Lecture 2. Exercise 11

Setup

$$P(X_j \in A \mid \mathcal{F}^X_{j-1} \vee \mathcal{F}^Y_{j-1}) = \int_A \Delta(X_{j-1}, du) \quad P\text{-a.s.}$$

$$P(X_0 \in A) = \int_A p(du)$$

$$Y_j = h(X_j) + \xi_j \qquad \mathcal{L}(\xi_j) \sim q(x)dx \quad q > 0.$$

We need to check that

$$\tilde{\pi}_j(dx) := \frac{\rho_j(dx)}{\int_{\mathbb{R}} \rho_j(du)} \qquad \text{verifies}$$

$$\tilde{\pi}_j(dx) = \frac{\int_{\mathbb{R}} \gamma(u, Y_j)\, \Delta(u, dx)\, \tilde{\pi}_{j-1}(du)}{\iint_{\mathbb{R}\,\mathbb{R}} \gamma(u, Y_j)\, \Delta(u, dx)\, \tilde{\pi}_{j-1}(du)}$$

where $\gamma(x, u)$ is such that

$$P(Y_j \in B \mid \mathcal{F}^X_j \vee \mathcal{F}^Y_{j-1}) = \int_B \gamma(X_{j-1}, u)\, \nu(du)$$

ν σ-finite measure on $\mathcal{B}(\mathbb{R})$.

Our assumption on Y_j yields that

$$P(Y_j \in B \mid \mathcal{F}^X_j \vee \mathcal{F}^Y_{j-1}) = P(h(X_j) + \xi_j \in B \mid \mathcal{F}^X_j \vee \mathcal{F}^Y_{j-1})$$

$$= \int_{B - h(X_j)} q(u)\, du \underset{v = u + h(X_j)}{=} \int_B q(v - h(X_j))\, dv$$

And a further assumption is that $\rho_j(dx)$ satisfies the following recurrence equation

$$\ell_j(du) = \int_{\mathbb{R}} \frac{g(Y_j - h(u))}{g(Y_j)} \Delta(s, du)\, \ell_{j-1}(ds)$$

$$\tilde{\pi}_j(dx) = \frac{\ell_j(dx)}{\int_{\mathbb{R}} \ell_j(du)} = \frac{\int_{\mathbb{R}} \frac{g(Y_j - h(u))}{\cancel{g(Y_j)}} \Delta(s, du)\, \ell_{j-1}(ds)}{\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{g(Y_j - h(u))}{\cancel{g(Y_j)}} \Delta(s, du)\, \ell_{j-1}(ds)}$$

$$= \frac{\int_{\mathbb{R}} \overbrace{g(Y_j - h(u))}^{=\gamma(u, Y_j)} \Delta(s, du)\, \ell_{j-1}(ds)}{\int\int_{\mathbb{R}\,\mathbb{R}} g(Y_j - h(u))\, \Delta(s, du)\, \ell_j(ds)} \cdot \frac{\int_{\mathbb{R}} \ell_{j-1}(du)}{\int_{\mathbb{R}} \ell_{j-1}(du)}$$

$$= \frac{\int_{\mathbb{R}} \gamma(u, Y_j)\, \Delta(s, du)\, \tilde{\pi}_{j-1}(dx)}{\int_{\mathbb{R}} \int_{\mathbb{R}} \gamma(u, Y_j)\, \Delta(s, du)\, \tilde{\pi}_{j-1}(dx)}$$