Lecture 2. Exercise 17

Let $X = (X_j)_{j\in\mathbb{Z}_+}$ Markov chain with values in a finite set $S = \{a_1, \dots, a_d\}$

$\Lambda = (\lambda_{ij})$ matrix of transition probabilities and vector of initial distributions γ

$$P(X_j = a_l \mid X_{j-1} = a_m) = \lambda_{ml}, \quad P(X_0 = a_l) = \gamma_l, \quad 1 \le l, m \le d$$

and $Y_j = h(X_j) + \xi_j$, $j \geq 0$, where $\xi = (\xi_j)_{j\geq 0}$ is an i.i.d sequence and $\mathcal{L}(\xi_1) \sim f(u)\,du$.

Write the recursion for the nonlinear filter in componentwise notation.

We have that

$$P(Y_j \in A \mid \mathcal{F}_j^X \vee \mathcal{F}_{j-1}^Y) = P(h(X_j) + \xi_j \in A \mid \mathcal{F}_j^X \vee \mathcal{F}_{j-1}^Y)$$

$$= E[\mathbb{1}_{\{a + \xi_j \in A\}}]\Big|_{a = h(X_j)} = \int_{A-a} f(u)\,du \Big|_{a = h(X_j)}$$

$$= \int_A f(z-a)\,dz \Big|_{a = h(X_j)} = \int_A f(z - h(X_j))\,dz$$

$$=: \int_A \tilde{f}_l(z)\,\nu(dz), \quad \text{where } \tilde{f}_l(z) = f(z - h(a_l)), \ l = 1, \dots, d.$$
$$\nu(dz) = dz$$

We need to write the following equation componentwise.

$$\pi_j = \frac{\mathrm{diag}(\tilde{f}(Y_j))\,\Lambda^T \pi_{j-1}}{|\mathrm{diag}(\tilde{f}(Y_j))\,\Lambda^T \pi_{j-1}|}, \quad \text{where } \tilde{f}(Y_j) = (\tilde{f}_1(Y_j), \dots, \tilde{f}_d(Y_j))^T$$

this yields

$$\pi_j(a_l) = \frac{\tilde{f}_l(Y_j) \sum_{m=1}^{d} \lambda_{lm} \pi_{j-1}(a_m)}{\sum_{l=1}^{d} \left| f_l(Y_j) \sum_{m=1}^{d} \lambda_{lm} \pi_{j-1}(a_m) \right|}$$

$$= \frac{f(Y_j - h(a_l)) \sum_{m=1}^{d} \lambda_{lm} \pi_{j-1}(a_m)}{\sum_{l=1}^{d} \left| f(Y_j - h(a_l)) \right| \sum_{m=1}^{d} \lambda_{lm} \pi_{j-1}(a_m) \Big|}$$