17^{th} November, 2023

MAT4750/9750

Mandatory assignment 1 of 1

Submission deadline

Thursday 2^{nd} NOVEMBER 2023, 14:30 in Canvas (canvas.uio.no).

Instructions

Note that you have one attempt to pass the assignment. This means that there are no second attempts.

For courses on bachelor level, you can choose between scanning handwritten notes or using a typesetting software for mathematics (e.g. LaTeX). Scanned pages must be clearly legible. For courses on master level the assignment must be written with a typesetting software for mathematics. It is expected that you give a clear presentation with all necessary explanations. The assignment must be submitted as a single PDF file. Remember to include any relevant programming code and resulting plots and figures, in the PDF-file.

All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, you may be asked to give an oral account.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline. Note that teaching staff cannot grant extensions.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

To pass the assignment you need a score of at least 50p. All questions have equal weight.

Problem 1. Describe the setup seen in class for the linear filtering problem in discrete time and its solution. That is, provide the assumptions on the signal and observation process as well as the Kalman-Bucy filter.

Problem 2. (taken from R. Kalman [1]) A number of particles leaves the origin at time j = 0 with random velocities; after j = 0, each particle moves with a constant (unknown velocity). Suppose that the position of one of these particles is measured, the data being contaminated by stationary, additive, correlated noise. What is the optimal estimate of the position and velocity of the particle at the time of the last measurement ? Let $x_1(j)$ be the position and $x_2(j)$ the velocity of the particle; $x_3(j)$ is the noise. The problem is then represented by the model:

$$\begin{aligned} x_1(j+1) &= x_1(j) + x_2(j), \\ x_2(j+1) &= x_2(j), \\ x_3(j+1) &= \varphi x_3(j) + u(j), \\ y(j) &= x_1(j) + x_3(j), \end{aligned}$$

and the additional conditions

$$\mathbb{E}\left[x_1^2(0)\right] = \mathbb{E}\left[x_2(0)\right] = 0, \quad \mathbb{E}\left[x_2(0)\right] = a^2 > 0.$$
$$\mathbb{E}\left[u(j)\right] = 0, \quad \mathbb{E}\left[u^2(j)\right] = b^2 > 0.$$

1. Derive the Kalman-Bucy filter equations for the signal
$$X_j = \left(x_1(j), x_2(j), x_3(j)\right)^T.$$
2. Derive the Kalman-Bucy filter equations for the signal
$$X_j = \left(x_2(j), x_3(j)\right)^T,$$
using the obvious relation $x_1(j) = jx_2(j) = jx_2(0)$.
3. Solve the Riccati equation from 2. explicitly.
4. Show that for $\varphi \neq 1$ (both $|\varphi| < 1$ and $|\varphi| > 1$), the mean square errors of the velocity and position estimates converge to 0 and b^2 respectively. Find the convergence rate for the velocity error.
5. Show that for $\varphi = 1$ the mean square error for the estimate of the position diverges.

6. Define the new observation sequence

$$\delta y(j+1) = y(j+1) - \varphi y(j), \quad j \geq 0$$

   and $\delta y\,(0) = y\,(0)$. Then,

$$\overline{\operatorname{span}}\,\{\delta y\,(j)\,, 0 \leq j \leq n\} = \overline{\operatorname{span}}\,\{y\,(j)\,, 0 \leq j \leq n\}\,.$$

   Derive the Kalman-Bucy filter for the signal $X_j = x_2\,(j)$ and observations δy_j. Verify your answer in 5.

Problem 3. Consider a signal/observation pair $(\theta, \xi_j)_{j\geq 1}$, where θ is a random variable distributed uniformly on $[0,1]$ and (ξ_j) is a sequence generated by:

$$\xi_j = \theta U_j,$$

where $U = (U_j)_{j\geq 1}$ is a sequence of i.i.d. random variables with uniform distribution on $[0,1]$. θ and U are independent.

