SUGGESTED SOLUTIONS, PART II

(5) Based on earlier exercises, let us guess a solution of the form

$$(*) \quad S(t) = S(0)\exp\left((\mu - \tfrac{1}{2}\sigma^2)t + \sigma B(t)\right)$$

Itô's Formula using $f(t,x) = S(0)\exp\left((\mu-\tfrac{1}{2}\sigma^2)t + \sigma x\right)$ yields

$$dS(t) = df(t,B(t)) = \frac{\partial}{\partial t} f(t,B(t))\,dt + \frac{\partial}{\partial x} f(t,B(t))\,dB(t) + \frac{1}{2}\frac{\partial^2}{\partial x^2} f(t,B(t))\left(dB(t)\right)^2$$

$$= (\mu - \tfrac{1}{2}\sigma^2) f(t,B(t))\,dt + \sigma f(t,B(t))\,dB(t) + \tfrac{1}{2}\sigma^2 f(t,B(t))\,dt$$

$$= \mu S(t)\,dt + \sigma S(t)\,dB(t)$$

$$\Rightarrow \frac{dS(t)}{S(t)} = \mu\,dt + \sigma\,dB(t)$$

Hence, $(*)$ is a solution.

Introduce a stochastic process

$$dW(t) = \theta\,dt + dB(t)$$

where θ is some constant.

Substituting in the dynamics of S:

$$\frac{dS(t)}{S(t)} = \mu\,dt + \sigma\left(dW(t) - \theta\,dt\right)$$
$$= (\mu - \sigma\theta)\,dt + \sigma\,dW(t)$$

Choose θ so that $\mu - \sigma\theta = r$, i.e.

$$\boxed{\theta = \frac{\mu - r}{\sigma}}$$

Girsanov's theorem ensures that there exists a $\mathbb{Q} \sim \mathbb{P}$ such that W is a $\mathbb{Q}$-Brownian motion.

From Girsanov, we also know that

$\mathbb{Q}(A) = E[\mathbb{1}_A Z]$, where Z is

$$Z = \exp\left(\int_0^T \frac{\mu - r}{\sigma}\,dB(s) - \frac{1}{2}\int_0^T \left(\frac{\mu - r}{\sigma}\right)^2 ds\right)$$
$$= \exp\left(\frac{\mu - r}{\sigma} \cdot B(T) - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 T\right)$$

By the definition in the exercise:

$$F(t,T) = E_{\mathbb{Q}}\left[S(T) \mid \mathcal{F}_t\right]$$

$$= E_Q\left[S(T) \mid S(t)\right]$$

We know that

$$S(T) = S(0)\exp\left((r - \tfrac{1}{2}\sigma^2)T + \sigma W(T)\right)$$

$$= S(t)\exp\left((r - \tfrac{1}{2}\sigma^2)(T-t) + \sigma(W(T) - W(t))\right)$$

$W(T) - W(t)$ is independent of $S(t)$, since $S(t)$ depends on $W(t)$, and W has independent increments.

$$\underline{F(t,T)} = E_Q\left[S(t)e^{(r-\frac{1}{2}\sigma^2)(T-t)} \cdot e^{\sigma(W(T)-W(t))} \mid S(t)\right]$$

$$= S(t)e^{(r-\frac{1}{2}\sigma^2)(T-t)} E_Q\left[e^{\underbrace{\sigma(W(T)-W(t))}_{\sim N(0,T-t)}}\right]$$

$$= S(t)e^{(r-\frac{1}{2}\sigma^2)(T-t)} \cdot e^{\frac{1}{2}\sigma^2(T-t)}$$

$$= \underline{e^{r(T-t)}S(t)}$$

Notice that $F(t,T) = e^{rT} \cdot (e^{-rt}S(t))$

As a process in t, e^{rT} is a constant.

So $dF(t,T) = e^{rT} d(e^{-rt} S(t))$.

We use Ito's Formula for function

$f(t,x) = e^{-rt} \cdot x$, and process $X = S(t)$

with $\frac{dS(t)}{S(t)} = r\,dt + \sigma\,dW(t)$.

Q-dynamics of F then becomes

$$dF(t,T) = e^{rT} d(e^{-rt} S(t))$$

$$= e^{rT} \cdot d f(t, S(t))$$

$$= e^{rT} \left(\frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dS(t) + \frac{1}{2} \underbrace{\frac{\partial^2 f}{\partial x^2}}_{=0} (dS(t))^2 \right)$$

$$= e^{rT} \left((-r) e^{-rt} S(t)\, dt + e^{-rt} dS(t) \right)$$

$$= -r F(t,T)\, dt + r F(t,T)\, dt$$

$$+ \sigma F(t,T)\, dW(t)$$

$$\Rightarrow \underline{dF(t,T) = \sigma F(t,T)\, dW(t)}$$

⑥ From the exercise we have that $r = 0.0433$, $S = 86$ and $F = 81$.

