\int_t^\tau \varphi_{\mathbb{Q}}\left(-e^{-b(\tau-s)}\right)ds\right\}$$

$$= \exp\left\{\int_0^{\tau-t}\left(\varphi\left(1-e^{-bs}\right)-\varphi\left(1\right)\right)ds\right\}.$$