Exercise 1.c Find $\hat{X}_j$ and a_j for.

$$\begin{cases} X_j = a X_{j-1} + b Y_j + \varepsilon_j , & j \geq 1 \\ Y_j = c X_{j-1} + \xi_j \end{cases}$$

We can rewrite the previous system as

$$X_j = a X_{j-1} + bc X_{j-1} + \varepsilon_j + b \xi_j$$
$$= (a+bc) X_{j-1} + \varepsilon_j + b \xi_j$$
$$Y_j = c X_{j-1} + \xi_j$$

then

$a_0 = 0, \; a_1 = a+bc, \; a_2 = 0, \; b_1 = 1, \; b_2 = b$

$A_0 = 0, \; A_1 = c, \; A_2 = 0, \; B_1 = 0, \; B_2 = 1$

and we have that

$$b \circ b = 1 + b^2 , \quad b \circ B = b , \quad B \circ B = 1$$

$$a_1 P_{j-1} A_1^T = (a+bc) P_{j-1} c = c(a+bc) P_{j-1}$$

$$A_1 P_{j-1} A_1^T = c^2 P_{j-1}$$

$$a_1 P_{j-1} a_1^T = (a+bc)^2 P_{j-1}$$

therefore

$$\hat{X}_j = (a+bc) \hat{X}_{j-1}$$
$$+ \left(c(a+bc) P_{j-1} + b \right) \left(c^2 P_{j-1} + 1 \right)^{-1} \left(Y_j - c \hat{X}_{j-1} \right)$$

$$P_j = (a+bc)^2 P_{j-1} + 1 + b^2 - \left(c(a+bc) P_{j-1} + b \right)^2 \left(c^2 P_{j-1} + 1 \right)^{-1}$$

$$\hat{X}_0 = E[X_0] + \mathrm{cov}(X_0, Y_0)\,\mathrm{cov}(Y_0)^{(+)}\,(Y_0 - E[Y_0]) = 0$$

$$P_0 = \mathrm{cov}(X_0) - \mathrm{cov}(X_0, Y_0)\,\mathrm{cov}(Y_0)^{(+)}\,\mathrm{cov}(X_0, Y_0)^T = 0$$

Exercise 2.c

We have that

$$\begin{cases} X_j = \varepsilon_j / \xi(X_{j-1}) \\ Y_j = X_j + \zeta_j \end{cases}$$

$\xi(x) \geq c > 0.$

ζ and ε i.i.d.

$$P(X_j \in B / \mathcal{F}^X_{j-1} \vee \mathcal{F}^Y_{j-1}) = P(\varepsilon_j / \xi(X_{j-1}) \in B / \mathcal{F}^X_{j-1} \vee \mathcal{F}^Y_{j-1})$$

$$= E[\mathbb{1}_{\{\varepsilon_j / \xi(X_{j-1}) \in B\}} / \mathcal{F}^X_{j-1} \vee \mathcal{F}^Y_{j-1}]$$

$$= E[\mathbb{1}_{\{\varepsilon_j / \xi(a) \in B\}}] \Big|_{a = X_{j-1}}$$

$$= P(\varepsilon_j / \xi(a) \in B) \Big|_{a = X_{j-1}} = (*)$$

Let $Z_j := \varepsilon_j / \xi(a)$, $P(Z_j \leq z) = P(\varepsilon_j \leq z\,\xi(a))$

$$= \int_{-\infty}^{z\xi(a)} p(u)\,du \Rightarrow f_{Z_j}(z) = \frac{d}{dz} P(Z_j \leq z)$$

$$= p(z\,\xi(a))\,\xi(a)$$

$$(*) = \int_B p(u\,\xi(X_{j-1}))\,\xi(X_{j-1})\,du$$

$$\Rightarrow \Lambda(X_{j-1}, du) = p(u\,\xi(X_{j-1}))\,\xi(X_{j-1})\,du$$

$$P(Y_j \mid \mathcal{F}_j^X \vee \mathcal{F}_{j-1}^Y) = P(X_j + \xi_j \in B \mid \mathcal{F}_j^X \vee \mathcal{F}_{j-1}^Y)$$

$$= E[\mathbb{1}_{\{X_j+\xi_j \in B\}} \mid \mathcal{F}_j^X \vee \mathcal{F}_{j-1}^Y] = E[\mathbb{1}_{\{a+\xi_j \in B\}}]\big|_{a = X_j}$$

$$= P(a+\xi_j \in B)\big|_{a=X_j} = (**)$$

Let $W_j := a + \xi_j$

$$P(W_j \leq z) = P(a+\xi_j \leq z) = P(\xi_j \leq z-a)$$

$$= \int_{-\infty}^{z-a} q(u)\,du$$

$$\Rightarrow f_{W_j}(z) = \frac{d}{dz} P(W_j \leq z) = q(z-a)$$

$$(**) = \int_B q(u - X_j)\,du =: \int_B \Gamma(X_j, du)$$

$$\Rightarrow \gamma(x,u) = q(u-x) \qquad \nu(du) = du$$

then by the previous result

$$\pi_j(dx) = \frac{\int_{\mathbb{R}} \gamma(x, Y_j)\,\Lambda(u,dx)\,\pi_{j-1}(du)}{\int_{\mathbb{R}}\int_{\mathbb{R}} \gamma(x,Y_j)\,\Lambda(u,dx)\,\pi_{j-1}(du)}$$

$$= \frac{\int_{\mathbb{R}} q(Y_j - x)\,p(x\,\xi(u))\,\xi(u)\,dx\,\pi_{j-1}(du)}{\int_{\mathbb{R}}\int_{\mathbb{R}} q(Y_j - x)\,p(x\,\xi(u))\,\xi(u)\,dx\,\pi_{j-1}(du)}$$

$$= \frac{\int_{\mathbb{R}} q(Y_j - x)\, p(x\,\xi(u))\, \xi(u)\, \pi_{j-1}(du)}{\iint_{\mathbb{R}\,\mathbb{R}} q(Y_j - x)\, p(x\,\xi(u))\, \xi(u)\, \pi_{j-1}(du)}\, dx$$

$$=: f(Y_j, x)\, dx$$

Exercise 3.

