Lecture 3. Exercise 1

(1) Let $a = \{a_t\}_{t\geq 0}$ and $b = \{b_t\}_{t\geq 0}$ a pair of deterministic functions. Find the differential of

$$X_t = \exp\{\int_0^t a_s\,ds\}\{x + \int_0^t \exp(-\int_0^s a_u\,du)\, b_s\, dW_s\}$$

where $x \in \mathbb{R}$. Show that the mean $m_t = E[X_t]$, variance $V_t = E[(X_t - m_t)^2]$ and covariance $K(t,s) = E[(X_t - m_t)(X_s - m_s)]$ functions satisfy the equations

$$\dot{m}_t = a_t m_t, \quad m_0 = x$$

$$\dot{V}_t = 2a_t V_t + b_t^2, \quad V_0 = 0$$

$$K(t,s) = \exp\{\int_s^t a_u\,ds\} V_s \qquad t \geq s.$$

Minimal assumptions on a and b are $a \in L^1([0,t])$ and $b \in L^2([0,t])$ for all $t \geq 0$.

Let $Y_t = \exp\{\int_0^t a_s\,ds\} \Rightarrow dY_t = a_t Y_t\,dt \Rightarrow \dot{Y}_t = a_t Y_t$

$$Z_t = x + \int_0^t \exp(-\int_0^s a_u\,du)\, b_s\, dW_s$$

$$\Rightarrow dZ_t = \exp(-\int_0^t a_u\,du)\, b_t\, dW_t = \frac{b_t}{Y_t}\, dW_t$$

Y finite variation and continuous $\Rightarrow d\langle Y, Z\rangle_t \equiv 0$

$X_t = Y_t Z_t$,

$$dX_t = Y_t\,dZ_t + Z_t\,dY_t + d\langle Y, Z\rangle_t$$

$$= \cancel{Y_t}\frac{b_t}{\cancel{Y_t}}\,dW_t + Z_t a_t Y_t\,dt$$

$$= b_t\,dW_t + Z_t a_t Y_t\,dt \qquad \text{or} \qquad (\text{Note that } X_0 = x)$$

$$X_t = x + \int_0^t b_s\,dW_s + \int_0^t Z_s a_s Y_s\,ds$$

$$* \quad m_t = E[X_t] = x + E\left[\overset{=0}{\cancel{\int_0^t b_s\, dW_s}}\right] + E\left[\int_0^t Z_s a_s Y_s\, ds\right]$$

$$= x + \int_0^t E[Z_s]\, a_s Y_s\, ds = x(1 + Y_t - 1)$$

$$= x Y_t$$

$$\left.\begin{cases} \dot m_t = x \dot Y_t = x a_t Y_t\, dt = a_t m_t \\ m_0 = x Y_0 = x. \end{cases}\right\} \Rightarrow m_t = x \exp\left(\int_0^t a_s\, ds\right)$$

$$* \quad V_t = E[(X_t - m_t)^2] = E[(Y_t Z_t - x Y_t)^2] = Y_t^2 E[(Z_t - x)^2]$$

$$= Y_t^2\, E\left[\left(\int_0^t \exp\left(-\int_0^s a_u\, du\right) b_s\, dW_s\right)^2\right]$$

$$= Y_t^2 \int_0^t \exp\left(-2\int_0^s a_u\, du\right) b_s^2\, ds = Y_t^2 \int_0^t \frac{b_s^2}{Y_s^2}\, ds$$

$$\dot V_t = 2 Y_t \dot Y_t \int_0^t \frac{b_s^2}{Y_s^2}\, ds + Y_t^2 \frac{b_t^2}{Y_s^2}$$

$$= 2 Y_t Y_t a_t \int_0^t \frac{b_s^2}{Y_s^2}\, ds + b_t^2$$

$$= 2 a_t V_t + b_t^2$$

Let $t \geq s$, then

$$K(t,s) = E[(X_t - m_t)(X_s - m_s)]$$

$$= Y_t Y_s E[(Z_t - x)(Z_s - x)]$$

$$= Y_t Y_s E\left[\int_0^t \frac{b_u}{Y_u}\, dW_u \int_0^s \frac{b_v}{Y_v}\, dW_v\right]$$

$$= Y_t Y_s \int_0^s \frac{b_u^2}{Y_u^2}\, du = \frac{Y_t}{Y_s} Y_s^2 \int_0^s \frac{b_u^2}{Y_u^2}\, du$$

$$= \exp\left(\int_s^t a_u\, du\right) V_s$$