Complex Analysis

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CHAPTER 1

Fundamental Properties of Holomorphic Functions

1. Basic definitions

DEFINITION 1.1. Let $\Omega \subset \mathbb{C}$ be an open set, and let $f = u + iv \in \mathcal{C}^1(\Omega)$, where u, v are real functions. We say that f is *holomorphic* on Ω is for any point $a \in \Omega$ we have that

$$\frac{\partial u}{\partial x}(a) = \frac{\partial v}{\partial y}(a) \text{ and } \frac{\partial u}{\partial y}(a) = -\frac{\partial v}{\partial x}(a). \tag{1.1}$$

The equations (1.1) are called the Cauchy-Riemann equations. We define the following differential operators:

DEFINITION 1.2. Let $\Omega \subset \mathbb{C}$ be an open set, and let $f = u + iv$ be differentiable at every point of Ω. We set

$$\frac{\partial f}{\partial z}(a) := \frac{1}{2}(\frac{\partial f}{\partial x}(a) - i\frac{\partial f}{\partial y}(a)) \text{ and } \frac{\partial f}{\partial \overline{z}}(a) := \frac{1}{2}(\frac{\partial f}{\partial x}(a) + i\frac{\partial f}{\partial y}(a)) \tag{1.2}$$

We see that the condition that $\frac{\partial f}{\partial \overline{z}}(a) = 0$ is satisfied is equivalent to the conditions (1.1) being satisfied.

LEMMA 1.3. *Let $f \in \mathcal{C}^1(\Omega)$ and let $a \in \Omega$. Then*

$$f(z) = f(a) + \frac{\partial f}{\partial z}(a) \cdot (z - a) + \frac{\partial f}{\partial \overline{z}}(a) \cdot \overline{(z - a)} + O(|z|^2). \tag{1.3}$$

PROOF. This follows from Taylor's Theorem for maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ writing it on complex form. □

Using this it is not hard to see that a $\mathcal{C}^1$-smooth function f on Ω is holomorphic if and only if the limit

$$\lim_{\delta \to 0} \frac{f(a + \delta) - f(a)}{\delta} \tag{1.4}$$

exists for all $a \in \Omega$.

If a function f is holomorphic on an open set Ω the expression $\frac{\partial f}{\partial z}(a)$ is called the (complex) derivative of f at $a \in \Omega$, and we denote this also by $f'(a)$. We denote the set of holomorphic functions on Ω by $\mathcal{O}(\Omega)$, and we note that $\mathcal{O}(\Omega)$ is an algebra, *i.e.*, if $f, g \in \mathcal{O}(\Omega)$ then $f + g, f - g, f \cdot g$ are holomorphic on Ω. If f is non-zero on Ω then $1/f$ is holomorphic on Ω. Moreover, the usual rules of differentiation hold: $(f + g)' = f' + g', (f \cdot g)' =$

$f \cdot g' + f' \cdot g, (1/g)' = -g'/g^2$. If $f \in \mathcal{O}(\Omega_1)$ and $g \in \mathcal{O}(\Omega_2)$ and $f(a) \in \Omega_2$, then the composition $g \circ f$ is holomorphic at a, and $(g \circ f)' = g'(f(a)) \cdot f'(a)$.

EXAMPLE 1.4. A polynomial $P(z) = a_n \cdot z^n + \cdots + a_1 \cdot z + a_0$ is holomorphic.

2. Integration and Integral formulas

We will start by proving the fundamental result that if $f \in \mathcal{C}^1(\overline{\mathbb{D}})$ and if f is holomorphic on $\mathbb{D}$ then

$$f(a) = \frac{1}{2\pi i} \int_{b\mathbb{D}} \frac{f(a)}{z-a} dz, \tag{2.1}$$

for all $a \in \mathbb{D}$. Actually we will prove the more general result that if Ω is a bounded $\mathcal{C}^1$-smooth domain and if $f \in \mathcal{C}^1(\overline{\Omega})$, then

$$f(a) = \frac{1}{2\pi i} \int_{b\Omega} \frac{f(z)}{z-a} dz - \frac{1}{\pi} \int\int_{\Omega} \frac{\frac{\partial f}{\partial \overline{z}}(z)}{z-a} dxdy. \tag{2.2}$$

DEFINITION 2.1. Let $f, g \in \mathcal{C}^k(\Omega)$. We call the expression $\omega(z) = f(z) \cdot dx + g(z) \cdot dy$ a *differentiable 1-form* of class $\mathcal{C}^k$. We denote the vector space of differentiable 1-forms on Ω by $\mathcal{E}^1(\Omega)$.

DEFINITION 2.2. We set

$$dz := dx + i \cdot dy \text{ and } d\overline{z} := dx - i \cdot dy. \tag{2.3}$$

DEFINITION 2.3. Let $\omega \in \mathcal{E}^1(\Omega)$ be a continuous 1-form, and let $\gamma : [0,1] \to \Omega$ be a $\mathcal{C}^1$-smooth map. We set

$$\int_\gamma \omega := \int_0^1 f(\gamma(t)) \cdot \gamma_1'(t) + g(\gamma(t)) \cdot \gamma_2'(t) dt. \tag{2.4}$$

PROPOSITION 2.4. *Let $\gamma, \sigma : [0,1] \to \Omega$ be two parametrizations of the same curve,* i.e., *$\gamma = \sigma \circ \phi$ where $\phi : [0,1] \to [0,1]$ is a strictly increasing $\mathcal{C}^1$-smooth function, $\phi(0) = 0, \phi(1) = 1$. Then for $\omega \in \mathcal{E}^1(\Omega)$ continuous, we have that*

$$\int_\gamma \omega = \int_\sigma \omega. \tag{2.5}$$

PROOF. We have that

$$\begin{aligned}
\int_0^1 f(\gamma(t)) \cdot \gamma_1'(t) dt &= \int_0^1 f(\sigma \circ \phi(t)) \cdot (\sigma_1 \circ \phi)'(t) dt \\
&= \int_0^1 f(\sigma \circ \phi(t)) \cdot \sigma_1'(\phi(t)) \cdot \phi'(t) dt \\
&= \int_0^1 f(\sigma(t)) \cdot \sigma_1'(t) dt,
\end{aligned}$$

where the last equation follows from the change of variables formula in one variable, and the same holds if you exchange f by g and γ_1, σ_1 by γ_2, σ_2. $\square$

This allows us to define integration of 1-forms on *oriented* curves in $\mathbb{C}$, and furthermore on the boundaries of (piecewise) $\mathcal{C}^1$-smooth domains in $\mathbb{C}$, as long as we orient the boundary components. The following theorem is fundamental, and is the basic ingredient to prove (2.2):

THEOREM 2.5. *(Stokes) Let Ω be a bounded (piecewise) $\mathcal{C}^1$-smooth domain in $\mathbb{C}$ and let $\omega = f dx + g dy \in \mathcal{E}^1(\overline{\Omega})$ be of class $\mathcal{C}^1$. Then*

$$\int_{b\Omega} \omega = \int_{\Omega} (\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}) dxdy. \tag{2.6}$$

