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SOLUTIONS, EXAM JUNE 2018, MAT4770

(1)

a) Esscher transform $\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = \exp(\theta I(t) - \varphi(\theta)t)$

Where $\varphi(\theta) = \ln E[e^{\theta I(1)}]$.

Esscher is defined when $E[e^{\theta I(1)}] < \infty$.

$$E[e^{\theta I(1)}] = E\left[E\left[e^{\theta \sum_{k=1}^{N(1)} \partial_k} \mid N(1)\right]\right]$$

$$= e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} E[e^{\theta \partial}]^n$$

$$= \exp(\lambda(E[e^{\theta\partial}] - 1))$$

$E[e^{\theta I(1)}] < \infty$ if and only if $E[e^{\theta \partial}] < \infty$.

We assume this to hold. I is CPP under Q, as we know from lectures/theory.

$$E_Q[e^{xI(1)}] = E\left[e^{xI(1)} \left.\frac{dQ}{dP}\right|_{\mathcal{F}_1}\right]$$

$$= e^{-\varphi(\theta)} E[e^{(x+\theta)I(1)}]$$

$= \exp(\varphi(x+\theta) - \varphi(\theta))$, for all x such that $E[e^{(x+\theta)\partial}] < \infty$.

$$\varphi_Q(x) = \varphi(x+\theta) - \varphi(\theta) = \lambda\left(E\left[e^{(x+\theta)J}\right] - 1 - E\left[e^{\theta J}\right] + 1 \right)$$

$$= \lambda\, E\left[e^{\theta J}\left(e^{xJ} - 1\right)\right]$$

$$= \lambda \int_{\mathbb{R}} \left(e^{xz} - 1\right) e^{\theta z}\, p(z)\, dz$$

i.e.,

$$\lambda_Q = \lambda \int_{\mathbb{R}} e^{\theta z}\, p(z)\, dz, \qquad J \overset{Q}{\sim} \frac{e^{\theta z}}{E\left[e^{\theta J}\right]}\, p(z)\, dz$$

↑

Q-distributed

b) Finding first the solution of $X(t)$:

$$d\left(e^{bt} X(t)\right) = b e^{bt} X(t)\, dt + e^{bt}\, dX(t)$$

$$= ab\, dt + e^{bt}\, dI(t)$$

$$\underline{X(t)} = e^{-bt} X(0) + ab \int_0^t e^{-b(t-s)}\, ds + \int_0^t e^{-b(t-s)}\, dI(s)$$

$$= e^{-bt} X(0) + a\left(1 - e^{-bt}\right) + \int_0^t e^{-b(t-s)}\, dI(s)$$

$$E[S(t)] = \exp\left(e^{-bt} X(0) + a\left(1 - e^{-bt}\right)\right) \underbrace{E\left[\exp\left(\int_0^t e^{-b(t-s)}\, dI(s)\right)\right]}_{= \exp\left(\int_0^t \varphi\left(e^{-bs}\right) ds\right)}$$

$$\varphi(x) = \ln E\left[e^{x\,\mathcal{I}(1)}\right] \overset{\text{from a)}}{=} \lambda\Big(\underbrace{\int_{\mathbb{R}} e^{xz}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}\,dz}_{=e^{\frac{1}{2}x^2}} - 1\Big)$$

$$= \lambda\left(e^{1/2x^2} - 1\right)$$

$$\underline{\underline{E[S(t)] = \exp\left(e^{-bt}\Sigma(0) + a(1-e^{-bt}) + \lambda\int_0^t \left(e^{\frac{1}{2}e^{-2bs}} - 1\right)ds\right)}}$$

$$\int_0^t \left(e^{\frac{1}{2}e^{-2bs}} - 1\right)ds = \sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)^n \int_0^t (e^{-2bs})^n ds}{n!} = \sum_{n=1}^{\infty} \frac{2^{-n}}{n!}(-2nb)^{-1}\left(e^{-2nbt} - 1\right)$$

$$\xrightarrow{t\to\infty} \frac{1}{2b}\sum_{n=1}^{\infty}\frac{2^{-n}}{n!\,n} < \infty$$

