SUGGESTED SOLUTIONS TO PARTS OF MANDATORY ASSIGNMENT: MAT 4770

1a

We only do the argument with Itô's Formula. The other approach has been done in previous exercises.

Choose a function $g(t,x) = e^{\alpha t} \cdot x$, and use $x = X(t)$, where $dX(t) = -\alpha X(t)dt + \sigma dB(t)$:

$$d g(t, X(t)) = \frac{\partial g}{\partial t}(t, X(t))\, dt + \frac{\partial g}{\partial x}(t, X(t))\, dX(t) + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, X(t))\, (dX(t))^2$$

$$\left(\frac{\partial g}{\partial t} = \alpha e^{\alpha t} x = \alpha g, \quad \frac{\partial g}{\partial x} = e^{\alpha t}, \quad \frac{\partial^2 g}{\partial x^2} = 0 \right)$$

$$= \alpha g(t, X(t))\, dt + e^{\alpha t}\, dX(t)$$

$$= \underbrace{\alpha g(t, X(t))\, dt + e^{\alpha t}(-\alpha X(t))}_{=0}\, dt$$

$$+ e^{\alpha t} \sigma \, dB(t)$$

$$= \sigma e^{\alpha t} \, dB(t)$$

Integrating both sides from 0 to t yields

$$e^{\alpha t} X(t) - \underbrace{e^{\alpha 0}}_{=1} X(0) = \int_0^t \sigma e^{\alpha v} \, dB(v)$$

$$\Rightarrow \quad \underline{X(t) = e^{-\alpha t} X(0) + \sigma e^{-\alpha t} \int_0^t e^{\alpha v} \, dB(v)}$$

For $s \geq t$, we have

$$X(s) = e^{-\alpha s} X(0) + \sigma e^{-\alpha s} \int_0^s e^{\alpha v} \, dB(v)$$

$$= e^{-\alpha(s-t)} \cdot e^{-\alpha t} X(0) + \sigma e^{-\alpha(s-t)} e^{-\alpha t} \int_0^s e^{\alpha v} \, dB(v)$$

$$\int_0^s e^{\alpha v} \, dB(v) = \int_0^t e^{\alpha v} \, dB(v) + \int_t^s e^{\alpha v} \, dB(v)$$

$$= e^{-\alpha(s-t)} \left(e^{-\alpha t} X(0) + \sigma e^{-\alpha t} \int_0^t e^{\alpha v} \, dB(v) \right)$$

$$+ \sigma e^{-\alpha(s-t)} e^{-\alpha t} \int_t^s e^{\alpha v} \, dB(v)$$

$$= e^{-\alpha(s-t)} X(t) + \sigma e^{-\alpha s} \int_t^s e^{\alpha v}\, dB(v)$$

Note that $\int_t^s e^{\alpha v}\, dB(v)$ is an integral of a deterministic function $e^{\alpha v}$ with respect to Brownian motion. We know that such integrals are Gaussian random variables. We also know that

$$E\left[\int_t^s e^{\alpha v}\, dB(v)\right] = 0$$

$$E\left[\left(\int_t^s e^{\alpha v}\, dB(v)\right)^2\right] = \int_t^s e^{2\alpha v}\, dv$$

Hence $\int_t^s e^{\alpha v}\, dB(v) \sim N\left(0, \int_t^s e^{2\alpha v}\, dv\right)$

$$\Rightarrow X(s)\big|_{X(t)} \sim N\left(e^{-\alpha(s-t)} X(t),\ \sigma^2 e^{-2\alpha s} \int_t^s e^{2\alpha v}\, dv\right)$$

from the rules of Gaussian random variables

$$\left(Z \sim N(0, b^2) \Rightarrow m + cZ \sim N(m, c^2 b^2)\right)$$

Since $X(t)$ by definition is given by

$\int_0^t e^{\alpha v} dB(v)$, it depends on Brownian motion $B(v)$, for $0 \leq v \leq t$, ie, on B only up to time t. As $\int_t^s e^{\alpha v} dB(v) \sim \sum_i e^{\alpha v_i} \Delta B(v_i)$ this integral depends on the increments of B from t to s, ie, increments <u>after</u> time t. Brownian motion has independent increments by definition, and hence $\int_t^s e^{\alpha v} dB(v)$ is independent of $X(t)$ for $s \geq t$.