PROOF. We give a complete proof only when Ω is the square $[0,1]\times[0,1]$.

$$\begin{aligned}\int_{\Omega}(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y})dxdy &= \int_0^1 (\int_0^1 \frac{\partial g}{\partial x}(x,y)dx)dy - \int_0^1 (\int_0^1 \frac{\partial f}{\partial y}(x,y)dy)dx \\ &= \int_0^1 g(1,y) - g(0,y)dy - \int_0^1 f(x,1) - f(x,0)dx \\ &= \int_{b\Omega} \omega.\end{aligned}$$

□

Note that if ω is a 1-form $\omega(z) = f(z)\cdot dz$, then (2.6) reads

$$\int_{b\Omega} \omega = 2i \cdot \int\int_{\Omega} \frac{\partial f}{\partial \overline{z}} dxdy \tag{2.7}$$

In particular we get the following result:

PROPOSITION 2.6. *Let $\Omega \subset \mathbb{C}$ be a bounded $\mathcal{C}^1$-smooth domain and let $\omega \in \mathcal{E}^1(\overline{\Omega})$ be $\mathcal{C}^1$-smooth* holomorphic *on Ω,* i.e., *$\omega(z) = f(z)\cdot dz$, $f \in \mathcal{O}(\Omega)$. Then*

$$\int_{b\Omega} \omega = 0. \tag{2.8}$$

THEOREM 2.7. *(Genrealized Cauchy Integral Formula) Let $\Omega \subset \mathbb{C}$ be a bounded $\mathcal{C}^1$-smooth domain and let $f \in \mathcal{C}^1(\overline{\Omega})$. Then for each $a \in \Omega$ we have that*

$$f(a) = \frac{1}{2\pi i}\int_{b\Omega} \frac{f(z)}{z-a} dz - \frac{1}{\pi}\int\int_{\Omega} \frac{\frac{\partial f}{\partial \overline{z}}(z)}{z-a} dxdy. \tag{2.9}$$

PROOF. We prove first that the last integral is well defined:

LEMMA 2.8. $\int\int_{\mathbb{D}^*} \frac{1}{|z|} = 2\pi$.

PROOF. $\int\int_{\mathbb{D}^*} \frac{1}{|z|} = \int_0^1 \int_0^{2\pi} \frac{1}{|re^{it}|} \cdot r dt d\theta = 2\pi$. □

Now for ϵ small enough such that $\mathbb{D}_\epsilon(a) \subset\subset \Omega$ we set $\Omega_\epsilon = \Omega \setminus \overline{\mathbb{D}}_\epsilon(a)$. Stoke's Theorem gives us that

$$\frac{1}{2\pi i}\int_{b\Omega_\epsilon} \frac{f(z)}{z-a} dz = \frac{1}{\pi}\int\int_{\Omega_\epsilon} \frac{\frac{\partial f}{\partial \overline{z}}}{z-a} dxdy, \tag{2.10}$$

and furthermore

$$\int_{b\Omega_\epsilon} \frac{f(z)}{z-a} dz = \int_{b\Omega} \frac{f(z)}{z-a} dz - \int_{b\mathbb{D}_\epsilon(a)} \frac{f(z)}{z-a} dz, \tag{2.11}$$

and we see that the last rightmost integral approaches $2\pi i f(a)$ uniformly as $\epsilon \to 0$. □

3. Some consequences of the integral formulas

PROPOSITION 3.1. *Let $\Omega \subset \mathbb{C}$ be a bounded $\mathcal{C}^1$-smooth domain and let $f \in \mathcal{C}(b\Omega)$. Then the function*

$$\tilde{f}(\zeta) = \frac{1}{2\pi i} \int_{b\Omega} \frac{f(z)}{z-\zeta} dz \tag{3.1}$$

is holomorphic on Ω. Moreover, $\tilde{f}$ is $\mathcal{C}^\infty$-smooth, $\tilde{f}'$ is holomorphic, and we have that

$$\tilde{f}^{(k)}(\zeta) = \frac{k!}{2\pi i} \int_{b\Omega} \frac{f(z)}{(z-\zeta)^{k+1}} dz \tag{3.2}$$

PROOF. This follows by differentiating under the integral sign. □

PROPOSITION 3.2. *Let $f_j \in \mathcal{O}(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$ for $j \in \mathbb{N}$, and assume that $f_j \to f$ uniformly on $\overline{\mathbb{D}}$ as $j \to \infty$. Then $f \in \mathcal{O}(\mathbb{D})$, and $f_j^{(k)} \to f^{(k)}$ uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$.*

PROOF. Note that f is given by a integral formula as in the previous proposition. □

DEFINITION 3.3. We say that a function f on $\mathbb{D}_r(0)$ is *analytic* if $f(z) = \sum_{j=0}^\infty c_j \cdot z^j$ for all $z \in \mathbb{D}_r$.

PROPOSITION 3.4. *If f is analytic on $\mathbb{D}_r$ then $f \in \mathcal{O}(\mathbb{D}_r)$.*

PROOF. Fix a $0 < t < s < r$, and note that there exists $M > 0$ such that $|c_j \cdot s^j| < M$ for all $j \in \mathbb{N}$. Then for all $z \in \overline{\mathbb{D}}_t$ we have that

$$|\sum_{j=N}^\infty c_j \cdot z^j| \leq \sum_{j=N}^\infty |c_j \cdot s^j| \cdot (\frac{t}{s})^j \leq M \sum_{j=N}^\infty (\frac{t}{s})^j. \tag{3.3}$$

By the convergence of geometric series, this shows that f is the limit of a sequence of polynomials on $\overline{\mathbb{D}}_t$ for all $t < r$, hence f is holomorphic on $\mathbb{D}_r$ by Proposition 3.4 □

PROPOSITION 3.5. *(Cauchy Estimates) Let $f \in \mathcal{O}(\mathbb{D}_r) \cap \mathcal{C}(\overline{\mathbb{D}}_r)$. Then*

$$|f^{(k)}(0)| \leq \frac{k! \cdot \|f\|_{b\mathbb{D}_r}}{r^k}. \tag{3.4}$$

PROOF. By (3.2) we have that

$$\begin{aligned}|f^{(k)}(0)| &\leq \frac{k!}{2\pi}\left|\int_{b\mathbb{D}_r}\frac{f(z)}{z^{k+1}}dz\right| \\ &= \frac{k!}{2\pi}\left|\int_0^{2\pi}\frac{f(re^{it})}{(re^{it})^{k+1}}ire^{it}dt\right| \\ &\leq \frac{k!\cdot\|f\|_{b\mathbb{D}_r}}{r^k}.\end{aligned}$$

□

COROLLARY 3.6. *(Simple Maximum principle for a disk) Let* $f \in \mathcal{O}(\mathbb{D}_r)\cap \mathcal{C}(\overline{\mathbb{D}}_r)$. *Then* $|f(0)| \leq \|f\|_{b\mathbb{D}_r}$.

