SOLUTIONS EXERCISE SET II - FEBRUARY

Exercise 1

Let t be current time ($=$April 1), and

s be Easter Eve ($=$ April 8).

$T(s) = -2 + X(s)$, where

$$dX(s) = -\alpha X(s)\,ds + \sigma\, dB(s),$$

or

$$X(s) = e^{-\alpha(s-t)} X(t) + \int_t^s \sigma e^{-\alpha(s-u)}\, dB(u)$$

This last we know from previous exercises.

The HDD-contract will have a forward price (when assuming zero risk premium), given by

$$F_{HDD}(t,s) = E\left[\max\left(5 - T(s), 0\right) \mid \mathcal{F}_t\right]$$

$$= E\left[\max\left(5 - (-2 + X(s)), 0\right) \mid \mathcal{F}_t\right]$$

$$= E\Big[\max\Big(7 - \underbrace{e^{-\alpha(s-t)}X(t)}_{\mathcal{F}_t\text{-adapted}} - \underbrace{\int_t^s \sigma e^{-\alpha(s-u)}\,dB(u)}_{\substack{\mathcal{F}_t\text{-independent since} \\ \text{Brownian motion has} \\ \text{independent increments}}}, 0\Big) \Big| \mathcal{F}_t\Big]$$

Use "freezing lemma"

$$= E\Big[\max\Big(7 - x - \underbrace{\int_t^s \sigma e^{-\alpha(s-u)}\,dB(u)}_{\sim N\left(0,\ \underbrace{\int_t^s \sigma^2 e^{-2\alpha(s-u)}\,du}_{\substack{\text{calculate yourself, we} \\ \text{denote its square root by } v}}\right)}, 0\Big)\Big]_{x = e^{-\alpha(s-t)}X(t)}$$

$$= E\Big[\max\Big(7 - x - v\underbrace{Y}_{\sim N(0,1)}, 0\Big)\Big]_{x = e^{-\alpha(s-t)}X(t)}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \max(7 - x - vy, 0)\, e^{-\frac{1}{2}y^2}\, dy$$

Continue this calculation yourself. Insert then $s - t = 7$, $\sigma = 2$, $\alpha = 0.2$ to have a price F_{t00D}.

Imagine the ski resort enters a such forwards, long or short (to be seen!)
Their income on Easter Eve will be

$$\underbrace{k \max(5-T(s),0)}_{\substack{\text{income from running} \\ \text{the resort. } k=10.000}} + a\,\underbrace{\left(F_{HDD}(t,s) - \max(5-T(s),0)\right)}_{\substack{\text{profit/loss from selling (short)} \\ \text{1 contract.}}}$$

$$= (k-a)\max(5-T(s),0) + a\,F_{HDD}(t,s)$$

choose $\underline{\underline{a=k}}$!

$$= \underline{\underline{k\,F_{HDD}(t,s)}}$$

With this deal, we see that the ski resort has changed the <u>random</u> income $k \cdot \max(5-T(s),0)$ into the <u>fixed</u> income $k\,F_{HDD}(t,s)$. Entering 10.000 short positions in the HDD-contracts, leads to that the ski resort knows at time t (April 1) exactly how much they will earn on Easter Eve!

Notice that $F_{HDD} > 0$, so they will for sure earn money, while there is a chance that $\max(S - T(t), 0) = 0$. Without insurance, they can end up with no income. On the other hand, if Easter Eve is cold, they can earn a lot more than $h \cdot F_{HDD}(t,s)$ without the insurance.

Find out what the insurance company has as profit/loss in this deal.