Hence

$$\underline{\underline{\lim_{t\to\infty} E[S(t)] = \exp\left(a + \frac{1}{2b}\sum_{n=1}^{\infty}\frac{2^{-n}}{n!\,n}\right)}}$$

c) $f(t,T) = E_Q[S(T)\,|\,\mathcal{F}_t]$, $t \le T$

$$= E_Q\Big[\exp\Big(e^{-b(T-t)}\Sigma(t) + a(1-e^{-b(T-t)}) + \underbrace{\int_t^T e^{-b(T-s)}\,d\mathcal{I}(s)}_{\mathcal{F}_t\text{-independent}}\Big)\,\Big|\,\mathcal{F}_t\Big]$$

$$= \left(e^{\Sigma(t)}\right)^{e^{-b(T-t)}} e^{a(1-e^{-b(T-t)})} \underbrace{E_Q\left[e^{\int_t^T e^{-b(T-s)}\,d\mathcal{I}(s)}\right]}_{\text{Compute this}}$$

$$E_t\left[e^{\int_t^T e^{-b(T-s)}\, dZ(s)}\right] = \exp\left(\int_t^T \left(\varphi\left(\theta + e^{-b(T-s)}\right) - \varphi(\theta)\right) ds\right)$$

$$\overset{\theta=1}{=} \exp\left(\int_0^{T-t} \left(\varphi\left(1+e^{-bs}\right) - \varphi(1)\right) ds\right)$$

$$= \exp\left(\lambda \int_0^{T-t} \left(E\left[e^{(1+e^{-bs})J}\right] - E\left[e^{J}\right]\right) ds\right)$$

$$J \sim N(0,1)$$

$$= \exp\left(\lambda \int_0^{T-t} \left(e^{\frac{1}{2}(1+e^{-bs})^2} - e^{\frac{1}{2}}\right) ds\right)$$

Hence,

$$f(t,T) = S(t)^{e^{-b(T-t)}} \exp\left(a\left(1-e^{-b(T-t)}\right) + \lambda \int_0^{T-t} \left(e^{\frac{1}{2}(1+e^{-bs})^2} - e^{\frac{1}{2}}\right) ds\right)$$

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a) Let $A = \begin{bmatrix} 0 & 1 \\ -\alpha_2 & -\alpha_1 \end{bmatrix}$.

By Itô's Formula, we have

$$\vec{X}(t) = e^{At}\vec{X}(0) + \int_0^t \left(e^{A(t-s)}\vec{e}_2\right) \sigma\, dB(s)$$

$$\Rightarrow Y(t) = \vec{e}_1^{\,T} e^{At} \vec{X}(0) + \int_0^t \underbrace{\left(\vec{e}_1^{\,T} e^{A(t-s)} \vec{e}_2\right)}_{\text{deterministic function.}} \sigma\, dB(s)$$

We know that the stochastic integral w.r.t. a deterministic function is Gaussian with zero mean and variance given by the integrated square of the integrand:

$$\Sigma(t) \sim \mathcal{N}\left(\vec{e}_1^{\,T} e^{At} \vec{x}(0),\ \sigma^2 \int_0^t (\vec{e}_1^{\,T} e^{A(t-r)} \vec{e}_2)^2 \, dr\right)$$

$$E[\Sigma(t)] = \vec{e}_1^{\,T} e^{At} \vec{x}(0)$$

$$Var(\Sigma(t)) = \sigma^2 \int_0^t (\vec{e}_1^{\,T} e^{Au} \vec{e}_2)^2 \, du$$

b)

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ \alpha_2 & \lambda + \alpha_1 \end{vmatrix} = \lambda^2 + \alpha_1 \lambda + \alpha_2 = 0$$

$$\Rightarrow \lambda = -\frac{1}{2}\alpha_1 \pm \frac{1}{2}\sqrt{\alpha_1^2 - 4\alpha_2}$$

If $\alpha_1^2 > 4\alpha_2$, then we have two real eigenvalues and moreover, since $\sqrt{\alpha_1^2 - 4\alpha_2} \le \alpha_1$,

$$\lambda_{1,2} = -\frac{\alpha_1}{2} \pm \frac{1}{2}\sqrt{\alpha_1^2 - 4\alpha_2} < 0.$$

If $\alpha_1^2 < 4\alpha_2$, then $\sqrt{\alpha_1^2 - 4\alpha_2} = i\sqrt{4\alpha_2 - \alpha_1^2}$

with $i = \sqrt{-1}$, and $Re(\lambda) = -\frac{\alpha_1}{2} < 0$.

In conclusion, A has two eigenvalues with negative real part.