THEOREM 3.7. *(Montel) Let* $\Omega \subset \mathbb{C}$ *be an open set, and* $\mathcal{F}$ *be a family of holomorphic functions on* Ω *with the property that for each compact set* $K \subset \Omega$ *there exists a constant* $C_K > 0$ *such that* $\|f\|_K \leq C_K$ *for all* $f \in \mathcal{F}$. *Then for any sequence* $\{f_j\}_{j\in\mathbb{N}} \subset \mathcal{F}$ *there exists a subsequence* $\{f_{n(j)}\}$ *such that* $f_{n(j)} \to f \in \mathcal{O}(\Omega)$ *uniformly on compact subsets of* Ω.

PROOF. Let $A \subset \Omega$ be a dense sequence of points, and let $\{f_j\} \subset \mathcal{F}$ be a sequence such that $f_j(a) \to \tilde{a} \in \mathcal{C}$ for all $a \in A$. We claim that the sequence $\{f_j\}$ converges to a holomorphic function f uniformly on compact subsets of Ω. Choose an exhaustion of Ω by compact sets $K_j \subset K^\circ_{j+1}$. For any j we have that $\|f_i\|_{K_j} \leq M_j$ for all i. By the Cauchy estimates there is a constant N_j such that $\|f_i'\|_{K_j} < N_j$ for all i.

Now we fix K_j and show that $\{f_i\}|_{K_j}$ is a Cauchy sequence. Note that by the Mean Value Theorem we have for $z, z' \in K_{j+1}$ that $|f_i(z) - f_i(z')| \leq N_{j+1}|z - z'|$. Given any $\epsilon > 0$ we may choose a finite subset $\tilde{A} \subset K_{j+1}$ of A such that for any $z \in K_j$, there exists an $a \in \tilde{A}$ with $|z - a| < \frac{\epsilon}{4N_{j+1}}$. Furthermore, since $\{f_i\}|_{\tilde{A}}$ is Cauchy, we may find $N \in \mathbb{N}$ such that $|f_l(a) - f_m(a)| < \frac{\epsilon}{2}$ for all $m, n \geq N$. So given any $z \in K_j$ we may pick $a \in \tilde{A}$ to see that

$$\begin{aligned}|f_l(z) - f_m(z)| &\leq |f_l(z) - f_l(a)| + |f_l(a) - f_m(a)| + |f_m(a) - f_m(z)| \\ &\leq 2N_{j+2}|z - a| + \epsilon/2 < \epsilon,\end{aligned}$$

for all $l, m \geq N$, hence $\{f_i\}|_{K_j}$ is a Cauchy sequence. □

THEOREM 3.8. *Let* $f \in \mathcal{O}(\mathbb{D}_r)$. *Then we have that*

$$f(\zeta) = \sum_{j=0}^{\infty} c_j \cdot \zeta^j, \tag{3.5}$$

where

$$c_j = \frac{1}{2\pi i}\int_{b\mathbb{D}_r}\frac{f(z)}{z^{j+1}}dz. \tag{3.6}$$

Proof. Note that $\frac{1}{z-\zeta} = \frac{1}{z(1-\zeta/z)} = 1/z\sum_{j=0}^{\infty}(\frac{\zeta}{z})^j$ as long as $|\zeta| < |z|$, and plug this into Cauchy's Integral Formula. □

Proposition 3.9. *(Identity principle) Let* $f \in \mathcal{O}(\Omega)$. *If* $Z(f) = \{z \in \Omega : f(z) = 0\}$ *has non-empty interior, then* $f \equiv 0$ *on* Ω.

Proof. For each $a \in \Omega$ we have that $f(z) = \sum_{j=0}^{\infty} c_j(a)(z-a)^j$ on a small enough disk centered at a. By (3.6) we see that $c_j(a)$ is continuous in a for all j. So the set of points $\{a \in \Omega : c_j(a) = 0 \text{ for all } j \in \mathbb{N}\}$ is non-empty, open and closed in Ω. □

Proposition 3.10. *Let* $f \in \mathcal{O}(\Omega)$. *Then* $Z(f)$ *is discrete unless* f *is constantly equal to zero.*

Proof. We assume that f is not constant. Near a point $a \in \Omega$ with $f(0) = 0$ we have that $f(z) = \sum_{j=k}^{\infty} c_j(z-a)^j, k \geq 1, c_k \neq 0$, so we can write $f(z) = (z-a)^k(c_k + \sum_{j=1}^{\infty} c_{k+j}(z-a)^j)$. □

Theorem 3.11. *(Open Mapping Theorem) Let* $f \in \mathcal{O}(\mathbb{D})$ *be nonconstant. Then* $f(\mathbb{D})$ *is an open set.*

Proof. Assume that $f(0) = 0$ but that there are points $a_j \to 0$ such that $f(z) \neq a_j$ for all $j \in \mathbb{N}$. Set $g_j(z) := \frac{1}{f(z)-a_j}$. Choose $r > 0$ such that $f(z) \neq 0$ for all $|z| = r$. Then $|g_j|$ is uniformly bounded on bD_r but $g(0) \to \infty$ as $j \to 0$ which contradicts the simple maximum principle for a disk. □

Corollary 3.12. *(Maximum principle) Let* $\Omega \subset \mathbb{C}$ *be a domain, and let* $f \in \mathcal{O}(\Omega)$. *If* $|f(a)| = \sup_{z \in \Omega}\{|f(z)|\}, a \in \Omega$, *then* f *is constant.*

Proposition 3.13. *Let* $\Omega \subset \mathbb{C}$ *be a domain. Let* $f_j \in \mathcal{O}^*(\Omega)$ *for* $j = 1, 2, ...,$ *and assume that* $f_j \to f$ *uniformly on compact subsets of* Ω *as* $j \to \infty$. *Then either* $f \in \mathcal{O}^*(\Omega)$ *of* f *is constantly equal to zero.*

Proof. Same proof as for Theorem 3.11. □

Proposition 3.14. *Let* $\Omega \subset \mathbb{C}$ *be a bounded* $\mathcal{C}^1$*-smooth domain, let* $f \in \mathcal{O}(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$, *and assume that* $f(z) \neq 0$ *for all* $z \in b\Omega$. *Then*

$$2\pi i \sum_{a \in \Omega} ord_a(f) = \int_{b\Omega} \frac{f'(z)}{f(z)} dz. \tag{3.7}$$

Proof. Set $Z(f) = \{a_1, ..., a_m\}$, choose $\epsilon > 0$ such that the closure of the disks $D_\epsilon(a_j)$ are pairwise disjoint and contained in Ω. Then

$$\int_{b\Omega} \frac{f'(z)}{f(z)} dz = \sum_{j=1}^{m} \int_{bD_\epsilon(a_j)} \frac{f'(z)}{f(z)} dz. \tag{3.8}$$

For each j we may write $f(z) = (z-a_j)^{k(j)} \cdot g(z)$ with $g(a_j) \neq 0$, where $k(j) = ord_{a_j}(f)$. So $\frac{f'(z)}{f(z)} = \frac{k(j)}{(z-a_j)} + \frac{g'(z)}{g(z)}$, and so by possibly having to decrease ϵ we see that $\int_{bD_\epsilon(a_j)} \frac{f'(z)}{f(z)} dz = \int_{bD_\epsilon(a_j)} \frac{k(j)}{z-a_j} dz = 2\pi i k(j)$. □