Exercise 2:

Finding eigenvalues:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 \\ -\alpha_2 & -\alpha_1 - \lambda \end{bmatrix}\right)$$

$$= (-\lambda)(-\alpha_1 - \lambda) - 1(-\alpha_2)$$

$$= \lambda^2 + \alpha_1 \lambda + \alpha_2 \equiv 0.$$

Hence, we have solutions λ_1, λ_2 given by

$$\lambda = \frac{-\alpha_1 \pm \sqrt{\alpha_1^2 - 4 \cdot 1 \cdot \alpha_2}}{2 \cdot 1}$$

$$= -\frac{1}{2}\alpha_1 \pm \frac{1}{2}\sqrt{\alpha_1^2 - 4\alpha_2}$$

- $\underline{\alpha_1^2 > 4\alpha_2 > 0}$: $\alpha_1^2 - 4\alpha_2$ is real, but

$$0 < \sqrt{\alpha_1^2 - 4\alpha_2} < \alpha_1 \text{ , so}$$

$$-\frac{1}{2}\alpha_1 + \frac{1}{2}\sqrt{\alpha_1^2 - 4\alpha_2} < 0$$

and of course

$$-\frac{1}{2}\alpha_1 - \frac{1}{2}\sqrt{\alpha_1^2 - 4\alpha_2} < 0$$

In this case we have two real-valued <u>negative</u> eigenvalues.

- $\alpha_1^2 < 4\alpha_2$: $\sqrt{\alpha_1^2 - 4\alpha_2}$ is purely imaginary

ie, $\sqrt{\alpha_1^2 - 4\alpha_2} = \sqrt{4\alpha_2 - \alpha_1^2} \cdot i$

The two eigenvalues will be

$$\lambda_1 = -\frac{1}{2}\alpha_1 + \frac{1}{2}\sqrt{4\alpha_2 - \alpha_1^2} \cdot i$$

$$\lambda_2 = -\frac{1}{2}\alpha_1 - \frac{1}{2}\sqrt{4\alpha_2 - \alpha_1^2} \cdot i$$

$$\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) = -\frac{1}{2}\alpha_1 < 0$$

In this case we have two complex eigenvalues with <u>negative</u> real part.

Eigenvectors: Let v_i be the eigenvector associated with eigenvalue λ_i, $i=1,2$.

We have, for $v = \begin{bmatrix} x \\ y \end{bmatrix}$:

$$Av = \lambda v \Leftrightarrow \begin{bmatrix} 0 & 1 \\ -\alpha_2 & -\alpha_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -\alpha_2 x - \alpha_1 y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}$$

$y = \lambda x$. Choose $x=1$, and $y = \lambda$; ie, $v = \begin{bmatrix} 1 \\ \lambda \end{bmatrix}$

Eigenvectors:

$$v_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}.$$

with λ_1, λ_2 as calculated above!

From lectures we know that

this we know is bivariate Gaussian as integrand is deterministic.

$$X(t) = e^{At} X(0) + \int_0^t e^{A(t-s)} e_2 \, dB(s)$$

$$X(0) \overset{!}{=} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a v_1 + b v_2 = \begin{bmatrix} a + b \\ a\lambda_1 + b\lambda_2 \end{bmatrix}$$

$a + b = x_1 \Rightarrow a = x_1 - b.$

$a\lambda_1 + b\lambda_2 = x_2 \Rightarrow \lambda_1(x_1 - b) + b\lambda_2 = x_2$

$\Rightarrow b(\lambda_2 - \lambda_1) = x_2 - \lambda_1 x_1$

$$\Rightarrow \quad b = \frac{x_2 - \lambda_1 x_1}{\lambda_2 - \lambda_1}$$

$$a = x_1 - \frac{x_2 - \lambda_1 x_1}{\lambda_2 - \lambda_1}$$

With this choice of a and b, we find

$$e^{At}\,\underline{x}(0) = e^{At}(a v_1 + b v_2) = a e^{At} v_1 + b e^{At} v_2$$

$$= a e^{\lambda_1 t} v_1 + b e^{\lambda_2 t} v_2 \xrightarrow{t \to \infty} 0$$

Re(λ_1), Re$(\lambda_2) < 0$!