The buy-and-hold strategy would give that

$$F = S e^{r} \leftarrow r \cdot 1\text{year} = r \,!$$

But $Se^{r} = 86 \cdot e^{0.0433} > 81$, so there is apparently a clear arbitrage opportunity here.

According to the buy-and-sell strategy, you should sell spot (which is expensive), and buy the cheap forward. This is a turbulent time, where access to oil is an advantage. To have it physically is better than to have it as a

promise in the future (= forward contract)
This leads to something called the <u>convenience yield</u>, which is a yield implicitly given to those possessing a commodity physically. Think of Norway and our gas resources.

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$$
\begin{tikzcd}[row sep=small]
 & A_0(1+u) \\
A_0 \arrow[ur] \arrow[dr] & \\
 & A_0(1+d)
\end{tikzcd}
\qquad d < r < u
$$

Is there a way to invest in asset and bank to generate arbitrage? This means, if I buy/sell a assets and borrow/deposit b in bank, in total making a zero investment, can I generate money for sure?

$$V_0 = aA_0 + b = 0 \Rightarrow b = -aA_0$$

$$V_1 = aA_1 + b(1+r)$$

$$= aA_1 - aA_0(1+r)$$

$$= \begin{cases} aA_0(1+u-1-r) \\ aA_0(1+d-1-r) \end{cases} = aA_0 \begin{cases} u-r \\ d-r \end{cases}$$

If $a>0$, then $aA_0(d-r)<0$. If $a<0$, then $aA_0(u-r)<0$. So, any choice of $a \neq 0$ means that we have a risk of loosing money, $P(V_1<0)>0$!

$\Rightarrow$ The model is arbitrage-free!

$$A_0 \overset{!}{=} (1+r)^{-1} E_q[A_1] = (1+r)^{-1}\Big(q \cdot A_0(1+u)$$
$$+ (1-q)A_0(1+d)\Big)$$
$$= A_0(1+r)^{-1}\big((1+u)q + (1+d) - q(1+d)\big)$$

$\Leftrightarrow \quad 1+r = (u-d)\,q + (1+d)$

$$\Leftrightarrow \quad \underline{\underline{q = \frac{r-d}{u-d}}}$$

$q > 0$ since $r > d$ and $u > d$.

$q < 1$ since $r < u \Rightarrow r-d < u-d$

q is therefore a probability.

If $r < d$, then by repeating all above we find that by choosing any $a > 0$, we get $V_1 > 0$ no matter the asset price A_1. Hence, we can earn as much as we like by borrowing money to buy the asset.

We also see that $q<0$ in this case. Hence, we have a model for the asset which is <u>not</u> arbitrage-free, and which does not have any associated probability $q\in(0,1)$ such that its discounted expected price is the initial price.

Do the analysis of the case $r>u$ yourself.

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$$\int_0^t X(s)\,ds = \int_0^t X(0)e^{-\alpha s}\,ds + \int_0^t \sigma e^{-\alpha s}\int_0^s e^{\alpha u}\,dB(u)\,ds$$

$$= X(0)\,\frac{1}{\alpha}\left(1-e^{-\alpha t}\right) + \sigma\int_0^t\int_0^s e^{-\alpha s}e^{\alpha u}\,dB(u)\,ds$$

Consider now the last term, where we calculate using the stochastic version of Fubini's theorem:

$$\int_0^t \int_0^s e^{-\alpha s} e^{\alpha u} \, dB(u) \, ds$$

$$= \int_0^t \int_0^t 1_{[0,s]}(u) e^{-\alpha s} e^{\alpha u} \, dB(u) \, ds$$

$$\overset{\text{Fubini}}{=} \int_0^t \int_0^t \underbrace{1_{[0,s]}(u)}_{= \begin{cases} 1, & s \geq u \\ 0, & s < u \end{cases}} e^{-\alpha s} e^{\alpha u} \, ds \, dB(u)$$

$$= \int_0^t \int_u^t e^{-\alpha s} ds \, e^{\alpha u} \, dB(u)$$

$$= \int_0^t \frac{1}{\alpha}\left(e^{-\alpha u} - e^{-\alpha t}\right) e^{\alpha u}\, dB(u)$$

$$= \frac{1}{\alpha}\int_0^t dB(u) - \frac{1}{\alpha} e^{-\alpha t}\int_0^t e^{\alpha u}\, dB(u)$$

$$= \frac{1}{\alpha} B(t) - \frac{1}{\alpha} e^{-\alpha t}\int_0^t e^{\alpha u}\, dB(u)$$

