Lecture 4. Exercise 3

Let $Y_t = \int_0^t W_s \, ds + V_t$, where W and V are independent Wiener processes.

a) Find the optimal filter for $\hat{W}_t = \hat{E}[W_t \mid \mathcal{F}^Y_t]$

b) Find the explicit form of the optimal kernel $G(t,s)$, such that

$$\hat{W}_t = \int_0^t G(t,s) \, dY_s$$

c) Derive the equation for the linear estimate $\hat{V}_t = \hat{E}[V_t \mid \mathcal{F}^Y_t]$.

a) In the Kalman-Bucy model take

$a_t \equiv 0$, $b_t \equiv 1$, $A_t \equiv 1$, $B_t \equiv 1$.

Then $X_t \equiv W_t$. and the equations for the optimal filter are

$$\begin{cases} d\hat{W}_t = P_t \, (dY_t - \hat{W}_t \, dt) \\ \dot{P}_t = 1 - P_t^2 \end{cases}$$

$$\begin{cases} \hat{W}_0 = E[W_0] + \operatorname{cov}(W_0, Y_0) \operatorname{cov}(Y_0)^{\oplus} (Y_0 - E[Y_0]) \\ \quad = 0 \\ P_0 = \operatorname{Cov}(W_0) - \operatorname{cov}^2(W_0, Y_0) \operatorname{cov}(Y_0)^{\oplus} \end{cases}$$

$\hookrightarrow$ because $W_0 \equiv 0$ P-a.s.

Solving the equation for P_t gives

$$P_t = \frac{e^{2t} - 1}{e^{2t} + 1}$$

(You can use the formula given in class)

Hence

$$d\hat{W}_t = \frac{e^{2t}-1}{e^{2t}+1}\left(dY_t - \hat{W}_t\,dt\right)$$

$$= -\frac{e^{2t}-1}{e^{2t}+1}\hat{W}_t\,dt + \frac{e^{2t}-1}{e^{2t}+1}\,dY_t$$

b) In the proof of the Kalman-Bucy filter we saw that for $s \geq 0$ fixed and $t \geq s$.

$$\frac{\partial G}{\partial s}(t,s) = G(t,s)\left(a_t - \frac{A_t^2 P_t}{B_t^2}\right)$$

$$= -G(t,s)\,\frac{e^{2t}-1}{e^{2t}+1}$$

$$\Rightarrow \quad G(t,s) = G(s,s)\,\exp\left\{-\int_s^t \frac{e^{2u}-1}{e^{2u}+1}\,du\right\}$$

Moreover, in the proof is also shown that

$$G(s,s) = \frac{A_s P_s}{B_s^2} = P_s = \frac{e^{2s}-1}{e^{2s}+1}.$$

Hence,

$$\hat{W}_t = \int_0^t \frac{e^{2s}-1}{e^{2s}+1}\exp\left\{-\int_s^t \frac{e^{2u}-1}{e^{2u}+1}\,du\right\}dY_s$$

$$= \int_0^t P_s\,\exp\left\{-\int_s^t P_u\,du\right\}dY_s$$

We can check it.

$$\hat{W}_t = \exp\left\{-\int_0^t P_u\,du\right\}\int_0^t P_s\,\exp\left(\int_0^s P_u\,du\right)dY_s$$

$$=: L_t\,K_t$$

$$d\hat{W} = L_t\,dK_t + K_t\,dL_t + \overbrace{\cancel{d\langle L,K\rangle_t}}^{=0}$$

$$= \cancel{L_t}\, P_t \exp\Big(\cancel{\int_0^t P_u\, du}\Big)\, dY_t - \underbrace{\int_0^t P_s \exp\Big(\int_0^s P_u\, du\Big)\, dY_s \;\; L_t}_{\hat{W}_t} P_t\, dt$$

$$= P_t\, dY_t - \hat{W}_t P_t\, dt$$

c) Using the Kalman-Bucy formula in Theorem 4.4. we can write

$$d\begin{pmatrix} X^1_t \\ X^2_t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} dW_t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} dV_t$$

$$dY_t = (1 \;\; 0)\begin{pmatrix} X^1_t \\ X^2_t \end{pmatrix} dt + dV_t = X^1_t\, dt + dV_t$$

$$= W_t\, dt + dV_t$$

This implies $a_0 \equiv a_1 \equiv a_2 \equiv 0$, $b_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $b_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

and $A_0 \equiv A_2 \equiv B_1 \equiv 0$, $A_1 = (1 \; 0)$ and $B_2 = 1$.

Hence, $b \circ b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1 \;\; 0) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(0, 1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$b \circ B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot 0 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot 1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$B \circ B = 0 + 1 = 1$

$$d\begin{pmatrix} \hat{X}^1_t \\ \hat{X}^2_t \end{pmatrix} = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + P_t \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \cdot 1 \cdot \left(dY_t - (1 \;\; 0)\begin{pmatrix} \hat{X}^1_t \\ \hat{X}^2_t \end{pmatrix} dt\right)$$

or

$$d\hat{X}^1_t = P^{1,1}_t \left(dY_t - \hat{X}^1_t\, dt \right)$$

$$d\hat{X}^2_t = 1\, dt + P^{2,1}_t\, dY_t$$

Note that $X^1_0 = W_0 = 0$ and $X^2_0 = V_0 \equiv 0$

$\Rightarrow \hat{X}_0 = 0$ and $P_0 = 0$.

The equation for the error is given by

$$\dot{P}_t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + P_t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \cdot 1 \cdot \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + P_t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)^T$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} P_t^{1,1} \\ 1 + P_t^{2,1} \end{pmatrix} \left(P_t^{1,1} , 1 + P_t^{2,1} \right)$$

$$= \begin{pmatrix} 1 + (P_t^{1,1})^2 & (1 + P_t^{2,1}) P_t^{1,1} \\ (1 + P_t^{2,1}) P_t^{1,1} & 1 + (1 + P_t^{2,1})^2 \end{pmatrix}$$

Then $\hat{V}_t = E[V_t \mid \mathcal{F}^Y_t] = \hat{X}^2_t$ and one needs to solve "offline"

$$\dot{P}^{1,1}_t = 1 + (P^{1,1}_t)^2 , \quad P^{1,1}_0 = 0$$

and

$$\dot{P}^{2,1}_t = (1 + P^{2,1}_t) P^{1,1}_t \qquad P^{2,1}_0 = 0$$

and then compute

$$\hat{X}^2_t = t + \int_0^t P^{2,1}_s \, dY_s$$