Lecture 2 . Exercise 16

Find the filtering recursion for

$$X_j = g(X_{j-1}) + \varepsilon_j \quad , \quad j \geq 1$$

$$Y_j = f(X_j) + \xi_j \qquad j \geq 1$$

with $Y_0 = 0$ and random initial condition X_0 (independent of ε or ξ). g, f Borel measurable s.t. $E[|g(X_{j-1})|] < +\infty$ and $E[|f(X_j)|] < +\infty$ $j \geq 1$.

ε and ξ are independent and i.i.d. $\varepsilon_1 \sim p(u)du$ and $\xi_1 \sim q(u)du$.

Note that for all $B \in \mathcal{B}(\mathbb{R})$

$$P(X_j \in B \mid \mathcal{F}^X_{j-1} \vee \mathcal{F}^Y_{j-1}) = P(g(X_{j-1}) + \varepsilon_j \in B \mid \mathcal{F}^X_{j-1} \vee \mathcal{F}^Y_{j-1})$$

$$= P(\varepsilon_j \in B \setminus g(X_{j-1})) = \int_{B \setminus g(X_{j-1})} p(u)\, du$$

$$= \int_B p(u - g(X_{j-1}))\, du =: \int_B \Lambda(X_{j-1}, du)$$

Applying formula (2.19) obtained with the reference measure approach we get

$$\rho_j(du) = \int_{\mathbb{R}} \frac{q(Y_j - f(u))}{q(Y_j)} p(u - g(s))\, du\, \rho_{j-1}(ds)$$

$$= \frac{q(Y_j - f(u))}{q(Y_j)} \int_{\mathbb{R}} p(u - g(s))\, \rho_{j-1}(ds)\, du$$

$$\pi_j(du) = \frac{q(Y_j - f(u)) \int_{\mathbb{R}} p(u - g(s))\, \rho_{j-1}(ds)\, du}{\iint_{\mathbb{R}\,\mathbb{R}} q(Y_j - f(u))\, p(u - g(s))\, \rho_{j-1}(ds)\, du}$$

$$\pi_0 = p_{x_0}.$$