THEOREM 3.15. *(Rouchet) Let $\Omega \subset \mathbb{C}$ be a bounded $\mathcal{C}^1$-smooth domain, and let $f \in \mathcal{O}(\Omega) \cap \mathcal{C}^1(\overline{\Omega}), f(z) \neq 0, z \in b\Omega$. If $g \in \mathcal{O}(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$ and if $|g(z)| < |f(z)|$ for all $z \in b\Omega$, then*

$$\sum_{z \in \Omega} ord_z(f) = \sum_{z \in \Omega} ord_z(f+g). \tag{3.9}$$

PROOF. Note that the functions $h_t := f + t \cdot g, t \in [0,1]$ are all nonzero on $b\Omega$ by the assumption. By (3.7) the function $\varphi(t) = \sum_{z \in \Omega} ord_z(h_t)$ is continuous on $[0,1]$ and integer valued, hence the result follows. □

PROPOSITION 3.16. *Let $\Omega \subset \mathbb{C}$ be open, let $f \in \mathcal{O}(\Omega)$, and assume that f is injective. Then $f'(z) \neq 0$ for all $z \in \Omega$.*

PROOF. Fix $a \in \Omega$. Without loss of generality we assume $a = f(a) = 0$ and write $f(z) = \sum_{j \geq k} c_j (z-a)^j, c_k \neq 0$. Choose $r > 0$ such that $Z(f) \cap \overline{D}_r = \{0\}$ and such that $Z(f') \cap \overline{D}_r$ is non-empty or the origin. For $|c| < \|f\|_{bD_r}$ we have that $\sum_{z \in D_r} ord_z(f-c) = k$. So there are points $a_1, ..., a_k \in D_r$ with $f(a_j) = c$, where the a_j's a priori are not necessarily distinct. But $f'(a_j) \neq 0$ for each j, so by the inverse function theorem f is injective near a_j for each j. So the a_j's are all distinct, hence $k = 1$.

□

PROPOSITION 3.17. *Let Ω be a domain, let $f_j \in \mathcal{O}(\Omega)$, and assume that $f_j \to f$ uniformly on compacts in Ω. If each f_j is injective, then either f is constant or f is injective.*

PROOF. Assume to get a contradiction that there are two distinct points $a_1, a_2 \in \Omega$ with $f(a_1) = f(a_2) = 0$. Choose a smoothly bounded domain $\tilde{\Omega} \subset\subset \Omega$ with $f(z) \neq 0$ for all $z \in \tilde{\Omega}$. If j is large enough we have that $\sum_{z \in \tilde{\Omega}} ord_z(f_j) = \sum_{z \in \Omega} ord_z(f) > 1$. □

PROPOSITION 3.18. *Let $\Omega \subset \mathbb{C}$ be open, let $f \in \mathcal{O}(\Omega)$, and assume that f is injective. Then $f'(z) \neq 0$ for all $z \in \Omega$.*

PROOF. Fix $a \in \Omega$. Without loss of generality we assume $f(a) = 0$ and write $f(z) = \sum_{j \geq k} c_j (z-a)^j, c_k \neq 0$. If $r > 0$ is small enough we have that $|\sum_{j \geq k+1} c_j (z-a)^j| < c_k (z-a)^k$, and so $ord_a(f) = ord_a((z-a)^k)$. Since f is injective we have $k = 1$, and $f'(z) = c_1 \neq 0$. □

PROPOSITION 3.19. *Let $\Omega \subset \mathbb{C}$, let $f \in \mathcal{O}(\Omega)$, and assume that f is injective. Then $f^{-1} \in \mathcal{O}(\Omega)$.*

PROOF. For any point $a \in \Omega$ it follows from the inverse mapping theorem that $df^{-1}(a) = \frac{1}{f'(a)} dz$. □

THEOREM 3.20. *(Laurent Series Expansion) Let $f \in \mathcal{O}(A(r,s))$ where $A(r,s) := \{\zeta \in \mathbb{C} : r < |\zeta| < s\}, 0 \leq r < s \leq \infty$. Then there is a unique sequence $c_j \in \mathbb{Z}$ such that*

$$f(\zeta) = \sum_{j=-\infty}^{\infty} c_j \zeta^j, \zeta \in A(r,s), \tag{3.10}$$

and for any $r < \rho < s$ we have that

$$c_j = \frac{1}{2\pi i}\int_{b\mathbb{D}_\rho} f(z)z^{-j-1}dz \tag{3.11}$$

PROOF. Choose $r < \rho_1 < \rho_2 < s$. We have that

$$f(\zeta) = \frac{1}{2\pi i}\int_{bD_{\rho_2}} \frac{f(z)}{z-\zeta}dz - \frac{1}{2\pi i}\int_{b\mathbb{D}_{\rho_1}} \frac{f(z)}{z-\zeta}dz. \tag{3.12}$$

We have considered the first integral in the proof that holomorphic functions are analytic, so lets look at the second. We have that $-1/(z-\zeta) = 1/\zeta(1-(z/\zeta)) = 1/\zeta\sum_{j=0}^\infty (z/\zeta)^j = \sum_{j=0}^\infty z^j\zeta^{-j-1}$ for $|\zeta| > |z|$. By uniform convergence as in the the proof that holomorphic functions are analytic we may interchange summation and integration and get that

$$-\frac{1}{2\pi i}\int_{b\mathbb{D}_{\rho_1}} \frac{f(z)}{z-\zeta}dz = \sum_{j=0}^\infty (\frac{1}{2\pi}\int_{bD_{\rho_1}} f(z)z^j)\zeta^{-j-1} \tag{3.13}$$

for $|\zeta| > \rho_1$. By Stoke's Theorem the formula for the c_j's is independent of the choiceof ρ_1, ρ_2. Uniqueness follows from the residue theorem. □

THEOREM 3.21. *Let $f \in \mathcal{O}(\mathbb{D}^*)$ and assume that f is bounded. Then f extends to a holomorphic function on $\mathbb{D}$.*

PROOF. We consider the Laurent series coefficients c_j for $j < 0$ and get that

$$2\pi i c_j = \int_{bD_\epsilon} f(z)z^{-j-1}dz = \int_0^{2\pi} f(\epsilon e^{it})i\epsilon e^{it}dt \underset{\epsilon\to 0}{\to} 0 \tag{3.14}$$

since $|f|$ is bounded. By Abel's Lemma we have that f extends to $\mathbb{D}$. □

CHAPTER 2

Runge's Theorem

1. Partitions of unity

This can be read in Narasimhan's book, Chapter 5., Section 1.

