1st February, 2023

MAT4770

Mandatory assignment 1 of 1

Submission deadline

Thursday 23rd February 2023, 14:30 in Canvas (canvas.uio.no).

Instructions

Note that you have **one attempt** to pass the assignment. This means that there are no second attempts.

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) no later than the same day as the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

In order to pass the assignment, all tasks must be answered.

Problem 1. In this problem you are going to analyse the so-called Schwartz-model, which is a stochastic process often used to model the dynamics of spot prices in commodity markets, as coffee, metals, oil etc... You will derive forward prices based on this model, as well as option prices on forwards. In this task, you have given a (filtered) probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{t\geq 0}$.

1a)

Let B be a Brownian motion, α and σ be given positive constants, and consider the Ornstein-Uhlenbeck process X defined by the dynamics

$$dX(t) = -\alpha X(t)dt + \sigma dB(t).$$

Show that, for $t \geq 0$,

$$X(t) = e^{-\alpha t}X(0) + \sigma e^{-\alpha t}\int_0^t e^{\alpha s}dB(s),$$

in *two different ways*: either by calculating $\int_0^t X(u)du$, or by resorting to Ito's Formula calculating the dynamics of $\exp(\alpha t)X(t)$. Show that for $s \geq t \geq 0$,

$$X(s) = e^{-\alpha(s-t)}X(t) + \sigma e^{-\alpha s}\int_t^s e^{\alpha v}dB(v)$$

Argue that $X(s)$ given $X(t)$ is a Gaussian random variable, with mean $\exp(-\alpha(s-t))X(t)$ and variance $\sigma^2\exp(-2\alpha s)\int_t^s \exp(2\alpha v)dv$ (the latter integral you can calculate). Why is $\int_t^s \exp(\alpha v)dB(v)$ *independent* of $X(t)$?

1b)

The Schwartz-model is defined by

$$S(t) = \exp(X(t)).$$

Use Ito's Formula to derive the dynamics $dS(t)$ for this stochastic process. Argue based on what you know from **1a)** that $S(t)$ is a lognormal random variable. What is the mean and variance of $S(t)$?

1c)

Let θ be a constant, and define the stochastic process

$$dW(t) = \theta dt + dB(t)$$

Argue using Girsanov's Theorem that there exists a probability $\mathbb{Q} \sim \mathbb{P}$ such that $W(t)$ is a $\mathbb{Q}$-Brownian motion for a fixed time interval $0 \leq t \leq \tau$. Calculate the forward price at time t, denoted $f(t, \tau)$, for a contract delivering the commodity at time $\tau \geq t$ using $\mathbb{Q}$ as the pricing measure, i.e., calculate

$$f(t, \tau) = \mathbb{E}_{\mathbb{Q}}[S(T) \mid \mathcal{F}_t]$$

Hint: conditioning on $\mathcal{F}_t$ will here mean that you condition on $X(t)$! Notice that you get a price in terms of $X(t)$. Re-express this as a price in terms of $S(t)$. Argue, by appealing to Ito's Formula that the $\mathbb{Q}$-dynamics of $f(t, \tau), t \leq \tau$ is

$$df(t, \tau) = f(t, \tau)\sigma e^{-\alpha(\tau - t)} dW(t)$$

This is a geometric Brownian motion with time-dependent volatility $\sigma \exp(-\alpha(\tau - t)), t \leq \tau$. What happens when *time-to-delivery* $\tau - t$ tends to zero (immediate delivery) and $\tau - t \to \infty$ (delivery in very far future)? Such a *volatility term structure* is know as the *Samuelson effect*.

1d)

Suppose you can trade in a call option on the forward price, with exercise time $T \leq \tau$ and strike price K. Assume the risk-free interest rate is r. Calculate the call option price at current time $t = 0$, i.e., calculate

$$C(T, \tau) = e^{-rT} \mathbb{E}_Q[\max(f(T, \tau) - K, 0)]$$

Express the call price $C(T, \tau)$ in terms of $f(0, \tau)$, the current forward price. You have found the *Black-76*-formula for call options on forwards with time-dependent volatility.

Problem 2. 2a)

In this exercise you will analyse futures prices on temperature, based on the CAR(3)-stochastic process on Stockholm temperature data found in Section 10.3 in the book. When you do the exercise, please also record the daily temperature over three consecutive days using yr.no, say (you read off the current temperature, and then next day and day after your use the average of maximum and minimum as the three consecutive days of "data" for the temperatures). You also fix the volatility σ to be constant equal to 2, i.e. $\sigma = 2$!

You want to enter a CAT-futures on Stockholm-temperature for June. Calculate the CAT-futures price when you consider today as $t = 0$? Hint 1: Notice that from the CAR(3)-dynamics, you have that $X_1'(t) = X_2(t)$ and $X_2'(t) = X_3(t)$, which means that $X_1''(t) = X_3(t)$. Hence, when you want to set values of the vector $X(t) = (X_1(t), X_2(t), X_3(t))$, you can use that $X_1(t) = T(t) - \Lambda(t)$, while $X_2(t) = T'(t) - \Lambda'(t)$ and $X_3(t) = T''(t) - \Lambda''(t)$. Use numerical differentiation to find the derivatives of the temperature, along with the yr.no-values. Hint 2: Matrix exponentials are best calculated numerically using Matlab or R, say.

2b)

In this task you are going to estimate a CAR(3)-dynamics on a temperature data series. Based on the daily average temperature data provided on the web-page of the course, estimate the seasonality function. Plot this function together with the data. Next, use statistical software (like R, or some packages in Matlab) to estimate an AR(3)-model to the data. Use the relations between AR(3) and CAR(3) to find the parameters of the A-matrix in the CAR(3)-model, and calculate the eigenvalues of A. Do these have a negative real part? Finally, on the deseasonalized data, plot the empirical autocorrelation function for the 100 first lags, as well as the partial autocorrelation function for the 10 first lags. Is a CAR(3)-model reasonable? Why/why not?