UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	MAT4770 — Stochastic Modelling in Energy and Commodity Markets
Day of examination:	Wednesday June 15, 2022
Examination hours:	09.00 – 13.00

This problem set consists of 3 pages.

Appendices:	None
Permitted aids:	All, but it is not allowed to communicate or collaborate with others

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Throughout this exam, unless otherwise stated, we have given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$.

Problem 1

Let $(B(t))_{t\geq 0}$ be a Brownian motion, A a general $p \times p$-matrix and $\vec{e}_p$ the pth unit vector in $\mathbb{R}^p$ (a column vector!), where $p \in \mathbb{N}$. Define the stochastic process $(X(t))_{t\geq 0}$ with values in $\mathbb{R}^p$ as

$$X(t) = \int_0^t e^{A(t-u)} \vec{e}_p dB(u).$$

a

Let $\vec{e}_1$ be the first basis vector in $\mathbb{R}^p$ and $\vec{e}_1^\top$ its transpose. How does the matrix A look like when $Y(t) = \vec{e}_1^\top X(t)$ is a CAR(p)-process? Show that the function $g(u) = \vec{e}_1^\top e^{A(t-u)} \vec{e}_p$ defined for $0 \leq u \leq t$ is Itô integrable on $[0, t]$ for any matrix A. (Hint: either use matrix norms, or you can suppose that A has p distinct eigenvalues with corresponding p eigenvectors).

b

Compute $\int_0^t AX(s)ds$ to show that

$$dX(t) = AX(t)dt + \vec{e}_p dB(t).$$

c

Suppose that $Y(t)$ models the temperature at time t. Assume further that there is a market for temperature forwards, where the forward is written on a temperature index measuring the time the temperature is above a threshold

(Continued on page 2.)

c in a given period $[T_1, T_2]$. That is, the buyer of this forward receives at time T_2 the cash amount

$$\int_{T_1}^{T_2} 1(Y(s) > c)ds$$

where $1(\cdot)$ is the indicator function. Argue why the forward price at time $t \leq T_1$, denoted $F(t, T_1, T_2)$, is given by

$$F(t, T_1, T_2) = \mathbb{E}[\int_{T_1}^{T_2} 1(Y(s) > c)ds \,|\, \mathcal{F}_t]$$

when the pricing measure $\mathbb{Q} = \mathbb{P}$. What is the risk premium in this case? Calculate $F(t, T_1, T_2)$.

Problem 2

a

Assume $\mathbb{Q} \sim \mathbb{P}$ is a pricing measure. Suppose the forward price $f_1(t, \tau)$ at time t with delivery at time $\tau \geq t$ of a commodity is given by

$$f_1(t, \tau) = \exp(at + \sigma_1 W_1(t) + I(t))$$

where a and σ_1 are constants, $\sigma_1 > 0$. Here, $(W_1(t))_{t \geq 0}$ a Brownian motion and $(I(t))_{t \geq 0}$ is a compound Poisson process, both with respect to $\mathbb{Q}$. State conditions such that $t \mapsto f_1(t, \tau)$ is a $\mathbb{Q}$-martingale for $t \leq \tau$.

b

Let the forward price $f_2(t, \tau)$ of another commodity be given by

$$f_2(t, \tau) = \exp(-\frac{1}{2}\sigma_2^2 t + \sigma_2 W_2(t))$$

where $(W_2(t))_{t \geq 0}$ is a $\mathbb{Q}$-Brownian motion and $\sigma_2 > 0$ is a constant. Show that $t \mapsto f_2(t, \tau)$ is a $\mathbb{Q}$-martingale for $t \leq \tau$. For $-1 < \rho < 1$, define

$$W_1(t) = \rho W_2(t) + \sqrt{1 - \rho^2} W(t)$$

for $(W(t))_{t \geq 0}$ being a $\mathbb{Q}$-Brownian motion independent of W_2. Show that $\mathbb{E}_{\mathbb{Q}}[W_1(t) W_2(t)] = \rho t$.

c

Consider a spread option on the two forwards which has payoff $\max(f_1(T, \tau) - f_2(T, \tau), 0)$ at time $T \leq \tau$. Assume that the risk-free interest rate is zero, $r = 0$. The price of this option at time zero is

$$P = \mathbb{E}_{\mathbb{Q}}[\max(f_1(T, \tau) - f_2(T, \tau), 0)].$$

Introduce a probability measure $\widetilde{\mathbb{Q}}$ which has Radon-Nikodym derivative for $t \leq T$

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}\Big|_{\mathcal{F}_t} := f_2(t, \tau)$$

(Continued on page 3.)

Argue that $\widetilde{W}(t) = W_2(t) - \sigma_2 t$ defines a $\widetilde{\mathbb{Q}}$-Brownian motion for $t \leq T$ which is independent of W. Show that

$$P = \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[\max\left(\exp\left((a + \frac{1}{2}\sigma_2^2 t + \sigma_1 W_1(T) - \sigma_2 W_2(T) + I(t)\right) - 1, 0\right)\right]$$

d

Use Fourier methods to find an integral expression for P. What additional assumption do you need to make on the compound Poisson process I?

Problem 3

In this exercise, assume that we have given a (very simple) model for a forward price dynamics of a commodity under the market probability $\mathbb{P}$,

$$f(t, \tau) = \exp(I(t))$$

for $0 \leq t \leq \tau$. Here, τ is the delivery time of the commodity.

Using the Esscher transform, find an equation for the moment-generating function such that there exists a pricing measure $\mathbb{Q} \sim \mathbb{P}$ for which $t \mapsto f(t, \tau)$ is a $\mathbb{Q}$-martingale, when either:

a

I is a compound Poisson process, or,

b

I is a symmetric normal inverse Gaussian (NIG) Levy process. Derive an explicit condition for this Esscher transform when the NIG Levy process has mean zero.

Hint: You may find it useful that a moment-generating function of $I(1)$ in this case is given by

$$\mathbb{E}\left[e^{uI(1)}\right] = \exp\left(\mu u + \delta(\alpha - \sqrt{\alpha^2 - u^2})\right)$$

for the four parameters μ, α and δ with $\alpha > 0$ and $\delta > 0$. The moment-generating function is defined only for u such that $-\alpha \leq u \leq \alpha$.

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