2. Smeared out Cauchy Integral Formula

THEOREM 2.1. *Let $\Omega \subset \mathbb{C}$ be a domain, let $f \in \mathcal{O}(K)$,* i.e.*, there exists an open neighborhood $U \subset \Omega$ of K with $f \in \mathcal{O}(U)$, and let $\alpha \in \mathcal{C}_0^\infty(U)$ with $\alpha \equiv 1$ near K. Then for all $\zeta \in K$ we have that*

$$f(\zeta) = -\frac{1}{\pi}\int\int_{\mathbb{C}} \frac{\frac{\partial\alpha}{\partial\bar{z}}(z)\cdot f(z)}{z-\zeta} dxdxy, z = x+iy. \tag{2.1}$$

PROOF. This is immediate from the generalized Cauchy Integral Formula, since $\alpha \cdot f$ is compactly supported in $\mathbb{C}$ and $\frac{\partial}{\partial\bar{z}}(\alpha\cdot f) = \frac{\partial\alpha}{\partial\bar{z}}\cdot f$ since f is holomorphic. □

3. Runge's Theorem

DEFINITION 3.1. Let $\Omega \subset \mathbb{C}$ be a domain, and let $U \subset \Omega$ be a subset. We say that U is relatively compact in Ω and write $U \subset\subset \Omega$, if $cl_\Omega(U)$ is compact.

LEMMA 3.2. *Let $\Omega \subset \mathbb{C}$ be a domain, and let $K \subset \Omega$ be compact. Let U be a connected component of $\Omega \setminus K$ and let $\tilde{U}$ be the connected component of $\mathbb{C} \setminus K$ containing U. Then the following are equivalent*

1. *$U \subset\subset \Omega$,*
2. *U is bounded and $U = \tilde{U}$, and*
3. *U is bounded and $b_{\mathbb{C}}U \subset K$.*

PROOF. (1) $\Rightarrow$ (2) Clearly U is bounded, and if $U \neq \tilde{U}$ there is a sequence of points $\{z_j\} \subset U$ converging to a point $z \in \tilde{U} \setminus U$, hence $cl_\Omega U$ is not compact.

(2) $\Rightarrow$ (3) Let $z \in bU$. If $z \notin K$ there exists a disk $D_r(z) \cap K = \emptyset$, hence $\tilde{U} \neq U$.

(3) $\Rightarrow$ (1) $b_{\mathbb{C}}U \subset K \Rightarrow cl_{\mathbb{C}}(U) \subset \Omega \Rightarrow cl_{\mathbb{C}}(U) = cl_\Omega(U)$, and we know that a closed and bounded set is compact. □

Definition 3.3. Let $\Omega \subset \mathbb{C}$ be a domain and let $K \subset \Omega$ be compact. We set

$$\widehat{K}_{\mathcal{O}(\Omega)} := \{z \in \Omega : |f(z)| \leq \|f\|_K \text{ for all } f \in \mathcal{O}(\Omega)\}. \tag{3.1}$$

The set $\widehat{K}_{\mathcal{O}(\Omega)}$ is called the *holomorphically convex hull* of K.

Lemma 3.4. *Let $\Omega \subset \mathbb{C}$ be a domain and let $K \subset \Omega$ be compact. Then $\widehat{K}_{\mathcal{O}(\Omega)}$ is compact.*

Proof. It is clear that $\widehat{K}_{\mathcal{O}(\Omega)}$ is a closed subset of Ω. Since $K \subset \mathbb{B}_R$ for a large enough $R > 0$ is also clear that $\widehat{K}_{\mathcal{O}(\Omega)}$ is bounded. To see that $\widehat{K}_{\mathcal{O}(\Omega)}$ is a closed subset of $\mathbb{C}$ let $a \in b\Omega$ and set $\delta := \mathrm{dist}(K, b\Omega)$. If $\Omega \neq \mathbb{C}$ we have that $0 < \delta < \infty$. For any point $a \in b\Omega$ we have that $f(z) = \frac{1}{z-a}$ satisfies $\|f_a\|_K \leq \frac{1}{\delta}$, and for any point in the set $S := \{z \in \Omega : \mathrm{dist}(z, b\Omega) \leq \frac{\delta}{2}\}$ there exists an f_a with $|f_a(z)| \geq \frac{2}{\delta}$. □

Proposition 3.5. *(Pushing poles) Let $K \subset \mathbb{C}$ be a compact set and let $U \subset \mathbb{C} \setminus K$ be a connected component. Let $a \in U$ be any point, and let $\tilde{U}$ denote the set of points $b \in U$ such that the function $f_a(z) = \frac{1}{z-a}$ may be approximated uniformly on K by functions of the form*

$$f_b(z) = \sum_{j=-N}^{N} c_j (z-b)^j. \tag{3.2}$$

Then $\tilde{U} = U$.

Proof. By assumption we have that $\tilde{U}$ is non-empty. It is not hard using Laurent series expansion to show that Ω is also open and closed (Do it!). □

Corollary 3.6. *Let $\Omega \subset \mathbb{C}$ be a domain, and let $K \subset \Omega$ be a compact set. Then $\widehat{K}_{\mathcal{O}(\Omega)}$ is the union of K and all the components of $\Omega \setminus K$ which are relatively compact in Ω.*

Proof. If $U \subset \Omega \setminus K$ is relatively compact in Ω it follows form Lemma 3.2 and the maximum principle that $U \subset \widehat{K}_{\mathcal{O}(\Omega)}$. On the other hand, let $U \subset \Omega \setminus K$ not be relatively compact in Ω and let $a \in U$. Then $U \setminus \{a\}$ is a connected component of $\Omega \setminus \{K \cup \{a\}\}$ which is not relatively compact. Hence, by Lemma 3.2 and Propositon 3.5 any $z \mapsto \frac{1}{z-b}, b \in U \setminus \{a\}$ my be approximated on K by functions holomorphic on Ω. Considering b close enough to a this shows that $a \notin \widehat{K}_{\mathcal{O}(\Omega)}$. □

Proposition 3.7. *Let $K \subset \mathbb{C}$ be a compact set, and let $f \in \mathcal{O}(K)$. For any $\epsilon > 0$ there exist $a_j, c_j \in \mathbb{C} \setminus K, j = 1, ..., N = N(\epsilon)$ such that the function*

$$r(z) = \sum_{j=1}^{N} \frac{c_j}{z - a_j}, \tag{3.3}$$

satisfies $\|r - f\|_K < \epsilon$.

PROOF. Let U be an open set containing K such that $f \in \mathcal{O}(U)$ and let $\alpha \in \mathcal{C}_0^\infty(U)$ with $\alpha \equiv 1$ near K. Write $g(z) := \frac{\partial\alpha}{\partial\bar{z}} \cdot f(z)$ and choose an open set S with $\overline{S}$ disjoint from K such that $Supp(g) \subset S$. It follows from Theorem 2.1 that

$$f(\zeta) = -\frac{1}{\pi}\int\int_S \frac{g(z)}{z-\zeta}dxdy, \tag{3.4}$$

for all $\zeta \in K$. Choose a sequence $\{\triangle_k^j\}, j \in \mathbb{N}, 1 \leq k \leq m(j)$ of disjoint open squares of radius $r(k,j) \leq 1/j$ whose closures cover $Supp(g)$ and pick a point z_{jk} in each square. For each fixed ζ we know that the sequence of Riemann sums

$$R_j(\zeta) := \sum_{k=1}^{m(j)} \frac{g(z_{jk})}{z_{jk}-\zeta} \tag{3.5}$$

converges uniformly to $f(\zeta)$ for $\zeta \in K$ as $j \to \infty$. By compactness of K and since $\overline{S}$ is disjoint from K it follows that the convergence is uniform independently of $\zeta \in K$. □

THEOREM 3.8. *(Runge's Theorem) Let* $\Omega \subset \mathbb{C}$ *be a domain and let* $K \subset \Omega$ *be compact. The following are equivalent:*

(1) $\mathcal{O}(\Omega)$ *is dense in* $\mathcal{O}(K)$,
(2) $\Omega \setminus K$ *has no relativly compact components in* Ω, *and*
(3) $\widehat{K}_{\mathcal{O}(\Omega)} = K$.