1. Consider the recursive filtering estimate $\left(\tilde{\theta}_j\right)_{j\geq 0}$defined by

$$\tilde{\theta}_j = \max\left(\tilde{\theta}_{j-1}, \xi_j\right), \quad \tilde{\theta}_0 = 0.$$

   Find the corresponding mean square error, $Q_j = \mathbb{E}\left[\left(\theta - \tilde{\theta}_j\right)^2\right]$.

2. Show that Q_j converges to 0 and find the rate of convergence, that is, find $r\,(j)$ such that $\lim_{j\to\infty} r\,(j)\,Q_j$ exists, is finite and positive.

3. Find the optimal estimate $\bar{\theta} = \mathbb{E}\left[\theta|\,\mathcal{F}_j^{\xi}\right]$.

[1] R.E. Kalman, A New Approach to Linear Filtering and Prediction Problems, *Trans. ASME Ser. D. J. Basic Engrg.* 82 1960 35–45.

Solution Problem 1

We consider a pair of processes $(X,Y)=(X_j,Y_j)_{j\geq 0}$, generated by the linear recursive equations

$$X_j = a_0(j) + a_1(j)X_{j-1} + a_2(j)Y_{j-1} + b_1(j)\varepsilon_j + b_2(j)\xi_j, \tag{1}$$
$$Y_j = A_0(j) + A_1(j)X_{j-1} + A_2(j)Y_{j-1} + B_1(j)\varepsilon_j + B_2(j)\xi_j, \tag{2}$$

where

- X_j and Y_j has values in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively.
- $\varepsilon = (\varepsilon_j)_{j\geq 1}$ and $(\xi_j)_{j\geq 1}$ are orthogonal discrete time white noises with values in R^l and R^k. That is,

$$\mathbb{E}[\varepsilon_j] = 0, \qquad \mathbb{E}\left[\varepsilon_j\varepsilon_i^T\right] = \begin{cases} I, & i=j \\ 0, & i\neq j \end{cases} \in \mathbb{R}^{l\times l},$$
$$\mathbb{E}[\xi_j] = 0, \qquad \mathbb{E}\left[\xi_j\xi_i^T\right] = \begin{cases} I, & i=j \\ 0, & i\neq j \end{cases} \in \mathbb{R}^{k\times k},$$

  and $\mathbb{E}\left[\varepsilon_j\xi_i^T\right] = 0$,for all $i,j\geq 0$.

- The coefficients $a_0(j), a_1(j)$, etc. are deterministic (known) sequences of matrices of appropriate dimensions. In what follows we will drop the dependence of the coeffcients on time, to lighten the notation.
- The equations are solved subject to possibly random initial conditions X_0 and Y_0, uncorrelated with the noises ε and ξ, whose means and covariances are known.

We denote the optimal linear estimate of X_j given $\mathcal{L}_j^Y = \overline{\text{span}}\{1, Y_1, ..., Y_j\}$ by $\hat{X}_j = E\left[X_j|\,\mathcal{L}_j^Y\right]$ and the corresponding error covariance matrix by

$$P_j = \mathbb{E}\left[\left(X_j - \hat{X}_j\right)\left(X_j - \hat{X}_j\right)^T\right].$$

Theorem. *The estimate $\hat{X}_j$ and the error covariance P_j satisfy the equations*

$$\hat{X}_j = a_0 + a_1\hat{X}_{j-1} + a_2Y_{j-1} + \left(a_1P_{j-1}A_1^T + b\circ B\right)\left(A_1P_{j-1}A_1^T + B\circ B\right)^{\oplus}\left(Y_j - A_0 - A_1\hat{X}_{j-1} - A_2Y_{j-1}\right), \tag{3}$$

and

$$
\begin{aligned}
P_j &= a_1 P_{j-1} a_1^T + b \circ b \\
&\quad - \left(a_1 P_{j-1} A_1^T + b \circ B\right) \left(A_1 P_{j-1} A_1^T + B \circ B\right)^{\oplus} \left(a_1 P_{j-1} A_1^T + b \circ B\right)^T , \qquad (4)
\end{aligned}
$$

