Lecture 4. Exercise 2

Show that the innovation process

$$\overline{W}_t = \int_0^t \frac{1}{B_s}\left(dY_s - A_s \hat{X}_s \, ds\right)$$

satisfies

a) $\hat{E}[\overline{W}_t \mid \mathcal{F}^Y_s] = \overline{W}_s$

b) $E[(\overline{W}_t - \overline{W}_s)^2] = t - s$

c) Derive the Kalman-Bucy equations assuming that $\overline{W}$ is a Wiener process (in the wide sense) and that $\hat{E}[X_t \mid \mathcal{F}^Y_t] = \int_0^t \Gamma(t,s)\, d\overline{W}_s$ for some $\Gamma(t,s)$. (Too long)

We are assuming the model

$$dX_t = a_t X_t \, dt + b_t \, dW_t$$
$$dY_t = A_t X_t \, dt + B_t \, dV_t$$

V and W two indep. B. m.

First note that

$$\overline{W}_t = \int_0^t \frac{1}{B_s}\left(dY_s - A_s\hat{X}_s \, ds\right)$$

$$= \int_0^t \frac{1}{B_s}\left\{dY_s - A_s X_s \, ds - A_s(\hat{X}_s - X_s)\, ds\right\}$$

$$= V_t + \int_0^t \frac{A_s}{B_s}(X_s - \hat{X}_s)\, ds$$

or

$$\overline{W}_t - \overline{W}_s = V_t - V_s + \int_s^t \frac{A_u}{B_u}(X_u - \hat{X}_u)\, du$$

Moreover,

$\hat{E}[X_u - \hat{X}_u \mid \mathcal{F}^Y_u] \equiv 0$ by the definition of orthogonal projection and using the tower law for orthogonal projections we get

$$\hat{E}\left[\int_s^t \frac{A_u}{B_u} (X_u - \hat{X}_u)\, du \mid \mathcal{F}^Y_s \right]$$

$$= \hat{E}\left[\int_s^t \frac{A_u}{B_u} \underbrace{\hat{E}[X_u - \hat{X}_u \mid \mathcal{F}^Y_u]}_{= 0} du \mid \mathcal{F}^Y_s \right] = 0$$

a)
$$\hat{E}[\overline{W}_t \mid \mathcal{F}^Y_s] = \overline{W}_s \iff \hat{E}[\overline{W}_t - \overline{W}_s \mid \mathcal{F}^Y_s] = 0$$

Linearity of $\hat{E}[\cdot \mid \mathcal{F}^Y_s]$ and $\overline{W}_s \in \mathcal{F}^Y_s$

By the preliminary reasonings we have that

$$\hat{E}[\overline{W}_t - \overline{W}_s \mid \mathcal{F}^Y_s] = E[V_t - V_s \mid \mathcal{F}^Y_s]$$

Now, for any r.v. Z, we have that

$$\hat{E}[Z \mid \mathcal{F}^Y_s] = 0 \Leftarrow E\left[\int_0^s \lambda_u \, dY_u \, Z\right] = 0,$$ for all deterministic and bounded process λ.

Therefore, we only need to check that

$$E\left[\int_0^s \lambda_u \, dY_u (V_t - V_s)\right] = 0 \quad \forall \lambda \in L^\infty([0,s]).$$

- $E\left[\int_0^s \lambda_u \, dY_u (V_t - V_s)\right] = E\left[\int_0^s \lambda_u A_u X_u \, du (V_t - V_s)\right]$

$$+ E\left[\int_0^s \lambda_u B_u \, dV_u (V_t - V_s)\right] = 1 + 2$$

$$① = E\left[\int_0^s \lambda_u X_u \, du\right] \cdot \underbrace{E[V_t - V_s]}_{0} = 0$$

$\uparrow$ X indep. of V $\qquad\qquad$ V is a B.m.

$$② = E\left[\int_0^s \lambda_u B_u \, dV_u \; \underbrace{E[V_t - V_s \mid \mathcal{F}_s^V]}_{0}\right] = 0$$

b) We have that

$$E[(\overline{W}_t - \overline{W}_s)^2] = E[(\overline{W}_t)^2] - 2E[\overline{W}_t \overline{W}_s] + E[(\overline{W}_s)^2]$$

and

$$E[\overline{W}_t \overline{W}_s] = E[(\overline{W}_t - \overline{W}_s)\overline{W}_s] + E[\overline{W}_s^2].$$

Hence,

$$E[(\overline{W}_t - \overline{W}_s)^2] = E[(\overline{W}_t)^2] - E[(\overline{W}_s)^2] - 2E[(\overline{W}_t - \overline{W}_s)\overline{W}_s]$$

and the result follows if we prove that

ⓐ $E[(\overline{W}_t)^2] = t$

ⓑ $E[(\overline{W}_t - \overline{W}_s)\overline{W}_s] = 0$

Since $\overline{W}_s \in h_s^Y$, ⓑ follows from the proof of a) where we have shown that $\overline{W}_t - \overline{W}_s \perp h_s^Y$

ⓐ Note that $\overline{W}_t$ is a semimartingale (or an Itô process) with differential

$$d\overline{W}_t = dV_t + \frac{A_t}{B_t}(X_t - \hat{X}_t)\,dt$$

and $d\langle \overline{W} \rangle_t = d\langle V \rangle_t = dt$

Hence, by integration by parts

$$d\bar{W}_t^2 = 2\bar{W}_t\,d\bar{W}_t + d\langle \bar{W}\rangle_t$$
$$= 2\bar{W}_t\,d\bar{W}_t + dt$$

Since $\bar{W}_t$ has orthogonal increments,

$$E\left[\int_0^t \bar{W}_s\,d\bar{W}_s\right] = 0$$ (Discretize the integral)

and we can conclude that $E[\bar{W}_t^2] = t$.