PROOF. The equivalense (2) $\Leftrightarrow$ (3) is Corollary 3.6. We show first (2) $\Rightarrow$ (1). By Proposition 3.7 it is enough to show that we may approximate the function $r(z) = \frac{1}{z-a}$ for any point $a \in \Omega \setminus K$. There are two posibilites: (i) a is in a bounded connected component U of $\mathbb{C} \setminus K$. By Lemma 3.2 there is a point in $U \setminus \Omega$ so we can use Proposition 3.5. (ii) a is in the unbounded component of $\mathbb{C} \setminus K$. By Proposition 3.5 we may assume that $|a| > \|z\|_K$, and then it follows by Taylor series expansion.

Finally we show (1) $\Rightarrow$ (2). Suppose that (2) does not hold. Then by Lemma 3.2 there is a connected component U of $\Omega \setminus K$ with $b_{\mathbb{C}}U \subset K$. Pick a point $a \in U$ and define $r(z) = \frac{1}{z-a}$. Suppose there exists a sequence $f_j \in \mathcal{O}(\Omega)$ such that $f_j \to r(z)$ uniformly on K. Then by the maximum principle $f_j \to f \in \mathcal{O}(U) \cap \mathcal{C}(\overline{U})$ with $f = r$ on bU. So $g = (z-a) \cdot f$ is a holomorphic function on U which is identically one on U, hence $g \equiv 1$. This is a contradiction since $g(a) = 0$.

□

3.0.1. *Runge's Theorem for non-vanishing holomorphic functions.*

THEOREM 3.9. *Let* $\Omega \subset \mathbb{C}$ *be a domain, and let* $K \subset \Omega$ *be a compact set,* $\widehat{K}_{\mathcal{O}(\Omega)} = K$. *Then for any* $f \in \mathcal{O}^*(K)$ *and any* $\epsilon > 0$ *there exists* $F \in \mathcal{O}^*(\Omega)$

with

$$\|F - f\|_K < \epsilon. \tag{3.6}$$

PROOF. Using Proposition 3.7 we will now assume that f is of the form

$$f(z) = \sum_{j=1}^{N} \frac{c_j}{z - a_j} = c \cdot \Pi_{j=1}^{M} (z - b_j)^{m_j}, \tag{3.7}$$

with $c, b_j \in \mathbb{C}, b_j \notin K, m_j \in \mathbb{Z}$. It is then enough to show that each function $(z - b_j)$ can be approximated arbitrarily well on K by non-zero holomorphic functions on Ω.

The idea is as follows: Ideally, if $\log(z - b_j)$ exists near K we could approximate $\log(z - b_j)$ by some function g using the ordinary Runge's Theorem, and then use e^g as an approximation of $\log(z - b_j)$ on K. Such a logarithm does not exist in general, so we will do the following: since K is holomorphically convex we are in one of two situations, either (i) there exists a point d_j in the same connected component of $\mathbb{C} \setminus K$ as b_j with $d_j \notin \Omega$ (Lemma 3.2), or (ii) there is a point d_j in the same connected component of $\mathbb{C} \setminus K$ as b_j with $|d_j| > \|z\|_K$. In any case, we may write

$$(z - b_j) = (\frac{z - b_j}{z - d_j}) \cdot (z - d_j). \tag{3.8}$$

So it is enough to show that we may approximate both factors in the right hand side product but non-vanishing holomorphic functions. In situation (i) the function $(z - d_j)$ is already non-vanishing on Ω, and in situation (ii) the function $\log(z - d_j)$ exists near K, so it is enough to show that the first factor may be approximated. For this it is enough to show that $\log(\frac{z-b_j}{z-d_j})$ exists near K. To see this we show the following:

LEMMA 3.10. *Let $f_t : K \to \mathbb{C}^*$ be a homotopy of continuous maps, $t \in [0, 1]$. If $arg(f_0)$ exists on K then $arg(f_t)$ exists on K for all $t \in [0, 1]$.*

PROOF. For each t and each $z \in K$ we define $arg(f_t(z))$ to be the angle you get by continuing arg from $arg(f_0(z))$. We need to show that this function is continuous for each t. By compactness there exists a $\delta > 0$ such that the following holds: for any $t \in [0, 1]$ and any $z \in K$, the difference $|arg(f_t(z)) - arg(f_{t'}(z))| < \pi/2$ for all $|t - t'| \leq \delta$, if $arg(f_{t'}(z))$ is the branch you get by continuing from $arg(f_t(z))$. It follows that $arg(f_{t'})$ is continuous if $arg(f_t)$ is: we get

$$\begin{aligned} |arg(f_{t'}(z)) - arg(f_{t'}(z'))| &\leq |arg(f_t'(z) - arg(f_t)(z))| \\ &+ |arg(f_t(z)) - arg(f_t(z'))| \\ &+ |arg(f_t(z')) - arg(f_t'(z'))| \\ &\leq |arg(f_t(z)) - arg(f_t(z'))| + \pi, \end{aligned}$$

from which the result follows since arg is always continuous modulo 2π. □

The claim that $\log(\frac{z-b_j}{z-d_j})$ exists now follows from the fact that b_j and d_j lie in the same path-connected component of $\mathbb{C} \setminus K$. $\square$

CHAPTER 3

Applications of Runge's Theorem

1. The $\overline{\partial}$-equation

PROPOSITION 1.1. *Let $\Omega \subset \mathbb{C}$ be a domian, and let $u \in \mathcal{C}_0^1(\Omega)$. Then there exists $f \in \mathcal{C}^1(\Omega)$ solving the equation*

$$\frac{\partial f}{\partial \overline{z}}(\zeta) = u(\zeta), z = x + iy. \tag{1.1}$$

for all $\zeta \in \Omega$.