Hence,

$$\alpha\int_0^t X(s)\,ds = X(0) \underbrace{- X(0)e^{-\alpha t} + \sigma B(t) - \sigma e^{-\alpha t}\int_0^t e^{\alpha u}\, dB(u)}_{= -X(t)}$$

$$= X(0) + \sigma B(t) - X(t)$$

Re-organizing terms, yields

$$X(t) = X(0) - \alpha \int_0^t X(s)\,ds + \sigma B(t)$$

which is what we wanted.

In differential form, we can write

$$dX(t) = -\alpha X(t)\,dt + \sigma\,dB(t)$$

(just "by-the-way")

In the expression of $X(t)$, only $\int_0^t e^{\alpha u}\,dB(u)$ is random. We argue now that this is a Gaussian random variable. We argue by showing that it has moment-generating function coinciding with that of a Gaussian. I.e, if $X \sim N(\mu, v^2)$, then

$$E[e^{\theta X}] = \exp(\theta\mu + \tfrac{1}{2}v^2\theta^2)$$

We do this by a trick appealing to Itô's Formula:

Let $X(t) = \int_0^t e^{\alpha u}\, dB(u)$

or, $dX(t) = e^{\alpha t}\, dB(t)$

Introduce the function $f(x) = e^{\sigma x}$

Then

$$d e^{\sigma X(t)} = d f(X(t)) = \sigma e^{\sigma X(t)}\, dX(t)$$

$$+ \frac{1}{2} \sigma^2 e^{\sigma X(t)} (dX(t))^2$$

$$= \sigma e^{\alpha t} e^{\sigma X(t)}\, dB(t) + \frac{1}{2} \sigma^2 e^{2\alpha t} e^{\sigma X(t)}\, dt$$

This is the differential version, which is notation and means

$$e^{\sigma X(t)} = e^{\sigma X(0)} + \int_0^t \frac{1}{2} \sigma^2 e^{2\alpha s} e^{\sigma X(s)}\, ds$$

$$+ \int_0^t \sigma e^{\alpha s} e^{\sigma X(s)}\, dB(s)$$

Notice that $\underline{X}(0) = \int_0^0 e^{\alpha u}\, dB(u) = 0$

and

$$E\left[e^{\sigma \underline{X}(t)}\right] = 1 + \frac{1}{2}\sigma^2 \int_0^t e^{2\alpha s} E\left[e^{\sigma \underline{X}(s)}\right] ds + \sigma \underbrace{E\left[\int_0^t e^{\alpha s} e^{\sigma \underline{X}(s)}\, dB(s)\right]}_{=0}$$

$= 0$. Expected value of integrals w.r.t. Brownian motion is zero!

Define the function (of time t),

$$g(t) := E\left[e^{\sigma \underline{X}(t)}\right].$$

We have

$$g(t) = 1 + \frac{1}{2}\sigma^2 \int_0^t e^{2\alpha s} g(s)\, ds$$

Differentiating w.r.t. t;

$$g'(t) = \frac{1}{2}\theta^2 e^{2\alpha t} \cdot g(t)$$

and $g(0) = 1$.

This is an ordinary differential equation, with solution

$$g(t) = \exp\left(\frac{1}{2}\theta^2 \int_0^t e^{2\alpha s}\, ds\right)$$

(show yourself!)

$$\Rightarrow E\left[e^{\theta X(t)}\right] = \exp\left(\frac{1}{2}\theta^2 \cdot \frac{1}{2\alpha}\left(e^{2\alpha t} - 1\right)\right)$$

We conclude that

$$\int_0^t e^{\alpha u}\, dB(u) \sim N\left(0, \frac{1}{2\alpha}\left(e^{2\alpha t} - 1\right)\right)$$

But this means that

$$\underline{X}(t) \sim N\left(\underline{X}(0)e^{-\alpha t}, \frac{\sigma^2}{2\alpha} e^{-2\alpha t}\left(e^{2\alpha t} - 1\right)\right)$$

$$= N\left(\underline{X}(0)e^{-\alpha t}, \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha t}\right)\right)$$

When $t \to \infty$, $\underline{X}(t) \longrightarrow N\left(0, \frac{\sigma^2}{2\alpha}\right)$

So, in distribution, $\underline{X}(t)$ converges to a normal distribution with time-independent mean and variance. We sometimes refer to this as the stationary distribution of $\underline{X}$ (although this is unprecise)