where

$$
b \circ b = b_1 b_1^T + b_2 b_2^T, \qquad b \circ B = b_1 B_1^T + b_2 B_2^T, \qquad B \circ B = B_1 B_1^T + B_2 B_2^T,
$$

and

$$
\begin{aligned}
\hat{X}_0 &= \mathbb{E}\left[X_0\right] + \operatorname{cov}\left(X_0, Y_0\right) \operatorname{cov}\left(Y_0\right)^{\oplus} \left(Y_0 - \mathbb{E}\left[Y_0\right]\right), \\
P_0 &= \operatorname{cov}\left(X_0\right) - \operatorname{cov}\left(X_0, Y_0\right) \operatorname{cov}\left(Y_0\right)^{\oplus} \operatorname{cov}\left(X_0, Y_0\right)^T .
\end{aligned}
$$

and $(\delta Y)_0 = Y_0$. Then,

$$\overline{\text{span}}\{(\delta Y)_j, 0 \le j \le n\} = \overline{\text{span}}\{Y_j, 0 \le j \le n\}.$$

Derive the Kalman-Bucy filter for the signal $X_j = (X_2)_j$ and observations δY_j. Verify your answer in 5.

Solution to Problem 2, 1. Moreover, let $\mathbb{E}[(X_3)_0^2] = 0$. This already implies, that $(X_3)_0 = 0$ almost surely. The same argument shows $(X_1)_0 = 0$. Moreover, one finds $Y_0 = (X_1)_0 + (X_3)_0 = 0$. The process X_3 shall represent noise later on. Therefore let $b > 0$ be the positive square root of b^2 and set $U_j := b\varepsilon_j$, such that $U_j = b\varepsilon_j \sim \mathcal{N}(0, b^2)$. Then, the linear recursive equations for the discrete Kalman-Bucy filter read as

$$X_j = \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \phi \end{pmatrix}}_{=(a_1)_j} X_{j-1} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}}_{=(b_1)_j} \varepsilon_j,$$

$$Y_j = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} X_j = \underbrace{\begin{pmatrix} 1 & 1 & \phi \end{pmatrix}}_{=(A_1)_j} X_{j-1} + \underbrace{b}_{=(B_1)_j} \varepsilon_j.$$

From $(X_1)_0 = (X_3)_0 = 0$ follows that the only non-zero entry of $\operatorname{cov}(X_0)$ is $\operatorname{cov}(X_2)_0 = \mathbb{E}[((X_2)_0 - \mathbb{E}[(X_2)_0])^2] = \mathbb{E}[(X_2)_0^2] = a^2 > 0$. Moreover, $\mathbb{E}[(X_1)_0] = \mathbb{E}[(X_3)_0] = 0$. With $\operatorname{cov}(Y_0)^{\oplus} = 0$ and $\mathbb{E}[(X_2)_0] = 0$, the initial values are

$$\hat{X}_0 = \mathbb{E}[X_0] + \operatorname{cov}(X_0, Y_0)\operatorname{cov}(Y_0)^{\oplus}(Y_0 - \mathbb{E}[Y_0]) = (0, 0, 0)^\top,$$

$$P_0 = \operatorname{cov}(X_0) - \operatorname{cov}(X_0, Y_0)\operatorname{cov}(Y_0)^{\oplus}\operatorname{cov}(X_0, Y_0)^\top = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