PROOF. We define

$$f(\zeta) := -\frac{1}{\pi}\int_{\mathbb{C}} \frac{u(z)}{z-\zeta} dxdy, z = x + iy. \tag{1.2}$$

for $\zeta \in \mathbb{C}$. This is well defined since u has compact support. Note that

$$f(\zeta) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{u(z+\zeta)}{z} dxdy \tag{1.3}$$

Differentiating with respect to the x-variable we consider real δ's, and we get that

$$\begin{aligned}
\lim_{\delta\to 0}\frac{f(\zeta+\delta)-f(\zeta)}{\delta} &= \lim_{\delta\to 0} -\frac{1}{\pi}\int_{\mathbb{C}} \frac{1}{\delta}\frac{u(z+\zeta+\delta)-u(z+\zeta)}{z}dxdy \\
&= -\frac{1}{\pi}\int_{\mathbb{C}} \lim_{\delta\to 0}\frac{1}{\delta}\frac{u(z+\zeta+\delta)-u(z+\zeta)}{z}dxdy \\
&= -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\frac{\partial u}{\partial x}(z)}{z-\zeta}dxdy.
\end{aligned}$$

Differentiating with respect to the y-variable can be computed similarly, and so we get that

$$\frac{\partial f}{\partial \overline{z}}(\zeta) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\frac{\partial u}{\partial \overline{z}}(z)}{z-\zeta}dxdy = u(\zeta), \tag{1.4}$$

where the last equality follows from the generalized Cauchy Integral Formula. □

THEOREM 1.2. *Let $\Omega \subset \mathbb{C}$ be a domain, and let $u \in \mathcal{C}^1(\Omega)$. Then there exists $f \in \mathcal{C}^1(\Omega)$ satisfying the equation*

$$\frac{\partial f}{\partial \overline{z}}(\zeta) = u(\zeta), \tag{1.5}$$

for all $\zeta \in \Omega$.

PROOF. Let $K_j \subset K_{j+1}$ be a normal exhaustion of Ω. For each j let $\alpha_j \in \mathcal{C}_0^\infty(\Omega)$ such that $\alpha_j \equiv 1$ near K_j. Let $u_j := \alpha_j \cdot u$. We will solve $\frac{\partial f_j}{\partial \bar{z}} = u_j$ by induction. Assume that we have solved $\frac{\partial f_k}{\partial \bar{z}} = u_k$. Let $\tilde{f}_{k+1}$ be a solution to the equation $\frac{\partial \tilde{f}_{k+1}}{\partial \bar{z}} = u_{k+1}$. Then $\tilde{f}_{k+1} - f_k \in \mathcal{O}(K_m)$ so by Runge's Theorem there exists $g_{k+1} \in \mathcal{O}(\Omega)$ with $\|\tilde{f}_{k+1} - g_{k+1} - f_k\|_{K_m} < (1/2)^{k+1}$. We set $f_{k+1} := \tilde{f}_{k+1} - g_{k+1}$. Now it is clear that $\{f_j\}$ is a Cauchy sequence on each K_i and so $f_j \to f \in \mathcal{C}(\Omega)$. For each fixed i we see that $f - f_i \in \mathcal{O}(K_i^\circ)$, and so $f \in \mathcal{C}^1(\Omega)$ and solves our equation. □

2. The theorems of Mittag-Leffler and Weierstrass

THEOREM 2.1. *(Mittag-Leffler) Let $\Omega \subset \mathbb{C}$ be a domain, let $A = \{a_j\}$ be a discrete set of points, and for each $j \in \mathbb{N}$ let p_j be a prescribed principle part at a_j*

$$p_j(z) = \sum_{j=1}^{m(j)} c_j \cdot (z - a_j)^{-j}. \tag{2.1}$$

Then there exists $f \in \mathcal{O}(\Omega \setminus A)$ such that $f - p_j$ is holomorphic near a_j for all j.

We will leave it as an excercise to prove this theorem in a similar manner as the previous theorem, and we give here two different proofs.

Proof no. 1 of Theorem 2.1: Choose pairwise disjoint disks $\overline{D}_{\delta(a)}(a) \subset \Omega, a \in A$, and let $\phi_a \in \mathcal{C}_0^\infty(D_{\delta(a)}(a))$ be constantly equal to 1 near a. Then $\tilde{f} := \sum_a \phi_a p_a$ is a smooth solution to our problem. Now $\frac{\partial \tilde{f}}{\partial \bar{z}}$ extends to a smooth function u on Ω and we may solve $\overline{\partial} g = u$ on Ω. So $f - g$ is a holomorphic solution.

□

LEMMA 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an open covering of Ω. Furthermore, let $f_{\alpha\beta} \in \mathcal{C}^\infty(U_\alpha \cap U_\beta)$ for all $\alpha, \beta \in I$ such that*

$$f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = 0 \text{ on } U_\alpha \cap U_\beta \cap U_\gamma \text{ for all } \alpha, \beta, \gamma \in I. \tag{2.2}$$

Then there exist $f_\alpha \in \mathcal{C}^\infty(U_\alpha)$ for all $\alpha \in I$ such that $f_{\alpha\beta} = f_\alpha - f_\beta$ on $U_\alpha \cap U_\beta$ for all $\alpha, \beta \in I$.

PROOF. Let $\{\phi_\alpha\}$ be a partition of unity with respect to the cover $\mathcal{U}$. We define $f_\alpha := \sum_{\gamma \in I} \phi_\gamma \cdot f_{\alpha\gamma}$ and note that $f_\alpha \in \mathcal{C}^\infty(U_\alpha)$ since $f_{\alpha\alpha} = 0$ by

(2.4). On $U_\alpha \cap U_\beta$ we have that

$$\begin{aligned} f_\alpha - f_\beta &= \sum_\gamma \phi_\gamma f_{\alpha\gamma} - \phi_\gamma f_{\beta\gamma} \\ &= \sum_\gamma \phi_\gamma (f_{\alpha\gamma} + f_{\gamma\beta}) \\ &= \sum_\gamma \phi_\gamma f_{\alpha\beta} \\ &= f_{\alpha\beta}. \end{aligned}$$

□

THEOREM 2.3. *Let $\Omega \subset \mathbb{C}$ be a domain, and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an open covering of Ω. Furthermore, let $f_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta)$ for all $\alpha, \beta \in I$ such that*

$$f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = 0 \text{ on } U_\alpha \cap U_\beta \cap U_\gamma \text{ for all } \alpha, \beta, \gamma \in I. \tag{2.3}$$

Then there exist $f_\alpha \in \mathcal{O}(U_\alpha)$ for all $\alpha \in I$ such that $f_{\alpha\beta} = f_\alpha - f_\beta$ on $U_\alpha \cap U_\beta$ for all $\alpha, \beta \in I$.