1b

Choose the function $g(t,x) = e^x$, and $x = X(t)$, $dX(t) = -\alpha X(t)dt + \sigma dB(t)$.
Using Itô's Formula we find:

$$dS(t) = dg(t, X(t))$$

$$= \frac{\partial g}{\partial t}(t, X(t)) + \frac{\partial g}{\partial x}(t, X(t))\, dX(t)$$

$$+ \frac{1}{2}\frac{\partial^2 g}{\partial t}(t, X(t))\,(dX(t))^2$$

$$\frac{\partial g}{\partial t} = 0, \quad \frac{\partial g}{\partial x} = e^{x} = g, \quad \frac{\partial^2 g}{\partial x^2} = g$$

$$(dX(t))^2 = (-\alpha X(t))^2 (dt)^2 + 2(-\alpha X(t))\,dt \cdot \sigma\, dB(t) + \sigma^2 (dB(t))^2$$

$$= \sigma^2 dt, \quad \text{using} \quad \underbrace{\begin{array}{l} (dt)^2 = 0 \\ (dt)(dB(t)) = 0 \\ (dB(t))^2 = dt \end{array}}_{\text{RULES.}}$$

$$= \overbrace{g(t, X(t))}^{=S(t)} \big((-\alpha X(t))\,dt + \sigma\, dB(t) \big)$$

$$+ \frac{1}{2} \underbrace{g(t, X(t))}_{=S(t)} \sigma^2 dt$$

$$= \Big(\frac{1}{2}\sigma^2 - \alpha \underset{\ln S(t)}{\underset{\shortparallel}{X(t)}}\Big) S(t)\, dt + \sigma S(t)\, dB(t)$$

$$dS(t) = \Big(\frac{1}{2}\sigma^2 - \alpha \ln S(t)\Big) S(t)\, dt + \sigma S(t)\, dB(t)$$

Since $X(t)$ is Gaussian, $S(t) = e^{X(t)}$ is lognormal by definition.

Recall that if $Z \sim N(a, b^2)$, then

$$\underline{\underline{E[e^Z]}} = \underline{\underline{e^{a+\frac{1}{2}b^2}}}$$

Moreover $2Z \sim N(2a, 4b^2)$, so

$$E[(e^Z)^2] = E[e^{2Z}] = e^{2a+\frac{1}{2}4b^2} = e^{2a+2b^2}.$$

Therefore;

$$\underline{\underline{Var(e^Z)}} = E[e^{2Z}] - E[e^Z]^2$$

$$= e^{2a+2b^2} - e^{2a+b^2} = \underline{\underline{e^{2a+b^2}(e^{b^2}-1)}}$$

From 1a, we know that $X(t) \sim N(a, b^2)$ with $a = e^{-\alpha t} X(0)$ and $b^2 = \sigma^2 e^{-2\alpha t} \int_0^t e^{2\alpha v} dv$

Inserting into above, yields $E[S(t)]$ and $Var(S(t))$!

1c.