With Theorem 2.5 the Kalman-Bucy filter equations are given as

$$\begin{aligned} \hat{X}_j = \underbrace{(a_0)_j}_{=0} + (a_1)_j \hat{X}_{j-1} + \underbrace{(a_2)_j}_{=0} Y_{j-1} + ((a_1)_j P_{j-1}(A_1)_j^\top + b \circ B) \cdot \\ ((A_1)_j P_{j-1}(A_1)_j^\top + B \circ B)^{\oplus}(Y_j - \underbrace{(A_0)_j}_{=0} - (A_1)_j \hat{X}_{j-1} - \underbrace{(A_2)_j}_{=0} Y_{j-1}), \end{aligned}$$

$$\begin{aligned} P_j = (a_1)_j P_{j-1}(a_1)_j^\top + b \circ b + ((a_1)_j P_{j-1}(A_1)_j^\top + b \circ B) \cdot \\ ((A_1)_j P_{j-1}(A_1)_j^\top + B \circ B)^{\oplus}((a_1)_j P_{j-1}(A_1)_j^\top + b \circ B)^\top. \end{aligned}$$

By plugging in the determined values, the Kalman-Bucy filter equations are

$$\hat{X}_j = \begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & \phi\end{pmatrix}\hat{X}_{j-1} + \left[\begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & \phi\end{pmatrix} P_{j-1}\begin{pmatrix}1\\ 1\\ \phi\end{pmatrix} + \begin{pmatrix}0\\ 0\\ b^2\end{pmatrix}\right]\cdot$$
$$\left[\begin{pmatrix}1 & 1 & \phi\end{pmatrix} P_{j-1}\begin{pmatrix}1\\ 1\\ \phi\end{pmatrix} + b^2\right]^{\oplus}(Y_j - \begin{pmatrix}1 & 1 & \phi\end{pmatrix}\hat{X}_{j-1}),$$
$$P_j = \begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & \phi\end{pmatrix} P_{j-1}\begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 0 & 0 & \phi\end{pmatrix} + \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & b^2\end{pmatrix} - \left[\begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & \phi\end{pmatrix} P_{j-1}\begin{pmatrix}1\\ 1\\ \phi\end{pmatrix} + \begin{pmatrix}0\\ 0\\ b^2\end{pmatrix}\right]\cdot$$
$$\left[\begin{pmatrix}1 & 1 & \phi\end{pmatrix} P_{j-1}\begin{pmatrix}1\\ 1\\ \phi\end{pmatrix} + b^2\right]^{\oplus}\left[\begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & \phi\end{pmatrix} P_{j-1}\begin{pmatrix}1\\ 1\\ \phi\end{pmatrix} + \begin{pmatrix}0\\ 0\\ b^2\end{pmatrix}\right]^{\top}$$

Solution to Problem 2, 2. *Claim:* $(X_1)_j = j(X_2)_{j-1}$ for $j \geq 1$.

Proof of Claim: Since $(X_1)_0 = 0$, it holds $(X_1)_1 = (X_1)_0 + (X_2)_0 = 1 \cdot (X_2)_0$. Suppose the statement is true for $j-1 \in \mathbb{N}$. Then, with using the equation $(X_2)_{j-1} = (X_2)_{j-2}$, it holds

$$(X_1)_j = (X_1)_{j-1} + (X_2)_{j-1} = (j-1)(X_2)_{j-2} + (X_2)_{j-1} = j(X_2)_{j-1}.$$

□

From this, one finds

$$Y_j = (X_1)_j + (X_3)_j = j(X_2)_{j-1} + \phi(X_3)_{j-1} + U_j,$$

such that the linear recursive equations read as

$$X_j = \underbrace{\begin{pmatrix}1 & 0\\ 0 & \phi\end{pmatrix}}_{=(a_1)_j} X_{j-1} + \underbrace{\begin{pmatrix}0\\ b\end{pmatrix}}_{=(b_1)_j}\varepsilon_j, \quad Y_j = \underbrace{\begin{pmatrix}j & \phi\end{pmatrix}}_{=(A_1)_j} X_{j-1} + \underbrace{b}_{=(B_1)_j}\varepsilon_j.$$