$$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = a v_1 + b v_2 = \begin{bmatrix} a + b \\ \lambda_1 a + \lambda_2 b \end{bmatrix}.$$

$$\Rightarrow \quad a + b = 0 \quad \Rightarrow \quad a = -b.$$

and $\lambda_1 a + \lambda_2 b = 1 \Rightarrow -\lambda_1 b + \lambda_2 b = 1$

$$\Rightarrow \quad b = \frac{1}{\lambda_2 - \lambda_1} \quad \text{and} \quad a = \frac{1}{\lambda_1 - \lambda_2}.$$

$$e^{At} e_2 = \frac{1}{\lambda_1 - \lambda_2} e^{\lambda_1 t} v_1 + \frac{1}{\lambda_2 - \lambda_1} e^{\lambda_2 t} v_2$$

$$= \begin{bmatrix} \frac{1}{\lambda_1-\lambda_2}\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right) \\ \frac{1}{\lambda_1-\lambda_2}\left(\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}\right) \end{bmatrix}$$

$$\int_0^t e^{\cancel{A}(t-s)} e_2 \, dB(s) = \begin{bmatrix} \frac{1}{\lambda_1-\lambda_2}\int_0^t \left(e^{\lambda_1(t-s)} - e^{\lambda_2(t-s)}\right) dB(s) \\ \frac{1}{\lambda_1-\lambda_2}\int_0^t \left(\lambda_1 e^{\lambda_1(t-s)} - \lambda_2 e^{\lambda_2(t-s)}\right) dB(s) \end{bmatrix}$$

This is a bivariate Gaussian random variable. We compute its variance/covariance matrix by finding (each of the two integrals have mean mean zero)

$$E\left[\left(\int_0^t \left(e^{\lambda_1(t-s)} - e^{\lambda_2(t-s)}\right) dB(s)\right)^2\right] = \int_0^t \left(e^{\lambda_1(t-s)} - e^{\lambda_2(t-s)}\right)^2 ds$$

$$E\left[\left(\int_0^t \left(\lambda_1 e^{\lambda_1(t-s)} - \lambda_2 e^{\lambda_2(t-s)}\right) dB(s)\right)^2\right] = \int_0^t \left(\lambda_1 e^{\lambda_1(t-s)} - \lambda_2 e^{\lambda_2(t-s)}\right)^2 ds$$

$$E\left[\int_0^t \left(e^{\lambda_1(t-s)} - e^{\lambda_2(t-s)}\right) dB(s) \int_0^t \left(\lambda_1 e^{\lambda_1(t-s)} - \lambda_2 e^{\lambda_2(t-s)}\right) dB(s)\right]$$

$$= \int_0^t \left(e^{\lambda_1(t-s)} - e^{\lambda_2(t-s)}\right)\left(\lambda_1 e^{\lambda_1(t-s)} - \lambda_2 e^{\lambda_2(t-s)}\right) ds$$

Compute these right-hand side integrals yourself, and then find the limits when $t \to \infty$. This will give you the variance/covariance of $\underline{X}(t)$ when $t \to \infty$. The means will be zero when $t \to \infty$.
The limit of Gaussians is Gaussian.

Exercise 3

By definition of the determinant, we see that

$$\underbrace{\begin{vmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ \vdots & & & & 1 \\ -\alpha_p & & \cdots & & -\alpha_1 \end{vmatrix}}_{p \times p\text{-matrix}} = (-1) \underbrace{\begin{vmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ \vdots & & & & 1 \\ -\alpha_p & -\alpha_{p-2} & & \cdots & -\alpha_1 \end{vmatrix}}_{(p-1) \times (p-1)\text{ matrix}}$$

This matrix, we recognize as a CAR-matrix A, but with coefficients $\alpha_p, \alpha_{p-2}, \ldots, \alpha_1$

Thus, $\det(A_p) = (-1) \det(A_{p-1})$

We prove the claim on the determinant by induction:

$\underline{p=2}$ $\begin{vmatrix} 0 & 1 \\ -\alpha_2 & -\alpha_1 \end{vmatrix} = 0\cdot(-\alpha_1) - 1\cdot(-\alpha_2) = \alpha_2$

$= (-1)^2 \alpha_2$. ok!

