4.4 a) $\tan : \mathbb{C} \to \mathbb{P}^1$ is a local homeomorphism

Proof: $f(z) = \tan z = \frac{\sin z}{\cos z}$ has poles at $E = \{(k+\frac{1}{2})\pi \mid k \in \mathbb{Z}\}$.
$\tan z$ is π periodic.
In $\mathbb{C} \setminus E$ we have $f'(z) = \frac{1}{\cos^2 z} \neq 0$, hence f is a local homeom. for all $z \in \mathbb{C} \setminus E$. In local coordinates around $\infty \in \mathbb{P}^1$, we have

$$f(z) = \frac{\cos z}{\sin z}, \text{ so } f'(z) = \frac{-1}{\sin^2 z} = -1 \text{ for } z \in E.$$

Hence f is also a local homeom. at all points in E.

b) $\tan(\mathbb{C}) = \mathbb{P}^1 \setminus \{\pm i\}$ and $\tan : \mathbb{C} \to \mathbb{P}^1 \setminus \{\pm i\}$ is a covering map.

Proof: We need to solve the equation $\tan z = w$.
For $w \in \mathbb{C}$ this is the equation

$$e^{iz}(1 - iw) - e^{-iz}(1 + iw) = 0$$

If $1 - iw = 0$ or $1 + iw = 0$, i.e. $w = \pm i$, this has no solution. For $w \neq \pm i$, the equation is

$$\left(e^{iz}\right)^2 = \frac{1 + iw}{1 - iw} =: w_1^2$$

This second degree equation has solutions $e^{iz} = \pm w_1$, i.e.
$z_k = (k\pi + \arg w_1) - i \log |w_1|$.

Hence if $w \in \mathbb{C} \setminus \{\pm i\}$, there is a nbh V_0 of z_0 which is mapped biholomorphically to a nbh. W of w, i.e. $\tan : V_0 \to W$ is biholom. We can assume that if $V_k = V_0 + k\pi$, then $V_0 \cap V_k = \emptyset$ for all $k \neq 0$. It follows that $\tan : V_k \to W$ is biholomorphic and that

$$\tan^{-1}(W) = \bigcup_{k \in \mathbb{Z}} V_k.$$

If $w = \infty$, then $\tan^{-1}(w) = E = \{z_k = (k+\frac{1}{2})\pi \mid k \in \mathbb{Z}\}$.
In local coordinates $\frac{1}{w}$ around ∞ tan is given by

$$\tan z = \frac{\cos z}{\sin z}$$

so $(\tan z)' = \frac{-1}{\sin^2 z} = -1$ for $z = z_k$. As above there is a nbh V_0 of z_0 which is mapped biholomorphically to a nbh W of ∞ and $\tan^{-1}(W) = \bigcup_{k \in \mathbb{Z}} V_k$, a disjoint union

c) $X = \mathbb{C} \setminus \{it;\ t \in \mathbb{R}, |t| \geq 1\}$. For every $k \in \mathbb{Z}$ there is a unique holomorphic function $\arctan_k : X \to \mathbb{C}$ with $\arctan_k(0) = k\pi$

Proof: We have $\tan^{-1}(0) = \{k\pi;\ k \in \mathbb{Z}\}$. X is simply connected, hence for every $k \in \mathbb{Z}$ there is a

$$\begin{array}{ccc} & & \mathbb{C} \\ & \overset{\arctan_k}{\nearrow} & \downarrow \tan \\ X & \xrightarrow{i} & \mathbb{P}^1 \setminus \{\pm i\} \end{array}$$

unique lifting $\arctan_k$ of the inclusion map $X \hookrightarrow \mathbb{P}^1 \setminus \{\pm i\}$ with $\arctan_k(0) = k\pi$. The lifting is holomorphic.

<u>4.5</u> The ramification points of the map $f : \mathbb{C} \to \mathbb{P}^1$ given by $f(z) = \frac{1}{2}(z + \frac{1}{z})$ are ± 1.

Proof: The ramification points in $\mathbb{C}^*$ are given by $0 = f'(z) = \frac{1}{2}(1 - \frac{1}{z^2})$, so $z = \pm 1$. We have to check that 0 is not a ramification point. In local coordinates $\frac{1}{w}$ around $\infty = f(0)$, we have $f(z) = \frac{2z}{1+z^2} = 2(z - z^3 + z^5 - \dots)$

so $f'(0) = 2 \neq 0$. Hence 0 is not a ramification point.