The initial values are similarly computed as before and given as

$$\hat{X}_0 = \mathbb{E}[X_0] + \operatorname{cov}(X_0, Y_0)\operatorname{cov}(Y_0)^{\oplus}(Y_0 - \mathbb{E}[Y_0]) = (0,0)^{\top},$$
$$P_0 = \operatorname{cov}(X_0) - \operatorname{cov}(X_0, Y_0)\operatorname{cov}(Y_0)^{\oplus}\operatorname{cov}(X_0, Y_0)^{\top} = \begin{pmatrix}a^2 & 0\\ 0 & 0\end{pmatrix}.$$

With Theorem 2.5 the discrete Kalman-Bucy filter equations are given as

$$\hat{X}_j = \underbrace{(a_0)_j}_{=0} + (a_1)_j \hat{X}_{j-1} + \underbrace{(a_2)_j}_{=0} Y_{j-1} + ((a_1)_j P_{j-1} (A_1)_j^\top + b \circ B) \cdot$$
$$((A_1)_j P_{j-1} (A_1)_j^\top + B \circ B)^\oplus (Y_j - \underbrace{(A_0)_j}_{=0} - (A_1)_j \hat{X}_{j-1} - \underbrace{(A_2)_j}_{=0} Y_{j-1}),$$
$$P_j = (a_1)_j P_{j-1} (a_1)_j^\top + b \circ b + ((a_1)_j P_{j-1} (A_1)_j^\top + b \circ B) \cdot$$
$$((A_1)_j P_{j-1} (A_1)_j^\top + B \circ B)^\oplus ((a_1)_j P_{j-1} (A_1)_j^\top + b \circ B)^\top.$$

By plugging in the determined values, the Kalman-Bucy filter equations are

$$\hat{X}_j = \begin{pmatrix} 1 & 0 \\ 0 & \phi \end{pmatrix} \hat{X}_{j-1} + \left[\begin{pmatrix} 1 & 0 \\ 0 & \phi \end{pmatrix} P_{j-1} \begin{pmatrix} j \\ \phi \end{pmatrix} + \begin{pmatrix} 0 \\ b^2 \end{pmatrix} \right] \cdot$$
$$\left[\begin{pmatrix} j & \phi \end{pmatrix} P_{j-1} \begin{pmatrix} j \\ \phi \end{pmatrix} + b^2 \right]^\oplus (Y_j - \begin{pmatrix} j & \phi \end{pmatrix} \hat{X}_{j-1}),$$
$$P_j = \begin{pmatrix} 1 & 0 \\ 0 & \phi \end{pmatrix} P_{j-1} \begin{pmatrix} 1 & 0 \\ 0 & \phi \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & b^2 \end{pmatrix} - \left[\begin{pmatrix} 1 & 0 \\ 0 & \phi \end{pmatrix} P_{j-1} \begin{pmatrix} j \\ \phi \end{pmatrix} + \begin{pmatrix} 0 \\ b^2 \end{pmatrix} \right] \cdot$$
$$\left[\begin{pmatrix} j & \phi \end{pmatrix} P_{j-1} \begin{pmatrix} j \\ \phi \end{pmatrix} + b^2 \right]^\oplus \left[\begin{pmatrix} 1 & 0 \\ 0 & \phi \end{pmatrix} P_{j-1} \begin{pmatrix} j \\ \phi \end{pmatrix} + \begin{pmatrix} 0 \\ b^2 \end{pmatrix} \right]^\top$$

Solution to Problem 2, 3. The Riccati equation for P_j can be solved explicitly. By calculating the first two or three values P_1, P_2, P_3 one can recognize a reoccurring scheme. Then one has to claim the formula and prove it by induction.

