17th March, 2022

MAT4770

Mandatory assignment 1 of 1

Submission deadline

Thursday 31^{rst} March 2022, 14:30 in Canvas (canvas.uio.no).

Instructions

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. Students who fail the assignment, but have made a genuine effort at solving the exercises, are given a second attempt at revising their answers. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Exercise 1

Fix a time horizon $T > 0$. Consider the model for the energy spot price $(S(t))_{t\in[0,T]}$ given by

$$S(t) := \Lambda(t)e^{Y_t},$$

where $\Lambda : [0,T] \to \mathbb{R}$ is a continuous seasonal function and $(Y(t))_{t\in[0,T]}$ is a non-Gaussian Ornstein-Uhlenbeck process with dynamics

$$dY(t) = -\alpha Y(t) + \sigma dL(t).$$

Here $\alpha > 0$ and $(L(t))_{t\in[0,T]}$ is a Lévy process with characteristic triplet $(0, 1, \nu)$, where ν is given by

$$\nu(dx) = \lambda \frac{1}{\sqrt{2\pi\rho^2}} e^{\frac{-x^2}{2\rho^2}} dx.$$

We assume that interest rates are $r = 0$. We denote by $(\mathcal{F}_t)_{t\in[0,T]}$ the filtration induced by $(S(t))_{t\in[0,T]}$.

(a) Prove that $(S(t))_{t\in[0,T]}$ belongs to the class of geometric spot price models. More precisely, prove that there is a Brownian motion $(B(t))_{t\geq 0}$ and constants $\alpha_1, \alpha_2, \sigma_1, \sigma_2 > 0$, such that $Y(t) = \hat{X}(t) + \hat{Y}(t)$ for $t \in [0,T]$ where $(X(t))_{t\in[0,T]}$ is an Ornstein-Uhlenbeck process with dynamics

$$d\hat{X}(t) = -\alpha_1 \hat{X}(t)dt + \sigma_1 dB(t)$$

and $(Y(t))_{t\geq 0}$ is an Ornstein-Uhlenbeck process of the form

$$d\hat{Y}(t) = -\alpha_2 \hat{Y}(t)dt + \sigma_2 dI(t),$$

where $(I(t))_{t\in[0,T]}$ is a pure-jump Lévy process.

(b) Specify the distribution of the pure-jump process $(I(t))_{t\in[0,T]}$ that we derived in (a). What kind of Lévy process is this?

(c) We introduce a pricing measure $Q^\theta \sim \mathbb{P}$ by the Escher transform, where $\theta = (\hat{\theta}, \tilde{\theta}) \in \mathbb{R}^2$. For which $\theta \in \mathbb{R}^2$ is the Escher transform well defined? Derive for such θ the dynamics of $(S(t))_{[0,T]}$ under Q^θ. How does the Escher transform change the distribution of jump sizes?

(d) Determine the price of a forward $f(t,T)$ by using the formula

$$f(t,T) = \mathbb{E}_{\mathbb{Q}^\theta}[S(T)|\mathcal{F}_t].$$

(e) Simulate paths of the model (e.g. with R). Try out some parameter specifications. Start out with $n = 365$ simulation points (daily observations during a year) and the parameter specifications: $\Lambda(t) = 20$ constant, $\alpha = 0.285$, $\sigma = 0.5$ $\rho = 0.67$ and $\lambda = 8.5$.

Do the simulated paths suggest that this model captures some stylised facts of energy prices?

Exercise 2

We assume that $(L(t))_{t\geq 0}$ is a Variance-Gamma process. I.e. it is a Lévy process given by

$$L(t) = \delta T(t) + \sigma W(T(t)) \quad t \geq 0,$$

where $\delta \in \mathbb{R}$, $\sigma > 0$, $(W(t))_{t\geq 0}$ is a standard Brownian motion and $(T(t))_{t\geq 0}$ is a Gamma-process, such that $T(t) \sim \Gamma(t\alpha, \alpha)$ for some $\alpha > 0$. Recall that the Gamma distribution $\Gamma(\beta, \alpha)$ for $\alpha, \beta > 0$ has the density $f_{\alpha,\beta}(x) = \mathbb{I}_{x\geq 0}\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$.

(a) Calculate the characteristic function $\phi_{L(1)}(\zeta) := \mathbb{E}[e^{i\zeta L(1)}]$. (Hint: Use that we already derived the characteristics of the Gamma process)

(b) Derive from the expression of the characteristic function of $(L(t))_{t\geq 0}$ that it is in distribution equal to the difference of two independent Gamma processes. I.e. show that

$$L(t) \overset{d}{=} T_1(t) - T_2(t), \quad t \geq 0,$$

where $(T_1(t))_{t\geq 0}$ and $(T_2(t))_{t\geq 0}$ are two independent Gamma processes. From that, derive an exact form for the Lévy measure ν of $(L(t))_{t\geq 0}$.

(c) Argue why the Variance-Gamma process has paths of finite variation.