θ random parameter and observations given by

$$dY_t = \theta M_t \, dt + N_t \, dW_t$$

M, N deterministic and W a Brownian motion.

a) Find equations for $\hat{\theta} = \hat{E}[\theta \mid \mathcal{F}^Y_t]$ and $P_t = E[(\theta - \hat{\theta})^2]$.

b) Prove that

$$\hat{\theta}_t = \frac{\hat{\theta}_0 P_0^{-1} + \int_0^t M_s N_s^{-2} \, dY_s}{P_0^{-1} + \int_0^t M_s^2 N_s^{-2} \, ds}$$

a) The equations for the signal and observation process are

$$d\theta_t = 0$$

$$dY_t = \theta_t M_t \, dt + N_t \, dB_t$$

This yields

$$a_t \equiv 0, \quad b_t \equiv 0 \qquad A_t \equiv M_t \quad \& \quad B_t \equiv N_t$$

and using the Kalman-Bucy filter formula

$$d\hat{X}_t = a_t \hat{X}_t \, dt + P_t \frac{A_t}{B_t^2} (dY_t - A_t \hat{X}_t \, dt)$$

$$= P_t \frac{M_t}{N_t^2} (dY_t - M_t \hat{X}_t \, dt) \quad \left(\text{substitute } \hat{X}_t \text{ by } \hat{\theta}_t \right)$$

$$\dot{P}_t = 2 a_t P_t + b_t^2 - \frac{A_t^2 P_t^2}{B_t^2}$$

$$= - \frac{M_t^2}{N_t^2} P_t^2$$

b) Let's check that $P_t = \left(P_0^{-2} + \int_0^t M_s^2 N_s^{-2} ds \right)^{-2}$

$$\frac{d}{dt} P_t = -\left(P_0^{-2} + \int_0^t M_s^2 N_s^{-2} ds \right)^{-2} \frac{M_t^2}{N_t^2} dt$$

$$= -\left(\frac{M_t}{N_t} P_t \right)^2 dt \qquad \checkmark$$

Hence,

$$d\hat{\theta}_t = \left(P_0^{-2} + \int_0^t M_s^2 N_s^{-2} ds \right)^{-2} \frac{M_t}{N_t^2} \left(dY_t - M_t \hat{X}_t dt \right)$$

We can rewrite the previous equation as

$$\left(P_0^{-2} + \int_0^t M_s^2 N_s^{-2} ds \right) d\hat{\theta}_t = \frac{M_t}{N_t^2} dY_t - \frac{M_t^2}{N_t^2} \hat{\theta}_t dt$$

$$\Updownarrow$$

$$\left(P_0^{-2} + \int_0^t M_s^2 N_s^{-2} ds \right) d\hat{\theta}_t + \frac{M_t^2}{N_t^2} \hat{\theta}_t dt = \frac{M_t}{N_t^2} dY_t$$

But the left hand side of the previous equation coincides with

$$d\left(\left(P_0^{-2} + \int_0^t M_s^2 N_s^{-2} ds \right) \hat{\theta}_t \right) \qquad \left(\text{or } d\left(P_t^{-2} \hat{\theta}_t \right) \right)$$

Let's check it. First,

$$d P_t^{-2} = -\frac{1}{P_t^2} dP_t = -\frac{1}{P_t^2} \left(-\frac{M_t^2}{N_t^2} P_t^2 dt \right)$$

$$= M_t^2 N_t^{-2} dt$$

then

$$d\left(P_t^{-2}\hat{\theta}_t\right) = P_t^{-2}\, d\hat{\theta}_t + \hat{\theta}_t\, dP_t^{-2} + \cancel{d\langle \hat{\theta}, P^{-2}\rangle_t}^{=0}$$

P^{-2} continuous and finite variation

$$= \left(P_0^{-2} + \int_0^t M_s^2 N_s^{-2}\, ds\right) d\hat{\theta}_t + \hat{\theta}_t M_t^2 N_t^{-2}\, dt \quad \checkmark$$

Integrating

$$d\left(\left(P_0^{-2} + \int_0^t M_s^2 N_s^{-2}\, ds\right)\hat{\theta}_t\right) = \frac{M_t}{N_t^2}\, dY_t$$

we get

$$\left(P_0^{-2} + \int_0^t M_s^2 N_s^{-2}\, ds\right)\hat{\theta}_t = \left(P_0^{-2} + \int_0^0 M_s^2 N_s^{-2} ds\right)\hat{\theta}_0 + \int_0^t M_s N_s^{-2}\, dY_s$$

$$= P_0^{-2}\hat{\theta}_0 + \int_0^t M_s N_s^{-2}\, dY_s$$

which yields

$$\hat{\theta}_t = \frac{P_0^{-2}\hat{\theta}_0 + \int_0^t M_s N_s^{-2}\, dY_s}{P_0^{-2} + \int_0^t M_s^2 N_s^{-2}\, ds}$$