SUGGESTED SOLUTIONS TO EXERCISES I

First part, exercises from 31/1.

1

$$(1+\hat{r})^t = e^{\ln(1+\hat{r})^t} = e^{t\ln(1+\hat{r})} \overset{!}{=} e^{rt}$$

$$\Leftrightarrow \quad \underline{\underline{r = \ln(1+\hat{r})}}$$

First expression as ordinary differential equation (ODE):

$$A'(t) = \frac{d}{dt} e^{t\ln(1+\hat{r})} = \ln(1+\hat{r})\, e^{t\ln(1+\hat{r})} = \ln(1+\hat{r}) A(t)$$

Second expression as ODE

$$A'(t) = \frac{d}{dt} e^{rt} = r A(t)$$

In principle no difference between the two, but continuously compounding interest rates give natural link between ODE and solution as geometric growth.

2

To qualify as a probability, $Q(A) \geq 0$, $Q(\phi) = 0$ and $Q(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} Q(A_k)$ for pairwise disjoint sets $A_k \subseteq \Omega$.

Since $X > 0$, we must have

$$Q(A) = E[1_A X] \geq 0 \;, \text{ since } 1_A \geq 0.$$

If $A = \phi$, the empty set, $1_\phi = 0$, so

$1_\phi X = 0$. $\Rightarrow$ $Q(\phi) = E[1_\phi X] = E[0] = 0$.

Let $A_k \subseteq \Omega$ be disjoint sets. Notice that if $\omega \in A_j$ for one j, then $\omega \notin A_i$ for all $i \neq j$ since the sets are disjoint. Hence

$1_{\bigcup_{k=1}^{\infty} A_k} = \sum_{k=1}^{\infty} 1_{A_k}$. Therefore

$$Q\left(\bigcup_{k=1}^{\infty} A_k\right) = E\left[1_{\bigcup_{k=1}^{\infty} A_k} \cdot X\right] = E\left[\sum_{k=1}^{\infty} 1_{A_i} \cdot X\right]$$

$$= \sum_{k=1}^{\infty} E[1_{A_k} \cdot X] = \sum_{k=1}^{\infty} Q(A_k)$$

Q is a probability!

If $A \subseteq \Omega$ is such that $P(A) = 0$, we have $P(A) = E[1_A] = 0$. $1_A \geq 0$, so only way that $E[1_A] = 0$ is that $1_A = 0$.

$\Rightarrow$ $1_A \cdot X = 0$ $\Rightarrow$ $Q(A) = E[1_A X] = 0$.

If $Q(A) = 0$, we have that $E[1_A X] = 0$.

Since $X > 0$ $\Rightarrow$ $1_A \cdot X = 0$ $\Rightarrow$ $1_A = 0$.

But then $P(A) = E[1_A] = 0$.

In conclusion $Q(A) = 0 \Leftrightarrow P(A) = 0$,

and $\underline{\underline{Q \sim P}}$

3

From previous exercise, we define

$Q(A) = E[1_A \cdot X]$, with X as given.

Since $X = \exp(\dots)$, $X > 0$, so this gives us an equivalent probability.

Let now $t > s$ and $T \geq t$.

$$W(t) - W(s) = \theta(t-s) + B(t) - B(s)$$

We calculate

$$E_Q[\exp(x(W(t)-W(s)))]$$

$$\overset{\text{(Hint)}}{=} E[\exp(x(W(t)-W(s))) \cdot X]$$

$$= E[e^{x\theta(t-s)} \cdot e^{x(B(t)-B(s))} \cdot e^{-\theta B(T)} \cdot e^{-\frac{1}{2}\theta^2 T}]$$

$$= e^{x\theta(t-s) - \frac{1}{2}\theta^2 T} \cdot E[e^{x(B(t)-B(s))} \cdot e^{-\theta(B(T)-B(t)+B(t)-B(s)+B(s))}]$$

B is Brownian motion, and thus has independent increments. Hence, $B(T) - B(t)$ is independent of

$B(t)-B(s)$, which again is independent of $B(s)$)

$$= e^{x\sigma(t-s)-\frac{1}{2}\sigma^2 T} E\left[e^{(x-\sigma)(B(t)-B(s))}\right] \cdot E\left[e^{-\sigma B(s)}\right]$$

$$\cdot E\left[e^{-\sigma(B(T)-B(t))}\right]$$

Hint, normality of B

$$\overset{\text{Hint, normality of } B}{=} e^{x\sigma(t-s)-\frac{1}{2}\sigma^2 T} \cdot e^{\frac{1}{2}(x-\sigma)^2(t-s)} \cdot e^{\frac{1}{2}(-\sigma)^2 s} \cdot e^{\frac{1}{2}(-\sigma)^2(T-t)}$$

$$= \underline{\underline{\exp\left(\tfrac{1}{2}x^2(t-s)\right)}}$$

This shows (from hint) that $W(t)-W(s)$ is normal, with mean zero and variance $t-s$.

Let $t=u$, $s=0$, and repeat the above to get

$$E_Q\left[e^{xW(u)}\right] = \exp\left(\tfrac{1}{2}x^2 u\right)$$

With $u=t-s$, we thus find

$$E_Q\left[e^{x(W(t)-W(s))}\right] = e^{\frac{1}{2}x^2(t-s)} = E_Q\left[e^{xW(t-s)}\right]$$

Hence, $W(t-s) \sim N(0, t-s)$, and we see that $W(t)$ is normal with stationary increments under Q.

