18$^{\text{th}}$ October, 2023

MAT4790/9790

Mandatory assignment 1 of 1

Submission deadline

Thursday 2$^{\text{nd}}$ NOVEMBER 2023, 14:30 in Canvas (canvas.uio.no).

Instructions

Note that you have one attempt to pass the assignment. This means that there are no second attempts.

For courses on bachelor level, you can choose between scanning handwritten notes or using a typesetting software for mathematics (e.g. LaTeX). Scanned pages must be clearly legible. For courses on master level the assignment must be written with a typesetting software for mathematics. It is expected that you give a clear presentation with all necessary explanations. The assignment must be submitted as a single PDF file. Remember to include any relevant programming code and resulting plots and figures, in the PDF-file.

All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, you may be asked to give an oral account.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline. Note that teaching staff cannot grant extensions.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

To pass the assignment you need a score of at least 50p. All questions have equal weight.

Problem 1. Describe the setup seen in class for the linear filtering problem in discrete time and its solution. That is, provide the assumptions on the signal and observation process as well as the Kalman-Bucy filter.

Problem 2. (taken from R. Kalman [1]) A number of particles leaves the origin at time j = 0 with random velocities; after j = 0, each particle moves with a constant (unknown velocity). Suppose that the position of one of these particles is measured, the data being contaminated by stationary, additive, correlated noise. What is the optimal estimate of the position and velocity of the particle at the time of the last measurement ? Let $x_1(j)$ be the position and $x_2(j)$ the velocity of the particle; $x_3(j)$ is the noise. The problem is then represented by the model:

$$\begin{aligned} x_1(j+1) &= x_1(j) + x_2(j), \\ x_2(j+1) &= x_2(j), \\ x_3(j+1) &= \varphi x_3(j) + u(j), \\ y(j) &= x_1(j) + x_3(j), \end{aligned}$$

and the additional conditions

$$\begin{aligned} \mathbb{E}\left[x_1^2(0)\right] = \mathbb{E}\left[x_2(0)\right] = 0, \quad \mathbb{E}\left[x_2(0)\right] = a^2 > 0. \\ \mathbb{E}\left[u(j)\right] = 0, \quad \mathbb{E}\left[u^2(j)\right] = b^2. \end{aligned}$$

1. Derive the Kalman-Bucy filter equations for the signal
$$X_j = \left(x_1(j), x_2(j), x_3(j)\right)^T.$$
2. Derive the Kalman-Bucy filter equations for the signal
$$X_j = \left(x_2(j), x_3(j)\right)^T,$$
using the obvious relation $x_1(j) = jx_2(j) = jx_2(0)$.
3. Solve the Riccati equation from 2. explicitly.
4. Show that for $\varphi \neq 1$ (both $|\varphi| < 1$ and $|\varphi| > 1$), the mean square errors of the velocity and position estimates converge to 0 and b^2 respectively. Find the convergence rate for the velocity error.
5. Show that for $\varphi = 1$ the mean square error for the estimate of the position diverges.

6. Define the new observation sequence

$$\delta y(j+1) = y(j+1) - \varphi y(j), \quad j \geq 0$$

and $\delta y\,(0) = y\,(0)$. Then,

$$\overline{\text{span}}\,\{\delta y\,(j)\,, 0 \leq j \leq n\} = \overline{\text{span}}\,\{\delta y\,(j)\,, 0 \leq j \leq n\}\,.$$

Derive the Kalman-Bucy filter for the signal $X_j = x_2\,(j)$ and observations δy_j. Verify your answer in 5.

Problem 3. Consider a signal/observation pair $(\theta, \xi_j)_{j\geq 1}$, where θ is a random variable distributed uniformly on $[0,1]$ and (ξ_j) is a sequence generated by:

$$\xi_j = \theta U_j,$$

where $U = (U_j)_{j\geq 1}$ is a sequence of i.i.d. random variables with uniform distribution on $[0,1]$. θ and U are independent.

1. Consider the recursive filtering estimate $\left(\tilde{\theta}_j\right)_{j\geq 0}$defined by

$$\tilde{\theta}_j = \max\left(\tilde{\theta}_{j-1}, \xi_j\right), \quad \tilde{\theta}_0 = 0.$$

Find the corresponding mean square error, $Q_j = \mathbb{E}\left[\left(\theta - \tilde{\theta}_j\right)^2\right]$.

2. Show that Q_j converges to 0 and find the rate of convergence, that is, find $r\,(j)$ such that $\lim_{j\to\infty} r\,(j)\,Q_j$ exists, is finite and positive.

3. Find the optimal estimate $\bar{\theta} = \mathbb{E}\left[\theta|\, \mathcal{F}_j^{\xi}\right]$.

[1] R.E. Kalman, A New Approach to Linear Filtering and Prediction Problems, *Trans. ASME Ser. D. J. Basic Engrg.* 82 1960 35–45.