UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT4790/9790 — Stochastic Filtering

Day of examination: Monday 20, November 2023

Examination hours: 9:00 AM – 13:00 AM

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Discrete time linear filtering.

a (weight 10p)

Describe the setup for the Kalman-Bucy filter and state the main result (No proof).

b (weight 10p)

Let $X \in \mathbb{R}^n$ and $Y \in \mathbb{R}^m$ be random vectors. Prove that

$$\widehat{\mathbb{E}}\left[X | \mathcal{L}^Y \right] = \mathbb{E}\left[X\right] + \operatorname{cov}\left(X, Y\right) \operatorname{cov}\left(Y\right)^{\oplus} \left(Y - \mathbb{E}\left[Y\right]\right).$$

Hint: You can use without proof the orthogonal projection theorem and you may assume that $\operatorname{cov}(Y)$ is non singular.

c (weight 10p)

Find the recursions for $\widehat{X}_j$, the best linear prediction of X_j given $\mathcal{L}_j^Y := \overline{\operatorname{span}}\{1, Y_1, ..., Y_j\}$, and its error $P_j := \mathbb{E}\left[\left(X_j - \widehat{X}_j\right)^2\right]$ when

$$\begin{aligned} X_j &= aX_{j-1} + bY_j + \varepsilon_j, \qquad j \geq 1 \\ Y_j &= cX_{j-1} + \xi_j, \qquad j \geq 1, \end{aligned}$$

where $\{\varepsilon\}_{j\geq 1}$ and $\{\xi\}_{j\geq 1}$ are two independent sequences of orthogonal real valued white noises and $X_0 = Y_0 = 0$.

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Problem 2

Discrete time nonlinear filtering.

a (weight 10p)

Let $(\Omega, \mathcal{F}, P)$ be a probability space carrying a random variable X and let $\mathcal{G}$ be a sub σ-algebra of $\mathcal{F}$. Assume that there exists a regular conditional probability measure $P(d\omega | X = x)$ on $\mathcal{G}$ and it has Radon-Nikodym density $\rho(\omega, x)$ with respect to a σ-finite measure λ (on $\mathcal{G}$):

$$P(d\omega | X = x) = \int_B \rho(\omega, x)\, \lambda(d\omega).$$

Prove that, then, for every $\varphi : \mathbb{R} \to \mathbb{R}$, such that $\mathbb{E}[|\varphi(X)|] < \infty$ we have that

$$\mathbb{E}[\varphi(X) | \mathcal{G}] = \frac{\int_{\mathbb{R}} \varphi(u)\, \rho(\omega, u)\, P_X(du)}{\int_{\mathbb{R}} \rho(\omega, u)\, P_X(du)},$$

where P_X is the law of X.

b (weight 10p)

Describe the setup for finding the nonlinear filter via the Bayes formula and state the main result.

c (weight 10p)

Let $X_j = \varepsilon_j / \varphi(X_{j-1})$ and $Y_j = X_j + \xi_j$ where φ is a function bounded away from 0 ($\varphi(x) \geq c > 0$). The sequences $(\varepsilon_j)_{j \geq 1}$ and $(\xi_j)_{j \geq 1}$ are independent of each other and i.i.d.. Moreover, ε_1 and ξ_1 have densities p and q with respect to the Lebesgue measure (and enough integrability of its moments). Find a recursion for the random measure $\pi_j(dx)$ satisfying

$$\int_{\mathbb{R}} f(x)\, \pi_j(dx) = \mathbb{E}\left[f(X_j) | \mathcal{F}_j^Y \right].$$

Problem 3

Continuous time linear filtering.

Suppose we want to estimate the value of a constant parameter θ, based on observations Y_t satisfying

$$dY_t = \theta M_t dt + N_t dB_t,$$

where M_t and N_t are known deterministic functions and B is a Brownian motion.

(Continued on page 3.)

a (weight 10p)

Find the equations for $\widehat{\theta}$, the best linear estimator of θ using the values of the process Y up to time t, and $P_t = \mathbb{E}\left[\left(\theta - \widehat{\theta}_t\right)^2\right]$.

b (weight 10p)

Check that $P_t = \left(P_0^{-1} + \int_0^t M_s^2 N_s^{-2} ds\right)^{-1}$ solves the equation for P_t and prove that

$$\widehat{\theta}_t = \frac{\widehat{\theta}_0 P_0^{-1} + \int_0^t M_s N_s^{-2} dY_s}{P_0^{-1} + \int_0^t M_s^2 N_s^{-2} ds}.$$

Problem 4

Continuous time nonlinear filtering.

a (weight 10p)

Describe the general setup for the nonlinear filtering problem in continuous time and the main result (No proof).

b (weight 10p)

State and sketch the proof of Zakai equation. By sketching I mean that there is no need to justify some applications of dominated convergence in the proof.