Girsanov's theorem says that W is a Brownian motion on (any) given interval

$0 \le t \le \tau$ with respect to the probability Q given by

$$\frac{dQ}{dP} = \exp(-\theta B(\tau) - \tfrac{1}{2}\theta^2$$

With respect to Q, X has the form

$$X(s) = e^{-\alpha(s-t)} X(t) + \sigma e^{-\alpha s} \int_t^s e^{\alpha v} \left(dW(v) + \theta dv\right)$$

$$= e^{-\alpha(s-t)} X(t) + \sigma e^{-\alpha s} \theta \int_t^s e^{\alpha v} dv + \sigma e^{-\alpha s} \int_t^s e^{\alpha v} dW(v)$$

$$= e^{-\alpha(s-t)} X(t) + \frac{\sigma\theta}{\alpha}\left(1 - e^{-\alpha(s-t)}\right)$$

$$+ \sigma e^{-\alpha s} \int_t^s e^{\alpha v} dW(v)$$

Hence, (typo in assignment, $T \to \tau$!)

$$f(t,\tau) = E_Q\left[\exp(X(\tau)) \mid \mathcal{F}_t\right]$$

$$= E_Q\Big[\exp\Big(\underbrace{e^{-\alpha(\tau-t)} X(t)}_{\mathcal{F}_t\text{-measurable}} + \underbrace{\frac{\sigma\theta}{\alpha}(1 - e^{-\alpha(\tau-t)})}_{\text{deterministic}} + \underbrace{\sigma e^{-\alpha\tau}\int_t^\tau e^{\alpha v} dW(v)}_{\mathcal{F}_t\text{-independent}}\Big) \mid \mathcal{F}_t\Big]$$

$$= \exp\left(e^{-\alpha(T-t)} X(t) + \frac{\sigma b}{\alpha}\left(1-e^{-\alpha(T-t)}\right)\right)$$

$$\cdot\, E_Q\Big[\underbrace{\exp\Big(\sigma e^{-\alpha T}\int_t^T e^{\alpha v}\, dW(v)\Big)}_{\sim N\left(0,\ \sigma^2 e^{-2\alpha T}\int_t^T e^{2\alpha v}\, dv\right)}\Big]$$

(*) $$= \exp\left(e^{-\alpha(T-t)} X(t) + \frac{\sigma b}{\alpha}\left(1-e^{-\alpha(T-t)}\right) + \frac{1}{2}\sigma^2 e^{-2\alpha T}\underbrace{\int_t^T e^{2\alpha v}\, dv}_{= \frac{1}{2\alpha}\left(e^{2\alpha T} - e^{2\alpha t}\right)}\right)$$

Since $X(t) = \ln S(t)$, we get

$$\exp\left(e^{-\alpha(T-t)} X(t)\right) = \exp\left(e^{-\alpha(T-t)} \ln S(t)\right)$$

$$= \left(\exp(\ln S(t))\right)^{e^{-\alpha(T-t)}}$$

$$= S(t)^{e^{-\alpha(T-t)}}$$

Thus

$$f(t,T) = S(t)^{e^{-\alpha(T-t)}} \cdot \exp\left(\frac{\sigma b}{\alpha}\left(1-e^{-\alpha(T-t)}\right) + \frac{\sigma^2}{4\alpha}\left(1-e^{-2\alpha(T-t)}\right)\right)$$

By definition of $f(t,T)$ as a conditional expectation, the process $t \mapsto f(t,T)$ for $t \leq T$ is a martingale under Q. As a martingale under Q, its dynamics can only depend on $dW(t)$ (the Q-Brownian motion), and not on dt.

Going back to the expression of $f(t,T)$, in (*) we can write it as

$$f(t,T) = \exp\left(e^{-\alpha(T-t)} X(t) + h(t,T)\right)$$

for a function $h(t,T)$ that we can read off from the expression (*)

Consider now Itô's formula used on

$$g(t,x) = \exp\left(e^{-\alpha(T-t)} x + h(t,T)\right) \text{ and}$$

$$x = X(t) \text{ with } dX(t) = -\alpha X(t)\,dt + \sigma\,dW(t) - \sigma\theta\,dt$$

From Itô we see

$$df(t,T) = \underbrace{\frac{\partial g}{\partial t}\, dt}_{dt} + \frac{\partial g}{\partial x}\, \underbrace{dX(t)}_{dt\ \&\ dW} + \frac{1}{2}\, \frac{\partial^2 g}{\partial x^2}\, \underbrace{(dX(t))^2}_{dt}$$

