1.a For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C})$, let $\varphi_A(z) = \frac{az+b}{cz+d}$.

Then φ_A can be extended to a meromorphic function on $\mathbb{P}^1$.

Proof: We have $ad - bc \neq 0$. (Note that if $ad-bc = 0$, then φ_A is constant). If $c = 0$, then $f(z) = \frac{a}{d}z + \frac{b}{d}$. This is holomorphic in $\mathbb{P}^1 \setminus \{\infty\} = \mathbb{C}$ and $\lim_{z\to\infty} |f(z)| = \infty$, hence has a pole at ∞, i.e. f is meromorphic on $\mathbb{P}^1$.

If $c \neq 0$, then $f \in \mathcal{O}(\mathbb{P}^1 \setminus \{\infty, -\frac{d}{c}\})$. We have $\lim_{z\to\infty} f(z) = \frac{a}{c}$,

so the singularity at ∞ is removable. Also

$$\varphi_A(z) = \frac{1}{c}\left(a - \frac{ad-bc}{cz+d}\right) \xrightarrow[z \to -\frac{d}{c}]{} \infty$$

so φ_A has a pole at $-\frac{d}{c}$, hence is meromorphic. ∎

$\varphi_A : \mathbb{P}^1 \to \mathbb{P}^1$ is biholomorphic, i.e. $\varphi_A \in \mathrm{Aut}(\mathbb{P}^1)$.

Proof: A short computation gives

$$\varphi_A \circ \varphi_B = \varphi_{AB}$$

where it is defined, and therefore on $\mathbb{P}^1$ by the identity theorem. Then $\varphi_A^{-1} = \varphi_{A^{-1}}$, so $\varphi_A \in \mathrm{Aut}(\mathbb{P}^1)$. ∎

If $f \in \mathrm{Aut}(\mathbb{P}^1) \Rightarrow f = \varphi_A$ for some $A \in GL(2, \mathbb{C})$.

Proof: Enough to prove in the case $f(0) = 0$ and $f(\infty) = \infty$ (check this). Then $f \in \mathcal{O}(\mathbb{C})$, hence $f(z) = \sum_{n=1}^{\infty} a_n z^n$

If $a_n \neq 0$ for infinitely many n, then $f(\frac{1}{z}) = \sum a_n z^{-n}$

and f has an essential singularity at ∞, which is impossible. Hence f is a polynomial of degree k. But then f is $k-1$, so we must have $k=1$, i.e. $f = a_1 z$.

<u>1.4</u> $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, $\Gamma' = \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2'$ two lattices in $\mathbb{C}$. Then $\Gamma = \Gamma'$ if and only if there is $A \in M(2,\mathbb{Z})$, $\det A = \pm 1$ such that
$$\begin{pmatrix}\omega_1'\\ \omega_2'\end{pmatrix} = A\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}$$

Proof: "If". If there is such A, then $\omega_1', \omega_2' \in \Gamma$, hence $\Gamma' \subset \Gamma$. But $\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = A^{-1}\begin{pmatrix}\omega_1'\\ \omega_2'\end{pmatrix}$ and $A^{-1} \in M(2,\mathbb{Z})$, so $\Gamma \subset \Gamma'$.
"Only if". If $\Gamma = \Gamma'$ we have $\begin{pmatrix}\omega_1'\\ \omega_2'\end{pmatrix} = A\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}$ and $\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = B\begin{pmatrix}\omega_1'\\ \omega_2'\end{pmatrix}$, with $A, B \in M(2,\mathbb{Z})$. But then
$$\begin{pmatrix}\omega_1'\\ \omega_2'\end{pmatrix} = AB\begin{pmatrix}\omega_1'\\ \omega_2'\end{pmatrix}$$

and we must have $AB = I$, since ω_1', ω_2' are lin. ind. over $\mathbb{R}$. Hence $1 = \det AB = \det A \cdot \det B$. Since $\det A$, $\det B$ both are integers, we must have $\det A = \pm 1$.

<u>1.5</u> a) $\Gamma, \Gamma' \subset \mathbb{C}$ two lattices, $\alpha \in \mathbb{C}^*$, $\alpha\Gamma \subset \Gamma'$. Then the map $\mathbb{C} \to \mathbb{C}$, $z \to \alpha z$ induces a holomorphic map $\mathbb{C}/\Gamma \to \mathbb{C}/\Gamma'$ which is biholomorphic iff $\alpha\Gamma = \Gamma'$.