From this we can conclude that $\Psi(t)$ has a limiting distribution which is normal:

$$\Psi(t) \xrightarrow{d} N\left(0, \sigma^2 \int_0^\infty (\vec{e}_1^T e^{As} \vec{e}_p)^2 ds\right)$$

c)

Price of put is (with $r=0$)

$$P(t) = E_Q\left[\max(K - e^{\Psi(T)}, 0) \mid \mathcal{F}_t\right]$$

To avoid confusion between T being exercise time of option AND the transpose of a vector, I use from now on $\vec{e}_1'$ as the transpose!!!

$$\vec{X}(T) = e^{A(T-t)}\vec{X}(t) + \int_t^T e^{A(T-s)}\vec{e}_2\,\sigma\,dB(s)$$

and therefore

$$P(t) = \mathbb{E}\left[\max\left(K - \exp\left(\underbrace{\vec{e}_1^{\,\prime} e^{A(T-t)}\vec{X}(t)}_{\mathcal{F}_t\text{-meas.}} + \underbrace{\int_t^T \vec{e}_1^{\,\prime} e^{A(T-s)}\vec{e}_2\,\sigma\,dB(s)}_{\text{independent of } \mathcal{F}_t \text{ \& Gaussian}}\right), 0\right) \Big| \mathcal{F}_t\right]$$

$$= \mathbb{E}\left[\max\left(K - \exp(x + \Sigma Z), 0\right)\right]_{x = \vec{e}_1^{\,\prime} e^{A(T-t)}\vec{X}(t)}$$

where $Z \sim N(0,1)$ and $\Sigma^2 = \sigma^2 \int_t^T \left(\vec{e}_1^{\,\prime} e^{A(T-s)}\vec{e}_2\right)^2 ds$

I leave it to you to compute

$$\mathbb{E}\left[\max\left(K - e^{x+\Sigma Z}, 0\right)\right]$$

which is a derivation very similar to Black&Scholes derivation.

(3)

a) W and I are independent (I is pure-jump while W is BM).

σ must be measurable and $\int_0^T \sigma^2(u)\,du < \infty$.

When this holds, $\int_0^t \sigma(T-u)\,dW(u)$ is a Gaussian random variable, and it follows that

$$E_Q\left[\exp\left(\int_0^t \sigma(T-u)\,dW(u)\right)\right] < \infty$$

We must also have that $E_Q\left[\exp\left(\int_0^t y(T-u)\,dI(u)\right)\right] < \infty$

But

$$E_Q\left[\exp\left(\int_0^t y(T-u)\,dI(u)\right)\right] = \exp\left(\int_0^t \varphi_Q(y(T-u))\,du\right)$$

where $\varphi_Q(x) = \ln E_Q[e^{xI(1)}]$, if this is defined.

But $E_Q[e^{xI(1)}] = \lambda_Q \int_{-\infty}^{\infty}(e^{xz}-1)\,P_J^Q(dz) < \infty$

if and only if $E_Q[e^{xJ}] < \infty$.

So, γ must be measurable and so that

$$\underline{\underline{E_Q\left[e^{\gamma(T-u)}\right] < \infty}} \text{ for all } u \leq T,$$

AND $\underline{\underline{\int_u^t |\varphi_Q(\gamma(T-u))| du < \infty}}$ to ensure that

$$\int_0^t \varphi_Q(\gamma(T-u)) du \text{ is well-defined}.$$

Finally, a must be integrable, i.e. measurable

and $\underline{\underline{\int_0^T |a(T-u)| du < \infty}}$

Under all these conditions, we have that

$$E_Q[f(t,T)] < \infty \text{ for all } t \leq T.$$

b)

From a) we know that $E_Q[f(t,T)] < \infty$ for $t \leq T$. $t \mapsto f(t,T)$ is also $\mathcal{F}_t$-adapted as $t \mapsto \int_0^t \sigma(T-u) dW(u)$ and $t \mapsto \int_0^t \gamma(T-u) dI(u)$ are.

Let $t > s$;

$$E_Q\left[f(t,T) \mid \mathcal{F}_s\right] = f(0,T)\, e^{\int_0^t a(T-u)\,du}\, E_Q\left[e^{\int_0^t \sigma\, dW + \int_0^t \gamma\, dI} \,\Big|\, \mathcal{F}_s\right]$$

$$= f(0,T)\, e^{\int_0^t a(T-u)\,du}\, e^{\int_0^s \sigma(T-u)\,dW(u) + \int_0^s \gamma(T-u)\,dI(u)}$$

$$\cdot\, E_Q\left[e^{\int_s^t \sigma(T-u)\,dW(u)}\; e^{\int_s^t \gamma(T-u)\,dI(u)}\right]$$

independent

$$= f(s,T)\, e^{\int_s^t a(T-u)\,du} \cdot e^{\frac{1}{2}\int_s^t \sigma^2(T-u)\,du} \cdot e^{\int_s^t \Phi_Q(\gamma(T-u))\,du}$$