But all dt-terms must cancel since we know that we must end up with only a dW-term. Hence, we only need to consider $\frac{\partial g}{\partial x}$ and the dW-part of dX!

$$\underline{\underline{df(t,T)}} = \frac{\partial g}{\partial x}(t, X(t))\, \sigma\, dW(t) \qquad \frac{\partial g}{\partial x} = e^{-\alpha(T-t)} \cdot g$$

$$= e^{-\alpha(T-t)} g(t, X(t))\, \sigma\, dW(t)$$

$$= \underline{\underline{f(t,T)\, \sigma\, e^{-\alpha(T-t)}\, dW(t)}}$$

When $T-t \downarrow 0$, then $\sigma e^{-\alpha(T-t)} \to \sigma$

$T-t \to \infty$, then $\sigma e^{-\alpha(T-t)} \to 0$

1d

From above, $f(t,T)$ is a Geometric Brownian motion with time-dependent vol.

We guess that

$$(*) \qquad f(t,T) = f(0,T)\exp\left(\int_0^t \sigma e^{-\alpha(T-s)}dW(s) - \frac{1}{2}\int_0^t \sigma^2 e^{-2\alpha(T-s)}ds\right)$$

By letting $d\Sigma(t) = \sigma e^{-\alpha(T-t)}dW(t)$, and considering the function

$$g(t,y) = f(0,T)\exp\left(y - \frac{1}{2}\int_0^t \sigma^2 e^{-2\alpha(T-s)}ds\right)$$

we can show that $(*)$ is indeed the solution.

Hence, $t = T \leq T$

$$f(T,T) = f(0,T)\exp\left(\int_0^T \sigma e^{-\alpha(T-s)}dW(s) - \frac{1}{2}\int_0^T \sigma^2 e^{-2\alpha(T-s)}ds\right)$$

Notice that

$$\int_0^T \sigma e^{-\alpha(T-s)}dW(s) \sim N\left(0, \int_0^T \sigma^2 e^{-2\alpha(T-s)}ds\right)$$

Denote $\bar{\Sigma}^2 := \int_0^T \sigma^2 e^{-2\alpha(\tau-s)} ds = \frac{\sigma^2}{2\alpha}\left(e^{-2\alpha(\tau-T)} - e^{-2\alpha\tau}\right)$

Then, in distribution, we find

$$f(T,\tau) \overset{d}{=} f(0,\tau) \exp\left(\bar{\Sigma} Z - \tfrac{1}{2}\bar{\Sigma}^2\right)$$

where $Z \sim N(0,1)$ under Q.

We calculate $E_Q\left[\max(f(T,\tau)-K,0)\right]$.

$$E_Q\left[\max(f(T,\tau)-K,0)\right]$$
$$= E_Q\left[\max\left(f(0,\tau)e^{\bar{\Sigma} Z - \frac{1}{2}\bar{\Sigma}^2} - K, 0\right)\right]$$
$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \max\left(f(0,\tau)e^{\bar{\Sigma} z - \frac{1}{2}\bar{\Sigma}^2} - K, 0\right) e^{-z^2/2}\, dz$$

$$f(0,\tau)e^{\bar{\Sigma} z - \frac{1}{2}\bar{\Sigma}^2} > K \iff \bar{\Sigma} z - \tfrac{1}{2}\bar{\Sigma}^2 > \ln \frac{K}{f(0,\tau)}$$

$$\iff z > \frac{\ln \frac{K}{f(0,\tau)} + \frac{1}{2}\bar{\Sigma}^2}{\bar{\Sigma}} =: -d$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-d}^{\infty} \left(f(0,\tau)e^{\bar{\Sigma} z - \frac{1}{2}\bar{\Sigma}^2} - K\right) e^{-z^2/2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} f(0,\tilde{T}) e^{-\frac{1}{2}\bar{\Sigma}^2} \int_{-d}^{\infty} e^{\underbrace{\bar{\Sigma} z - \frac{1}{2} z^2}} \, dz$$