We show independence of increments:

It is sufficient to demonstrate that

$$E_Q\left[e^{x(W(t)-W(s))} \cdot e^{yW(s)}\right] =$$

(*)

$$E_Q\left[e^{x(W(t)-W(s))}\right] \cdot E_Q\left[e^{yW(s)}\right]$$

for $t > s$ and all x, y.

Consider left-hand side of (*):

$$x(W(t)-W(s)) + yW(s) = x\sigma(t-s) + x(B(t)-B(s)) + y\sigma s + yB(s)$$

$$= x\sigma(t-s) + y\sigma s + x(B(t)-B(s)) + yB(s)$$

$$E_Q\left[e^{x(W(t)-W(s))+yW(s)}\right] = \exp\left(x\sigma(t-s) + y\sigma s - \tfrac{1}{2}\sigma^2 T\right)$$

$$\cdot E\left[e^{x(B(t)-B(s))} \cdot e^{yB(s)} \cdot e^{-\sigma(B(T)-B(t)+B(t)-B(s)+B(s))}\right]$$

indep. incr.

$$= e^{x\sigma(t-s)+y\sigma s-\frac{1}{2}\sigma^2 T} \cdot e^{\frac{1}{2}(-\sigma)^2(T-t)} \, e^{\frac{1}{2}(x-\sigma)^2(t-s)} \, e^{\frac{1}{2}(y-\sigma)^2 s}$$

$$= \underbrace{\exp\left(\tfrac{1}{2}x^2(t-s)\right) \cdot \exp\left(\tfrac{1}{2}y^2 s\right)}$$

This is equal to right-hand side of (*), and we are done!

4

Define the function $f(t,x) = S(0)e^{\alpha t + \sigma x}$

Itô's Formula gives

$$\underline{\underline{dS(t)}} = df(t,B(t)) = \frac{\partial f}{\partial t}\cdot dt + \frac{\partial f}{\partial x}\, dB(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dB(t))^2$$

$$= \left(\alpha \cdot S(0) e^{\alpha t + \sigma B(t)}\right) dt + \left(\sigma S(0) e^{\alpha t + \sigma B(t)}\right) dB(t)$$

$$+ \frac{1}{2}\left(\sigma^2 S(0) e^{\alpha t + \sigma B(t)}\right) dt \leftarrow \text{using that } (dB)^2 = dt\,!$$

$$= \alpha S(t)\, dt + \sigma S(t)\, dB(t) + \frac{1}{2}\sigma^2 S(t)\, dt$$

$$= \underline{\underline{\left(\alpha + \frac{1}{2}\sigma^2\right) S(t)\, dt + \sigma S(t)\, dB(t)}}.$$

Sometimes we write

$$\frac{dS(t)}{S(t)} = \left(\alpha + \frac{1}{2}\sigma^2\right) dt + \sigma\, dB(t)$$

Notice: We could have used $f(x) = S(0)e^{x}$, with the stochastic process

$$dX(t) = \alpha\, dt + \sigma\, dB(t)$$

Then Itô's Formula would give us

$$dS(t) = f'(X(t))\, dX(t) + \frac{1}{2} f''(X(t)) (dX(t))^2$$

$f'(x) = f(x), \quad f''(x) = f(x)$

and $\quad (dX(t))^2 = \alpha^2 (dt)^2 + 2\alpha\sigma (dt)(dB(t))$ $=0$ $=0$

$+ \sigma^2 (dB(t))^2$ $= dt$

$= \sigma^2 dt$

$dS(t) = S(t)(\alpha dt + \sigma dB(t)) + \frac{1}{2}\sigma^2 S(t) dt$

$= (\alpha + \frac{1}{2}\sigma^2) S(t) dt + \sigma S(t) dB(t)$

$L(t) = \ln \frac{S(t+\Delta t)}{S(t)} = \underbrace{\alpha \Delta t + \sigma (B(t+\delta) - B(t))}_{\sim N(\alpha\Delta t, \sigma^2 \Delta t)}$

and independent since Brownian motion has indep. incr.

From above it follows immediately that

$\frac{dS(t)}{S(t)} = r\,dt + \sigma\, dB(t)$

when $\alpha := r - \frac{1}{2}\sigma^2$.

We know (also from above) that

$$S(t) = S(0)\, e^{(r-\frac{1}{2}\sigma^2)t + \sigma B(t)}$$

So, $$e^{-rt} S(t) = S(0) e^{-\frac{1}{2}\sigma^2 t + \sigma B(t)}$$

Moreover, $$e^{-rT} S(T) = S(0) e^{-\frac{1}{2}\sigma^2 T + \sigma B(T)}$$

$$= S(0) e^{-\frac{1}{2}\sigma^2 (T-t+t) + \sigma (B(T) - B(t) + B(t))}$$

$$= e^{-rt} S(t) \cdot \exp\left(-\tfrac{1}{2}\sigma^2 (T-t) + \sigma (B(T) - B(t))\right)$$

$$\underline{E\left[e^{-rT} S(T) \,|\, S(t)\right]} = e^{-rt} S(t)\, E\left[e^{-\frac{1}{2}\sigma^2 (T-t) + \sigma (B(T) - B(t))} \,|\, S(t)\right]$$

$B(T) - B(t)$ indep. of $S(t)$, which depends on $B(t)$!

$$= e^{-rt} S(t)\, e^{-\frac{1}{2}\sigma^2 (T-t)} \cdot \underbrace{E\left[e^{\sigma (B(T) - B(t))}\right]}_{= e^{\frac{1}{2}\sigma^2 (T-t)}}$$ (Ex 3!)

$$= \underline{e^{-rt} S(t)}$$