Lecture 4. Exercise 1

Let
$$dX_t = a_t X_t\,dt + b_t\,dW_t$$
$$dY_t = A_t X_t\,dt + B_t\,dV_t$$
X_0, Y_0 independent of W, V, and all the coefficients a, b, A, B being deterministic.

Assume that X_0, Y_0 are such that $\hat{E}[X_0 | Y_0] = 0$ P.a.s.

then $E[X_t] = 0$ and $\hat{E}[X_t | Y_0] = 0$ P.a.s.

Let $Z_t = \exp\{\int_0^t c_s\,ds\}$, Then $dZ_t = c_t Z_t\,dt$,

$d\langle X, Z\rangle_t \equiv 0$ and

$$d(X_t Z_t) = X_t\,dZ_t + Z_t\,dX_t + d\langle X, Z\rangle_t$$
$$= X_t c_t Z_t\,dt + Z_t a_t X_t\,dt + Z_t b_t\,dW_t$$
$$\underset{c_t := -a_t}{=} Z_t b_t\,dW_t$$
(c_t arbitrary function in $L^1([0,T])$)

$$\Rightarrow X_t Z_t = X_0 Z_0 + \int_0^t Z_s b_s\,dW_s$$
$$X_t = X_0 Z_t^{-1} + \int_0^t \frac{Z_s}{Z_t} b_s\,dW_s$$

By the properties of the orthogonal projection we have that
$$\hat{E}[X_t | Y_0] = 0 \Rightarrow E[X_t] = 0$$

Since
$$E[X_t] = E[\underbrace{\hat{E}[X_t | Y_0]}_{\overset{(*)}{0}}] = 0$$

On the other hand, we have that

$$\hat{E}[X_t \mid Y_0] = \hat{E}[X_0 Z_t^{-1} \mid Y_0] + \hat{E}\left[\int_0^t \frac{Z_s}{Z_t} b_s \, dW_s \mid Y_0\right]$$

$$= ① + ②$$

① $\hat{E}[X_0 Z_t^{-1} \mid Y_0] = Z_t^{-1} \underbrace{\hat{E}[X_0 \mid Y_0]}_{\overset{\text{By assumption}}{=} 0} = 0$

② We will show that $\int_0^t \frac{Z_s}{Z_t} b_s \, dW_s \perp \mathcal{F}^{Y_0}$

$$\Rightarrow E\left[\int_0^t \frac{Z_s}{Z_t} b_s \, dW_s \mid Y_0\right] = 0.$$

We have that

$$E\left[\left(\int_0^t \frac{Z_s}{Z_t} b_s \, dW_s\right)(c_0 + c_1 Y_0)\right] =$$

$$= \underbrace{E\left[\int_0^t \frac{Z_s}{Z_t} b_s \, dW_s\right]}_{0} E[c_0 + c_1 Y_0] = 0$$

Itô integral has 0 expectation.

$$\Rightarrow \int_0^t \frac{Z_s}{Z_t} b_s \, dW_s \perp \mathcal{F}^{Y_0}$$