Proof: We have commutative diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\alpha z} & \mathbb{C} \\ \downarrow & & \downarrow \\ \mathbb{C}/\Gamma & \xrightarrow{\alpha[z]} & \mathbb{C}/\Gamma' \end{array}$$

$\alpha[z] = [\alpha z]$ is well defined since $[z] = [w] \Leftrightarrow z - w \in \Gamma \Rightarrow \alpha(z-w) = \alpha z - \alpha w \in \Gamma'$
$\Leftrightarrow [\alpha z] = [\alpha w]$.

Pick $V \subset \mathbb{C}$ open such that $V \cap (V + \omega) = \emptyset$ for all $\omega \in \Gamma$ and if $V' = \alpha V$, then $V' \cap (V' + \omega') = \emptyset$ for all $\omega' \in \Gamma'$. (Show that this can be done!). If $U = \pi_\Gamma V$ and $U' = \pi_{\Gamma'} V'$, then in the local coordinates in U and U' we have that $\alpha[z]$ is given by αz which is holomorphic.

If $\alpha\Gamma = \Gamma'$, then $\Gamma' = \frac{1}{\alpha}\Gamma$ and $f([z]) = [\alpha z]$ has an inverse holomorphic map $f^{-1}([w]) = [\frac{1}{\alpha} w]$.

If $f([z]) = [\alpha z]$ is injective then $[\alpha z] = [0] \Leftrightarrow [z] = [0]$, i.e. $\alpha z \in \Gamma' \Leftrightarrow z \in \Gamma$. Hence if $w \in \Gamma'$, then $w = \alpha(\frac{w}{\alpha})$, hence $\frac{w}{\alpha} \in \Gamma$ and $w \in \alpha\Gamma$, so $\Gamma' \subset \alpha\Gamma$.

b) Every torus $X = \mathbb{C}/\Gamma$ is isomorphic to a torus of the form $X(\tau) := \mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$, where $\operatorname{Im}\tau > 0$.

Proof: If $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ then $\Gamma' = \frac{1}{\omega_1}\Gamma = \mathbb{Z} + \mathbb{Z}\frac{\omega_2}{\omega_1}$

and $\mathbb{C}/\Gamma$ is isomorphic to $\mathbb{C}/\Gamma'$. We have $\Gamma' = \mathbb{Z} + \mathbb{Z}\tau$ where $\tau = \frac{\omega_1}{\omega_2}$. Since ω_1 and ω_1 are lin. independent over $\mathbb{R}$ we must have $\operatorname{Im}\tau \neq 0$. If $\operatorname{Im}\tau > 0$, O.K. If not, use $-\tau$ instead. This generates the same lattice.

c) If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$, $\operatorname{Im}\tau > 0$ and $\tau' = \frac{a\tau+b}{c\tau+d}$

then $X(\tau)$ and $X(\tau')$ are isomorphic.

Proof: It is easy to see that $\operatorname{Im}\tau' > 0$. By 1.4 and 1.5 (b), two lattices $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $\Gamma' = \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2'$ are isomorphic if $\begin{pmatrix}\omega_1' \\ \omega_2'\end{pmatrix} = \alpha A'\begin{pmatrix}\omega_1 \\ \omega_2\end{pmatrix}$ for some $\alpha \in \mathbb{C}^*$, $A' \in SL(2,\mathbb{Z})$. In our case $\omega_1 = 1$, $\omega_2 = \tau$. Let $A' = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \in SL(2,\mathbb{Z})$

and $\alpha = \frac{1}{c\tau+d} \neq 0$. Then $\alpha A'\begin{pmatrix}\omega_1 \\ \omega_2\end{pmatrix} = \frac{1}{c\tau+d}\begin{pmatrix} d & c \\ b & a \end{pmatrix}\begin{pmatrix} 1 \\ \tau \end{pmatrix} =$

$\frac{1}{c\tau+d}\begin{pmatrix} c\tau+d \\ a\tau+b \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{a\tau+b}{c\tau+d} \end{pmatrix} = \begin{pmatrix} 1 \\ \tau' \end{pmatrix}$. Hence $X(\tau) \cong X(\tau')$.

2.1. $\Gamma = \{ n_1 w_1 + n_2 w_2 \mid n_1, n_2 \in \mathbb{Z} \}$ lattice

(a) $f(z) = -2 \sum_{w \in \Gamma} \frac{1}{(z-w)^3}$ is a doubly periodic function

with triple poles at all $w \in \Gamma$.

Pf: If $|w| > 2|z|$, then $|z-w| \geq \frac{1}{2}|w|$ so

$$\frac{1}{|z-w|^3} \leq \frac{8}{|w|^3}$$

It is therefore sufficient to prove that $\sum_{w \neq 0} \frac{1}{|w|^3} < \infty$.

