Lecture 2. Exercise 14

Changes we need to introduce to the nonlinear filtering recursion, when the Markov kernels are allowed to depend on j and on $\mathcal{F}^Y_{j-1} = \sigma(Y_0, \dots, Y_{j-1})$.

Let $Y_{[0,j]} := (Y_0, Y_1, \dots, Y_j)$.

We have

$$P(X_j \in B \mid \mathcal{F}^X_{j-1} \vee \mathcal{F}^Y_{j-1}) = \int_B \Lambda_j(X_{j-1}, Y_{[0,j-1]}, du)$$

and

$$P(Y_j \in B \mid \mathcal{F}^X_j \vee \mathcal{F}^Y_{j-1}) = \int_B \gamma_j(X_j, Y_{[0,j-1]}, u)\, \nu(du)$$

Note that now (X_j, Y_j) is NOT a Markov process.

The law of $Y_{[0,j]}$ given $\mathcal{F}^X_j$ is

$$P(Y_0 \in A_0, \dots Y_j \in A_j \mid \mathcal{F}^X_j) = E\Big[\mathbb{1}_{\{Y_0 \in A_0\}} \cdots \mathbb{1}_{\{Y_{j-1} \in A_{j-1}\}} \int_{A_j} \gamma_j(X_j, Y_{[0,j-1]}, u_j)\, d\nu(du_j)$$

$$= E\Big[\mathbb{1}_{\{Y_0 \in A_0\}} \cdots \mathbb{1}_{\{Y_{j-2} \in A_{j-2}\}}$$

$$\times \int_{A_{j-1}} \int_{A_j} \gamma_{j-1}(X_{j-1}, Y_{[0,j-2]}, u_{j-1})\, \gamma_j(X_j, (Y_{[0,j-2]}, u_{j-1}), u_j)\, \nu(du_{j-1})\, \nu(du_j)\Big]$$

$$= \dots$$

$$= \int_{A_0} \cdots \int_{A_j} \prod_{k=0}^{j} \underbrace{\gamma_k(X_k, u_{[0,k-2]}, u_{k-1})}_{\substack{\| \\ \gamma_k(X_k,\, u_{[0 \dots k-1]})}}\, \nu(du_0) \cdots \nu(du_j)$$

Abuse of notation

Then,

$$E[\varphi(X_j) \mid \mathcal{F}^Y_j] = \frac{\dot{E}[\varphi(X_j(\dot\omega)) \prod_{i=0}^{j} \gamma_i(X_i(\dot\omega), Y_{[0,i]})]}{\dot{E}[\prod_{i=0}^{j} \gamma_i(X_i(\dot\omega), Y_{[0,i]})]}$$

Introduce the relation

$$L_j(X(\dot\omega), Y) = \prod_{i=0}^{j} \gamma_i(X_i(\dot\omega), Y_{[0,i]})$$

and note that

$$\dot{E}[\varphi(X_j(\dot\omega)) L_j(X(\dot\omega), Y) \mid \mathcal{F}^Y_{j-1}] =$$

$$= L_{j-1}(X(\dot\omega), Y)\, \dot{E}[\varphi(X_j(\dot\omega))\, \gamma_j(X_j(\dot\omega), Y_{[0,j]}) \mid \mathcal{F}^Y_{j-1}]$$

$$= L_{j-1}(X(\dot\omega), Y) \int_{\mathbb{R}} \varphi(u)\, \gamma_j(u, Y_{[0,j]})\, \Lambda_j(X_{j-1}(\dot\omega), Y_{[0,j-1]}, du)$$

then,

$$E[\varphi(X_j) \mid \mathcal{F}^Y_j] = \frac{\dot{E}[\varphi(X_j(\dot\omega)) L_j(X(\dot\omega), Y)]}{\dot{E}[L_j(X(\dot\omega), Y)]}$$

$$= \frac{\dot{E}[L_{j-1}(X(\dot\omega), Y) \int_{\mathbb{R}} \varphi(u)\, \gamma_j(u, Y_{[0,j]})\, \Lambda_j(X_{j-1}(\dot\omega), Y_{[0,j-1]}, du)]}{\underbrace{\dot{E}[L_{j-1}(X(\dot\omega), Y) \int_{\mathbb{R}} \gamma_j(u, Y_{[0,j]})\, \Lambda_j(X_{j-1}(\dot\omega), Y_{[0,j-1]}, du)]}_{=:(*)}}$$

$$= (*)\, \frac{\dot{E}[L_{j-1}(X(\dot\omega), Y)]}{\dot{E}[L_{j-1}(X(\dot\omega), Y)]}$$

$$= \frac{E[\int_{\mathbb{R}} \varphi(u)\, \gamma_j(u, Y_{[0,j]})\, \Lambda_j(X_{j-1}(\dot\omega), Y_{[0,j-1]}, du) \mid \mathcal{F}^Y_{j-1}]}{E[\int_{\mathbb{R}} \gamma_j(u, Y_{[0,j]})\, \Lambda_j(X_{j-1}(\dot\omega), Y_{[0,j-1]}, du) \mid \mathcal{F}^Y_{j-1}]}$$

Let $\pi_j(dx)$ be the regular conditional distribution of X_j given $\mathcal{F}_j^Y$. Then, using Fubini

$$\int_{\mathbb{R}} \varphi(x)\,\pi_j(dx) = E[\varphi(X_j) \mid \mathcal{F}_j^Y]$$

$$= \frac{\overset{x}{\int_{\mathbb{R}}}\overset{u}{\int_{\mathbb{R}}} \varphi(u)\,\gamma_j(u, Y_{[0,j]})\,\Lambda_j(x, Y_{[0,j-1]}, du)\,\pi_{j-1}(dx)}{\int_{\mathbb{R}}\int_{\mathbb{R}} \gamma_j(u, Y_{[0,j]})\,\Lambda_j(x, Y_{[0,j-1]}, du)\,\pi_{j-1}(dx)}$$

$$= \overset{u}{\int_{\mathbb{R}}} \varphi(u)\, \frac{\overset{x}{\int_{\mathbb{R}}} \gamma(u, Y_{[0,j]})\,\Lambda_j(x, Y_{[0,j-1]}, du)\,\pi_{j-1}(dx)}{\int_{\mathbb{R}}\int_{\mathbb{R}} \gamma(u, Y_{[0,j]})\,\Lambda_j(x, Y_{[0,j-1]}, du)\,\pi_{j-1}(dx)}$$