$$= -\frac{1}{2}\left(z^2 - 2\bar{\Sigma} z + \bar{\Sigma}^2 - \bar{\Sigma}^2\right)$$

$$= -\frac{1}{2}\left(z^2 - 2\bar{\Sigma} z + \bar{\Sigma}^2\right) + \frac{1}{2}\bar{\Sigma}^2$$

$$= -\frac{1}{2}\left(z - \bar{\Sigma}\right)^2 + \frac{1}{2}\bar{\Sigma}^2$$

$$- K \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-d}^{\infty} e^{-z^2/2} \, dz}_{= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-z^2/2} \, dz = \Phi(d)}$$

Φ cumulative normal distr. function.

$$= f(0,\tilde{T}) \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-d}^{\infty} e^{-\frac{1}{2}(z-\bar{\Sigma})^2} \, dz}_{\text{change of variable } u := z - \bar{\Sigma}} - K \Phi(d)$$

$$= f(0,\tilde{T}) \frac{1}{\sqrt{2\pi}} \int_{-d-\bar{\Sigma}}^{\infty} e^{-\frac{1}{2}u^2} \, du - K \Phi(d)$$

$$= f(0,\tilde{T}) \Phi(d + \bar{\Sigma}) - K \Phi(d)$$

~~Hence~~

$$\boxed{C(T,\tilde{T}) = e^{-rT} f(0,\tilde{T}) \Phi(d + \bar{\Sigma}) - e^{-rT} K \Phi(d)}$$

where
$$d = \frac{\ln \frac{f(0,\tau)}{K} - \frac{1}{2}\Sigma^2}{\Sigma}$$
and
$$\Sigma^2 = \frac{\sigma^2}{2\alpha}\left(e^{-2\alpha(\tau - T)} - e^{-2\alpha T} \right)$$

Black76, almost as famous as Black & Scholes.
Here, Black76 with time-dependent vol!

2a

Comment on the initial values of X.

Since $X_1(t) = T(t) - \Lambda(t)$, you find that $X_1(0) = T(0) - \Lambda(0)$. Hence, you read off today's temperature $\hat{T}(0)$, subtract $\Lambda(0)$, and you have the initial value

$$X_1(0) = \hat{T}(0) - \Lambda(0).$$

Since

$X_2'(t) = T'(t) - \Lambda'(t)$, we approximate $T'(t)$ by its numerical derivative, which is given by (with daily time steps)

$$T'(t) \approx \frac{T(t) - T(t-1)}{1} = T(t) - T(t-1)$$

So, $\hat{T}'(0) = \underset{\substack{\uparrow \\ \text{today}}}{\hat{T}(0)} - \underset{\substack{\uparrow \\ \text{yesterday.}}}{\hat{T}(-1)}$

Hence,

$$X_2(0) = \hat{T}(0) - \hat{T}(-1) - \Lambda'(0)$$

this you calculate from your seasonality function.

Finally

$$X_3(t) = T''(t) - \Lambda''(t).$$

$$T''(t) \approx \frac{T(t) - 2T(t-1) + T(t-2)}{1^2}$$

$$= T(t) - 2T(t-1) + T(t-2)$$

Therefore

$$T''(0) \approx \underbrace{T(0)}_{\text{today}} - 2\underbrace{T(-1)}_{\text{yesterday}} + \underbrace{T(-2)}_{\text{2 days ago}}$$

$$\bar{X}_3(0) = T(0) - 2T(-1) + T(-2) - \Lambda''(0)$$

By reading off the temperature the last 3 days (today, yesterday and day before yesterday), we can initialize $\bar{X}(0) = (\bar{X}_1(0), \bar{X}_2(0), \bar{X}_3(0))$.