We have

$$\sum_{w \neq 0} \frac{1}{|w|^3} = \sum_{(n_1,n_2) \neq (0,0)} \frac{1}{|n_1 w_1 + n_2 w_2|^3} = \sum_{n=1}^{\infty} \sum_{|n_1|+|n_2|=n} \frac{1}{|n_1 w_1 + n_2 w_2|^3}$$

There are $4n$ pairs (n_1, n_2) with $|n_1| + |n_2| = n$.
We also have

(*) There exist $k > 0$ such that

$$|n_1 w_1 + n_2 w_2| \geq k(|n_1| + |n_2|) \text{ for all } (n_1, n_2) \in \mathbb{R}$$

(*) can be proved directly using the fact that $\operatorname{Im}\left(\frac{w_1}{w_2}\right) \neq 0$. An easier proof is to notice that $\|(n_1, n_2)\| = |n_1 w_1 + n_2 w_2|$ is a norm on $\mathbb{R}^2$. Since all norms are equivalent the inequality follows. The fact that $\|(n_1, n_2)\| = 0 \Rightarrow n_1 = n_2 = 0$ uses that $\operatorname{Im}\left(\frac{w_1}{w_2}\right) \neq 0$.

This gives
$$\sum_{\omega\neq 0} \frac{1}{|\omega|^3} \leq \sum_{n=1}^{\infty} 4n \cdot \frac{1}{(kn)^3} = \frac{4}{k^3}\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty .$$
■

The Weierstrass $\wp$-function with respect to Γ,
$$\wp_\Gamma = \frac{1}{z^2} + \sum_{\omega\in\Gamma\setminus\{0\}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$$

is a doubly periodic meromorphic function on $\mathbb{C}$ with double poles at all $\omega\in\Gamma$. The coefficients in the Laurent series expansion at any $\omega\in\Gamma$ are $c_{-2}=1$, $c_{-1}=c_0=0$.

Proof: An antiderivative of f is given by
$$\frac{1}{z^2} + \int_0^z \sum_{\omega\neq 0} -\frac{2}{(\zeta-\omega)^3}\, d\zeta$$
where we integrate over any path avoiding the lattice. The integral is independent of the path since all residues of f are 0. On such a path the series $\sum_{\omega\neq 0} -\frac{2}{(\zeta-\omega)^3}$ converge uniformly and we may integrate term by term:
$$\sum_{\omega\neq 0} \int_0^z -\frac{2}{(\zeta-\omega)^3} = \sum_{\omega\neq 0} (\zeta-\omega)^{-2}\Big|_0^z = \sum_{\omega\neq 0} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$$

Hence the antiderivative is $\wp_\Gamma$. Since $\wp_\Gamma' = f$ has periods ω_1, ω_2 it follows that $\wp_\Gamma(z+\omega_1) - \wp_\Gamma(z)$ and $\wp_\Gamma(z+\omega_2) - \wp_\Gamma(z)$ both are constant.

But f' is even, so choosing $z = -\frac{1}{2}w_1$ and $z = -\frac{1}{2}w_2$ proves that the constants are 0.

(b) If $f \in \mathcal{M}(\mathbb{C})$ is doubly periodic with respect to Γ with Laurent expansion $f(z) = \frac{1}{z^2} + \sum_{k \geq 1} c_k z^k$ around the origin, then $f = \wp_\Gamma$.

Proof: The Laurent series expansion of $\wp_\Gamma$ at the origin is also of this form $\wp_\Gamma = \frac{1}{z^2} + \sum_{k \geq 1} d_k z^k$. Then $f - \wp_\Gamma$ is a doubly periodic function with Laurent series $\sum_{k \geq 1} (c_k - d_k) z^k$ around 0, which is therefore a removable singularity. Then all the other singularities are removable too, so $g = f - \wp_\Gamma \in \mathcal{O}(\mathbb{C})$ and doubly periodic. This implies that g is constant (g may be regarded as a holomorphic function on the compact space $\mathbb{C}/\Gamma$) and since $g(0) = 0$, we have $g \equiv 0$.

2.2 If $f \in \mathcal{O}(X)$ is nonconstant, then $\operatorname{Re} f$ does not attain its maximum.

Proof: If $\operatorname{Re} f$ attains its maximum, then $g = e^f \in \mathcal{O}(X)$ is a nonconstant holomorphic function such that $|g| = e^{\operatorname{Re} f}$ attains its maximum, contradicting the maximum principle.

2.3 If $f \in \mathcal{O}(\mathbb{C})$ has a real part bounded from above, then f is constant.

Proof: If $\operatorname{Re} f \leq M$, then if $g = e^f$, we have $|g| \leq e^M$, hence g is constant by Liouville. But then f is constant.