Suppose the claim holds for $p-1$.

$$\begin{vmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ -\alpha_p & -\alpha_{p-1} & \cdots & & -\alpha_1 \end{vmatrix} \overset{\text{by above}}{=} (-1) \begin{vmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ -\alpha_p & -\alpha_{p-2} & \cdots & & -\alpha_1 \end{vmatrix}$$

$$= (-1)\cdot(-1)^{p-1}\alpha_p$$ by induction hypothesis, and the leading term in the $(p-1)\times(p-1)$ matrix is α_p!

$$= (-1)^p \alpha_p.$$

The result follows.

For the second claim on the inverse A^{-1}:

Introduce notation

$$\vec{c} = \left(-\frac{\alpha_{p-1}}{\alpha_p}, \dots, -\frac{\alpha_1}{\alpha_p}\right)^{T}, \quad \vec{a} = (\alpha_{p-1}, \dots, \alpha_1)^T$$

T = transpose!

$$A = \begin{bmatrix} \vec{0} & I_{p-1} \\ -\alpha_p & -\vec{a}^T \end{bmatrix}, \quad B \equiv \begin{bmatrix} \vec{c}^T & -\frac{1}{\alpha_p} \\ I_{p-1} & \vec{0} \end{bmatrix}$$

where $\vec{0} = (0, .., 0)^T \in \mathbb{R}^{p-1}$.

I_{p-1} is the $(p-1)\times(p-1)$ identity matrix.

Multiplication using block matrices;

$$AB = \begin{bmatrix} \vec{0} & I_{p-1} \\ -\alpha_p & -\vec{a}^T \end{bmatrix} \begin{bmatrix} \vec{c}^T & -\frac{1}{\alpha_p} \\ I_{p-1} & \vec{0} \end{bmatrix}$$

$$= \begin{bmatrix} \vec{0}\,\vec{c}^T + I_{p-1}^2 & \vec{0}\left(-\frac{1}{\alpha_p}\right) + I_{p-1}\vec{0} \\ (-\alpha_p)\vec{c}^T + (-\vec{a})^T I_{p-1} & (-\alpha_p)\left(-\frac{1}{\alpha_p}\right) + (-\vec{a})^T\vec{0} \end{bmatrix}$$

$$= \begin{bmatrix} I_{p-1} & \vec{0} \\ \vec{0}^T & 1 \end{bmatrix} = I_p .$$

$$BA = \begin{bmatrix} \vec{c}^T & -\frac{1}{\alpha_p} \\ I_{p-1} & \vec{c} \end{bmatrix} \begin{bmatrix} \vec{0} & I_{p-1} \\ -\alpha_p & -\vec{a}^T \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \vec{c}^T + \overbrace{\frac{1}{\alpha_p}\vec{a}^T}^{= -\vec{c}^T} \\ \vec{0} & I_{p-1} \end{bmatrix} = I_p .$$

Hence, $B = A^{-1}$, as we wanted to show!

Exercise 4

We integrate;

$$\int_{T_1-t}^{T_2-t} e_1^T e^{Ax} e_p \, dx = e_1^T \int_{T_1-t}^{T_2-t} e^{Ax} dx \; e_p$$

$$= e_1^T A^{-1} \left(e^{A(T_2-t)} - e^{A(T_1-t)} \right) e_p$$

$$= \Sigma_{CAT}(t, T_1, T_2)$$

$$g(0) = e_1^T e^{A0} e_p = e_1^T e_p$$

since $A \cdot 0 = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{bmatrix}$ and $A^n = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{bmatrix}$

for all $n \geq 1$, except $A^0 \equiv I$. Hence $e^{A \cdot 0} = I$.

But $e_1^T e_p = \begin{cases} 1, & p = 1 \\ 0, & p > 1 \end{cases}$.

So, $g(0) = \begin{cases} 1, & p = 1 \\ 0, & p > 1 \end{cases}$.