Claim: Define the recursive sequence

$$a_j := \frac{b^2}{\frac{b^2}{a_{j-1}} + (j - (j-1)\phi)^2}$$

for $j \geq 1$ with initial value $a_0 := a^2$. Then, for $j \geq 1$, it holds

$$P_j = a_j \begin{pmatrix} 1 & -j \\ -j & j^2 \end{pmatrix}.$$

Proof of Claim. For $j = 1$ the Riccati equation should read as

$$P_1 = \frac{b^2}{\frac{b^2}{a^2} + 1} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

3 Problem

3.1 Mean Square Error Formula

First, note that

$$\tilde{\theta}_0 = 0 \tag{57}$$
$$\tilde{\theta}_1 = \max\{0, \xi_1\} = \xi_1 \tag{58}$$
$$\tilde{\theta}_2 = \max\{\xi_1, \xi_2\} \tag{59}$$
$$\tilde{\theta}_3 = \max\{\max\{\xi_1, \xi_2\}, \xi_3\} = \max\{\xi_1, \xi_2, \xi_3\} \tag{60}$$

In general for $j \geq 1$

$$\tilde{\theta}_j = \max(\tilde{\theta}_{j-1}, \xi_j) = \max\{\xi_1, \ldots, \xi_j\} \tag{61}$$
$$= \theta \cdot \max\{U_1, \ldots, U_j\} \tag{62}$$

Using the independence of θ and U_j, we see that Q_j can be rewritten as:

$$Q_j = \mathbb{E}[(\theta - \xi_j)^2] = \mathbb{E}[(\theta - \theta \max\{U_1, \ldots U_j\})^2] \tag{63}$$
$$= \mathbb{E}[\theta^2] \cdot \mathbb{E}[(1 - \max\{U_1, \ldots U_j\})^2] \tag{64}$$

The first term is $\mathbb{E}[\theta^2] = \frac{1}{12} + \frac{1}{4} = \frac{4}{12} = \frac{1}{3}$, using basic formulas for mean and variance of $Uni[0, 1]$. The second term can be calculated manually by noticing that the maximum of independent uniform distributions has a nice distribution function:

$$\mathbb{P}(\max\{U_1, \ldots U_j\} < t) = t^j \tag{65}$$

meaning the density function for it (in the interval [0,1]) is

$$f(x) = j \cdot x^{j-1} \tag{66}$$

Now, we can calculate the expectation:

$$\int_0^1 (1-x)^2 \cdot jx^{j-1} \mathrm{d}x = \underbrace{\int_0^1 jx^{j-1} \mathrm{d}x}_{1} - \frac{2j}{j+1} \underbrace{\int_0^1 (j+1)x^j \mathrm{d}x}_{1} + \frac{j}{j+2} \underbrace{\int_0^1 (j+2)x^{j+1} \mathrm{d}x}_{1} \tag{67}$$

So, overall we get that

$$Q_j = \frac{1}{3} \cdot \left(1 - \frac{2j}{j+1} + \frac{j}{j+2}\right) \tag{68}$$

3.2 Rate of Convergence

One can see that this converges to 0, because

$$\lim_{j\to\infty} Q_j = \frac{1}{3} \cdot (1 - 2 + 1) = 0 \tag{69}$$

For the rate of convergence let's bring Q_j to a common denominator.

$$Q_j = \frac{1}{3} \cdot \frac{(j+1)(j+2) - 2j(j+2) + j(j+1)}{(j+1)(j+2)} \tag{70}$$

$$= \frac{1}{3} \cdot \frac{j^2 + 3j + 2 - 2j^2 - 4j + j^2 + j}{(j+1)(j+2)} \tag{71}$$

$$= \frac{1}{3} \cdot \frac{2}{j^2 + 3j + 2} \tag{72}$$

We can see that if we multiply Q_j by $r(j) = j^2$, then

$$\lim_{j\to\infty} Q_j r(j) = \lim_{j\to\infty} \frac{1}{3} \cdot \frac{2j^2}{j^2 + 3j + 2} \tag{73}$$

$$= \lim_{j\to\infty} \frac{1}{3} \cdot \frac{2}{1 + 3j^{-1} + 2j^{-2}} = \frac{2}{3} \tag{74}$$

The limit is finite and positive, so the rate of convergence is $r(j) = j^2$.

