## Lecture 2 . Exercise 4.

Let $X, Z$ a pair of indep. r.v. with densities $f_X, f_Z$

$E[X^2] < +\infty$.

Suppose $Y = X + Z$.

Prove that

$$E[X \mid Y](\omega) = \frac{\int_{\mathbb{R}} x f_X(x) f_Z(Y(\omega) - x)\,dx}{\int_{\mathbb{R}} f_X(x) f_Z(Y(\omega) - x)\,dx}$$

using the formula

$$E[\varphi(X) \mid \mathcal{G}] = \frac{\int_{\mathbb{R}} \varphi(x)\, n(Y(\omega), x)\, P_X(dx)}{\int_{\mathbb{R}} n(Y(\omega), x)\, P_X(dx)}$$

where $\mathcal{G} = \sigma(Y)$ and $P(Y \in A \mid X = x) = \int_A n(u, x)\,\nu(du)$

Clearly $\varphi(x) = x$ and we need to identify

$n(Y(\omega), x)\, P_X(dx)$ with $f_X(x) f_Z(Y(\omega) - x)\,dx$

We have that $P_X(dx) = f_X(x)\,dx$.

So we only need to prove that $n(y, x) = f_Z(y - x)$

$$P(Y \in A \mid X = x) = \int_A f_{Y \mid X}(y \mid x)\,dy$$

$$= \int_A \frac{f_{X,Y}(x, y)}{f_X(x)}\,dy = (*)$$

$(X, Y) = g(X, Z) = (X, X + Z) \Rightarrow g^{-1}(x, y) = (x, y - x)$

$$f_{X,Z}(x,z) = f_X(x)\, f_Z(z) \qquad (X \text{ and } Z \text{ are indep.})$$

$$f_{X,Y}(x,y) = f_{X,Z}(g^{-1}(x,y))\, |\operatorname{Jac} g^{-1}(x,y)|$$
$$= f_X(x)\, f_Z(y-x) \cdot 1$$

$$(*) = \int_A f_Z(y-x)\, dy \qquad \checkmark$$

Another way of proving it is to reason as follows.

$$Y \mid X = x \qquad \sim \quad (x + Z)$$

$$P(x + Z \le t) = P(Z \le t - x) = \int_{-\infty}^{t-x} f_Z(u)\, du$$
$$= \left\{ \begin{array}{l} v = u + x \\ dv = du \end{array} \right\} = \int_{-\infty}^{t} f_Z(v - x)\, dv$$
$$\Rightarrow \quad f_{Y|X}(y \mid x) = f_Z(y - x)$$