PROOF. We have seen that there exist $f_\alpha \in \mathcal{C}^\infty(U_\alpha)$ which solve the problem. Then $f_\alpha - f_\beta \in \mathcal{O}(U_\alpha \cap U_\beta)$ for all $\alpha, \beta \in I$ and so $u := \overline{\partial} f_\alpha$ is well defined on Ω. Let f solve $\overline{\partial} f = u$. Then $f_\alpha - f$ solves the problem. □

Proof no. 2 of Theorem 2.1: For each $a \in A$ let $U_a = \Omega \setminus (A \setminus \{a\})$. On $U_a \cap U_b$ set $f_{ab} = p_a - p_b$. Then $f_{ab} \in \mathcal{O}(U_a \cap U_b)$, and clearly $f_{ab} + f_{bc} + f_{ca} = 0$. Let $f_a \in \mathcal{O}(U_a)$ for all a such that $f_{ab} = f_a - f_b$. Define $f := p_a - f_a$ on U_a and note that f is now well defined. □

THEOREM 2.4. *(Weierstrass) Let $\Omega \subset \mathbb{C}$ be a domain, let $A = \{a_j\}$ be discrete, and for each $j \in \mathbb{N}$ let $m_j \in \mathbb{Z}$. Then there exists $f \in \mathcal{O}^*(\Omega \setminus A)$ such that $f \cdot (z - a_j)^{-m_j}$ is holomorphic and nonzero near a_j for all j.*

PROOF. Copy the proof of Theorem 2.1 based on the proof of Theorem 1.2 using products instead of sums, and Runge's theorem for non-zero approximation.

Alternatively, copy the second proof of Theorem 2.1 using Theorem 2.5 instead of Theorem 2.3. □

THEOREM 2.5. *Let $\Omega \subset \mathbb{C}$ be a domain, and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an open covering of Ω by simply connected domains. Furthermore, let $f_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ for all $\alpha, \beta \in I$ such that*

$$f_{\alpha\beta} \cdot f_{\beta\gamma} \cdot f_{\gamma\alpha} = 1 \text{ on } U_\alpha \cap U_\beta \cap U_\gamma \text{ for all } \alpha, \beta, \gamma \in I. \tag{2.4}$$

Then there exist $f_\alpha \in \mathcal{O}^*(U_\alpha)$ for all $\alpha \in I$ such that $f_{\alpha\beta} = f_\alpha / f_\beta$ on $U_\alpha \cap U_\beta$ for all $\alpha, \beta \in I$.

PROOF. From topology we know that there exists a continuous solution $\{g_\alpha\}$. For each α choose a branch $h_\alpha = \log g_\alpha$, and set $\tilde{f}_{\alpha\beta} = h_\alpha - h_\beta = \log f_{\alpha\beta}$. Then $\tilde{f}_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta)$ and the condition (2.4) is satisfied. By Theorem 2.3 there are functions $g_\alpha \in \mathcal{O}(U_\alpha)$ such that $\tilde{f}_{\alpha\beta} = g_\alpha - g_\beta$. So $f_\alpha := e^{g_\alpha}$ solves the problem. $\square$

CHAPTER 4

Piccard's Theorem

This can be read in Narasimhans book, Chapter 4.

CHAPTER 5

Riemann mapping theorem

This can be read in Narasimhans book, Chapter 7, pp 139–144.

CHAPTER 6

Some exercises fro Narasimhan/Nievergelt

48, 104, 105, 106, 117, 119, 126, 226, 241, 242, 248, 297.

CHAPTER 7

From Forster's Book, Lectures on Riemann Surfaces

1. Basics on Riemann surfaces, page 1–8.
2. Elementary properties of holomorphic mappings, page 10–13.
3. Branched and unbranched coverings, 20–30.
4. Sheaves, 40–43
5. Differential forms, 59–68.
6. Integration of differential forms, page 68–71, page 76–80.
7. Compact Riemann surfaces, 96–116.
8. The exact cohomology sequence, 118–126.
9. Riemann-Roch, 126–131.
10. Serre Duality, 132–140.
11. DeRham-Hodge Theorem, page 157 (with a more ad hoc proof).

1. DeRham-Hodge Theorem

The goal of this section is to show that the genus $g_X = dim H^1(X, \mathcal{O})$ of a compact Riemann surface X is a *topological invariant*, *i.e.*, if X is homeomorphic to X' then $g_X = g_{X'}$. This is done by finding a basis consisting of holomorphic and anti-holomorphic 1-forms for the first DeRham cohomology group. Consider the short exact sequence

$$0 \to \mathbb{C} \to \mathcal{E} \xrightarrow{d} \mathcal{E}^{(1)}_{cl} \to 0, \tag{1.1}$$

on X. Since $H^1(X, \mathcal{E}) = 0$ we have seen that

$$H^1(X, \mathbb{C}) \approx \mathcal{E}^{(1)}_{cl}(X)/d\mathcal{E}(X), \tag{1.2}$$

where the quotient on the right hand side is the first DeRham group, denoted by $Rh^1(X)$. By (1.2) the dimension of $Rh^1(X)$ is a topological invariant.

Recall that for a Riemann surface X, Ω denotes the sheaf of holomorphic 1-forms, and $\overline{\Omega}$ denotes the sheaf of anti-holomorphic 1-forms. We define the sheaf of *harmonic 1-forms* to be the direct sum

$$Harm^1 := \Omega \oplus \overline{\Omega}. \tag{1.3}$$

Note that by Serre duality we have that

$$dim Harm^1(X) = 2g. \tag{1.4}$$

THEOREM 1.1. *Let X be a compact Riemann surface. Then*

$$H^1(X,\mathbb{C}) \approx Rh^1(X) \approx Harm^1(X). \tag{1.5}$$

Note that by (1.4) and (1.2) the genus of X is a topological invariant.

PROOF. We have seen that $H^1(X,\mathcal{O}) \approx \mathcal{E}^{(0,1)}(X)/\overline{\partial}\mathcal{E}(X)$, so in particular, the dimension of the latter group is g. By Serre duality we have that $dim\overline{\Omega}(X) = g$. Moreover, no non-trivial form $\omega \in \overline{\Omega}$ is $\overline{\partial}$-exact: if $\overline{\partial} f = \omega \in \overline{\Omega}$ then $\partial\overline{\partial} f = 0$ and so f is harmonic. It follows that

$$\mathcal{E}^{(0,1)}(X) = \overline{\partial}\mathcal{E}(X) \oplus \overline{\Omega}(X). \tag{1.6}$$

By taking complex conjugates and considering the direct sum we get that

$$\mathcal{E}^{(1)}(X) = \partial\mathcal{E}(X) \oplus \overline{\partial}E(X) \oplus \Omega(X) \oplus \overline{\Omega}(X). \tag{1.7}$$

To describe $\mathcal{E}^{(1)}_{cl}(X)$ let $\omega = \partial f + \overline{\partial} g + \omega_1 + \overline{\omega}_2$ be a d-closed form decomposed according to (1.7). Since ω_1 and $\overline{\omega}_2$ are both d-closed it follows that $\partial f + \overline{\partial} g$ is d-closed, *i.e.*, $\overline{\partial}\partial f + \partial\overline{\partial} g = 0$. So $f - g$ is harmonic, hence constant. It follows that $df = \partial f + \overline{\partial} f = \partial f + \overline{\partial}(g + c) = \partial f + \overline{\partial} g$, hence $\omega = df + \omega_1 + \overline{\omega}_2$. It follows that

$$\mathcal{E}^{(1)}_{cl}(X) = d\mathcal{E}(X) \oplus \Omega(X) \oplus \overline{\Omega}(X). \tag{1.8}$$

Since no nontrivial element of $\Omega(X) \oplus \overline{\Omega}(X)$ is d-exact (why?) the proof is complete. ☐