3.3 Optimal Estimate

To find the optimal estimate, we use the Bayes formula for the conditional expectation of θ given the vector $(\xi_1, \dots, \xi_j)$ (Corollary 3.5 in [Chigansky, 2005]). It can be used because $\varphi(\theta) = \theta$ is between 0 and 1 a.s., so it is finite. "$P_X(\mathrm{d}u)$" is just $f_\theta(x)\mathrm{d}x$ and r is the conditional density function of the ξ's, given $\theta = x$.

$$\mathbb{E}[\theta \mid \mathcal{F}_j^\xi] = \int_{\mathbb{R}} s \frac{f_{\xi_1,\dots,\xi_j|\theta}(\xi_1,\dots,\xi_j; s) f_\theta(s)}{\int_{\mathbb{R}} f_{\xi_1,\dots,\xi_j|\theta}(\xi_1,\dots,\xi_j; x) f_\theta(x)\mathrm{d}x} \mathrm{d}s \tag{75}$$

- θ is just $Uni[0,1]$, so the density $f_\theta(s)$ is just 1, with support on $[0,1]$.
- If $\theta = s$, then the ξ_i values are just independent random variables on $Uni[0,s]$. Individually, they are $f_{\xi;\theta}(x; s) = \mathbb{1}\{x \in [0,s]\}s^{-1}$, so after multiplying them together we get:

$$f_{\xi_1,\dots,\xi_j|\theta}(\xi_1,\dots,\xi_j; s) = s^{-j} \prod_{i=1}^{j} \mathbb{1}\{\xi_i \in [0,s]\} \tag{76}$$

Combined, our equation is

$$\mathbb{E}[\theta \mid \mathcal{F}_j^\xi] = \int_0^1 \frac{s^{-j+1} \prod_{i=1}^j \mathbb{1}\{\xi_i \in [0,s]\}}{\int_0^1 x^{-j} \prod_{i=1}^j \mathbb{1}\{\xi_i \in [0,x]\}\mathrm{d}x} \mathrm{d}s \tag{77}$$

If any of the ξ_i are larger than s, then the product of indicators becomes 0. So, it has support when all ξ_i are less than s, in other words, the maximum of them is less than s. Let us denote $\xi_{max}^j = \max\{\xi_1, \dots, \xi_j\}$, then we only need to look at s greater than the maximum, but less than 1:

$$\mathbb{E}[\theta \mid \mathcal{F}_j^\xi] = \frac{\int_{\xi_{max}^j}^1 s^{-j+1}\mathrm{d}s}{\int_{\xi_{max}^j}^1 x^{-j}\mathrm{d}x} \tag{78}$$

If $j = 1$, then this is equal to

$$\mathbb{E}[\theta \mid \mathcal{F}_1^\xi] = \frac{\int_{\xi_1}^1 1\mathrm{d}s}{\int_{\xi_1}^1 x^{-1}\mathrm{d}x} = \frac{1 - \xi_1}{-\log(\xi_1)} \tag{79}$$

If $j = 2$, then this is equal to

$$\mathbb{E}[\theta \mid \mathcal{F}_2^\xi] = \frac{\int_{\xi_{max}^2}^1 s^{-1}\mathrm{d}s}{\int_{\xi_{max}^2}^1 x^{-2}\mathrm{d}x} = \frac{-\log(\xi_{max}^2)}{-1(1 - (\xi_{max}^2)^{-1})} \tag{80}$$

If $j \geq 3$, this type of integral can be calculated as follows, because $n \neq -1$

$$\int_a^b x^n \mathrm{d}x = \left[\frac{x^{n+1}}{n+1}\right]_a^b = \frac{b^{n+1} - a^{n+1}}{n+1} \tag{81}$$

So,

$$\tilde{\theta} = \mathbb{E}[\theta \mid \mathcal{F}_j^\xi] = \frac{1-j}{2-j} \cdot \frac{1 - (\xi_{max}^j)^{2-j}}{1 - (\xi_{max}^j)^{1-j}} \tag{82}$$