Hence, f is Q-martingale if and only if

$$\int_s^t a(T-u)\,du = -\frac{1}{2}\int_s^t \sigma^2(T-u)\,du - \int_s^t \Phi_Q(\gamma(T-u))\,du \quad \text{for all } s \le t.$$

In this case $E_Q\left[f(t,T)\right] = E_Q\left[E_Q\left[f(t,T) \mid \mathcal{F}_0\right]\right]$

$$= E_Q\left[f(0,T)\right] = f(0,T)$$

by the double-conditioning rule.

c) $\sigma = \rho = 0$: Then,

$$f(t,T) = f(0,T)\exp\left(\int_0^t a(T-u)\,du + \int_0^t \gamma(T-u)\,dI(u)\right)$$

with $\int_s^t a(T-u)\,du = -\int_s^t \Phi_\varphi(\gamma(T-u))\,du$ for all $s\le t$.

Price of call at time zero is

$$C = E_\varphi\left[\max(f(T,\tau)-K,0)\right]$$

$$= E_\varphi\left[\max\left(f(0,T)e^{\int_0^T a(T-u)\,du + \int_0^T \gamma(T-u)\,dI(u)} - K, 0\right)\right]$$

Define $g(x) := \max(he^{x} - K, 0)$, $x \in \mathbb{R}$.

with $h = f(0,T)\exp\left(\int_0^T a(T-u)\,du\right)$.

Notice that $g\left(\int_0^T \gamma(T-u)\,dI(u)\right) = \max(f(T,\tau)-K,0)$

As $g(x) \sim he^{x}$ when x is large, $g(x) \notin L^1(\mathbb{R})$

For $\alpha > 1$, define

$$g_\alpha(x) = e^{-\alpha x}\max(he^{x}-K,0)$$

If $x > \ln\frac{K}{h}$, $g_\alpha(x) = he^{-(\alpha-1)x} - Ke^{-\alpha x}$

and if $x < \ln\frac{K}{h}$, $g_\alpha(x) = 0$.

It is then easy to show that $\int_{-\infty}^{\infty} |g_\alpha(x)|\,dx < \infty$

and thus $g_\alpha \in L^1(\mathbb{R})$.

We find the Fourier transform:

$$\underline{\underline{\hat{g}_\alpha(y)}} = \int_{-\infty}^{\infty} g_\alpha(x) e^{-ixy}\,dx$$

$$= \int_{\ln\frac{K}{h}}^{\infty} he^{-(\alpha-1)x} e^{-ixy}\,dx - K\int_{\ln\frac{K}{h}}^{\infty} e^{-\alpha x} e^{-ixy}\,dx$$

$$= h\frac{1}{\alpha-1+iy} e^{-(\alpha-1+iy)\ln\frac{K}{h}} - K\frac{1}{\alpha+iy} e^{-(\alpha+iy)\ln\frac{K}{h}}$$

$$= \underline{\underline{\frac{h^\alpha K^{1-\alpha}}{(\alpha-1+iy)(\alpha+iy)} e^{iy\ln(\frac{K}{h})}}}$$

We see $|\hat{g}_\alpha(y)| \sim \frac{1}{|\alpha-1+iy||\alpha+iy|} \sim \frac{1}{y^2}$ when y is large.

$\Rightarrow$ $\hat{g}_\alpha(\cdot) \in L^1(\mathbb{R})$, and Fourier inversion formula is valid.

$$e^{-\alpha x} \max(he^{x} - K, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}_\alpha(y) e^{iyx} \, dy$$

We find

$$E_Q\left[\max\left(h e^{\int_0^T \eta(T-u)\, dL(u)} - K, 0\right)\right]$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}_\alpha(y) \, E_Q\left[e^{(\alpha+iy)\int_0^T \eta(T-u)\, dL(u)}\right] dy$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}_\alpha(y) \exp\left(\int_0^T \varphi_Q\left((\alpha+iy)\eta(T-u)\right) du\right) dy$$

Wrapping up;

$$C = \frac{K^{1-\alpha}}{2\pi} f^\alpha(0,T) \, e^{\alpha \int_0^T a(T-u)\, du} \int_{-\infty}^{\infty} \frac{e^{iy \ln\left(\frac{K}{h}\right)}}{(\alpha - 1 + iy)(\alpha + iy)} e^{\int_0^T \varphi_Q((\alpha+iy)\eta(T-u))\, du} \, dy$$

$$h = f(0,T) e^{\int_0^T a(T-u)\, du}$$