From the above exercises, we know that

$e_p = \sum_{j=1}^{p} a_j v_j$ where v_j is the i'th eigenvector

$$\Rightarrow g(x) = e_1^T e^{Ax} e_p = \sum_{j=1}^{p} a_j e^{\lambda_j x} (e_1^T v_j)$$

$$= \sum_{j=1}^{p} \underbrace{(a_j (e_1^T v_j))}_{\in \mathbb{C}} e^{\lambda_j x}$$

If $\mathrm{Re}(\lambda_j) < 0$ for all $j = 1, \dots, p$, then

$$e^{\lambda_j x} = e^{\mathrm{Re}(\lambda_j) \cdot x} \left(\cos(\mathrm{Im}(\lambda_j) x) + i \sin(\mathrm{Im}(\lambda_j) x) \right)$$

$$\xrightarrow{x \to \infty} 0$$

Thus; $g(x) \to 0$ when $x \to 0$.

Exercise 5

$M(t) = B(t)$:

$$E[B(T) | \mathcal{F}_t] = E[B(T) - B(t) + B(t) | \mathcal{F}_t]$$

$$= E[\underbrace{B(T) - B(t)}_{\substack{\text{independent of } \mathcal{F}_t \\ \text{due to independent} \\ \text{increments of } B!}} | \mathcal{F}_t] + E[\underbrace{B(t)}_{\mathcal{F}_t\text{-measurable}} | \mathcal{F}_t]$$

$$= \underbrace{E[B(T) - B(t)]}_{= 0} + B(t)$$

$= B(t).$

$\Rightarrow B$ is a martingale.

$\underline{M(t) = e^{\sigma B(t)}}$:

$$E\left[e^{\sigma B(T)} \mid \mathcal{F}_t\right] = E\left[e^{\sigma B(T) - \sigma B(t) + \sigma B(t)} \mid \mathcal{F}_t\right]$$

$$= E\Big[\underbrace{e^{\sigma(B(T)-B(t))}}_{\text{independent of } \mathcal{F}_t} \cdot \underbrace{e^{\sigma B(t)}}_{\mathcal{F}_t\text{-measurable}} \mid \mathcal{F}_t\Big]$$

Use "freezing lemma"

$$= E\left[e^{\sigma(B(T)-B(t))} \cdot x\right]_{x = e^{\sigma B(t)}}$$

$$= e^{\sigma B(t)} \cdot E\Big[e^{\sigma \underbrace{(B(T)-B(t))}_{\sim N(0, T-t)}}\Big]$$

$$= e^{\sigma B(t)} \cdot e^{\frac{1}{2}\sigma^2 (T-t)} \neq e^{\sigma B(t)}$$

$\Rightarrow e^{\sigma B(t)}$ is not a martingale.

$M(t) = e^{\sigma B(t) - \frac{1}{2}\sigma^2 t}$:

$$E\left[e^{\sigma B(T) - \frac{1}{2}\sigma^2 T} \mid \mathcal{F}_t\right] = e^{-\frac{1}{2}\sigma^2 T} E\left[e^{\sigma B(T)} \mid \mathcal{F}_t\right]$$

From above

$$= e^{-\frac{1}{2}\sigma^2 T} \cdot e^{\frac{1}{2}\sigma^2 (T-t)} \cdot e^{\sigma B(t)}$$

$$= e^{\sigma B(t) - \frac{1}{2}\sigma^2 t}$$

$\Rightarrow e^{\sigma B(t) - \frac{1}{2}\sigma^2 t}$ is a martingale.

Let $M(t)$ be a martingale. Then

$$E\left[M(t) \mid \mathcal{F}_0\right] = M(0)$$

by the martingale property (use $T = t$, $t = 0$ in definition!)

$$\Rightarrow E[M(0)] = E\left[E\left[M(t) \mid \mathcal{F}_0\right]\right] = E[M(t)]$$

new property, which is called the "tower law"