Some real analysis

o and O - notation

Suppose f is defined in a nbh V of $0 \in \mathbb{R}^n$, $f : V \to \mathbb{R}^m$

$$f = o(|x|^k) \iff \lim_{x \to 0} \frac{|f(x)|}{|x|^k} = 0 \qquad (k = 0 \text{ is called } o(1))$$

$$f = O(|x|^k) \iff \exists C > 0 \text{ s.t. } |f(x)| \leq C|x|^k, \ x \text{ small}$$

Def. f is differentiable at a if there is a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{x \to 0} \frac{|f(a + x) - f(a) - L(x)|}{|x|} = 0$$

$$\Updownarrow$$

$$f(a + x) = f(a) + L(x) + o(|x|)$$

- L is called the derivative of f at a and denoted df_a
- If f is differentiable at a, then the partial derivatives $\frac{\partial f_j}{\partial x_i}(a)$ exist and

$$df_a(v) = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(a) v_i \right) e_j$$

$$df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1} & & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \quad \text{Jacobian matrix.}$$

- If the partial derivatives $\frac{\partial f_j}{\partial x_i}$ exist in a nbh of a and are continuous at a, then f is differentiable at a.

$C(\Omega) = \{f : \Omega \to \mathbb{C} \; ; \; f \text{ is continuous}\}$

$C^1(\Omega) = \left\{ f : \Omega \to \mathbb{C} \; ; \; \frac{\partial f}{\partial x_i} \in C(\Omega), \; i = 1, \cdots, n \right\}$

$C^k(\Omega) = \left\{ f : \Omega \to \mathbb{C} \; ; \text{ all partial derivatives of order} \le k \text{ are cont} \right\}$.

Order does not matter.

$$\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n \quad \text{multiindex}$$

$$|\alpha| = \alpha_1 + \cdots + \alpha_n \, , \qquad \text{~~order the multiindex~~}$$

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$$

$$C^\infty(\Omega) = \bigcap_k C^k(\Omega)$$

Complex function of a complex variable, $\Omega \subset \mathbb{C}$

$$f : \Omega \to \mathbb{C} \quad , \; z = x + iy \quad , \; f = u + iv$$

$$f(z) = f(x,y) = u(x,y) + i\, v(x,y)$$

As a real function $f : \underset{\substack{\cap \\ \mathbb{R}^2}}{\Omega} \to \mathbb{R}^2$, $f = (u, v)$

Let $\lambda = \alpha + i\beta \in \mathbb{C} \cong \mathbb{R}^2$. What is $df(\lambda)$?

$$df(\lambda) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} \cdot \alpha + \frac{\partial u}{\partial y} \beta \\ \frac{\partial v}{\partial x} \cdot \alpha + \frac{\partial v}{\partial y} \beta \end{pmatrix} = \left(\frac{\partial u}{\partial x} \cdot \alpha + \frac{\partial u}{\partial y} \beta \right) + i \left(\frac{\partial v}{\partial x} \alpha + \frac{\partial v}{\partial y} \beta \right)$$

$$= \alpha \cdot \frac{\partial f}{\partial x} + \beta \cdot \frac{\partial f}{\partial y}$$

Want to express in terms of λ

$$\alpha = \operatorname{Re} \lambda = \frac{1}{2}(\lambda + \bar{\lambda}) \qquad \beta = \operatorname{Im} \lambda = \frac{1}{2i}(\lambda - \bar{\lambda})$$

$$df(\lambda) = \frac{1}{2}(\lambda+\bar\lambda)\frac{\partial f}{\partial x} + \frac{1}{2i}(\lambda-\bar\lambda)\frac{\partial f}{\partial y}$$

$$= \underbrace{\frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)\lambda}_{\substack{\text{Complex linear}\\ L(c\lambda)=cL(\lambda)}} + \underbrace{\frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)\bar\lambda}_{\substack{\text{Complex antilinear}\\ L(c\lambda)=\bar c L(\lambda).}} =: \frac{\partial f}{\partial z}\lambda + \frac{\partial f}{\partial \bar z}\bar\lambda$$

df is $\mathbb{C}$-linear $\Leftrightarrow \frac{\partial f}{\partial \bar z} = 0$.

- $\frac{\partial f}{\partial \bar z}=0$ is called the Cauchy-Riemann equations, i.e. $\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$

Real form:
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Exercise

a) Show that $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar z}$ satisfy Leibniz rule!

b) Suppose $L:\mathbb{C}^n \to \mathbb{C}^m$ is $\mathbb{R}$-linear. Show that

L is $\mathbb{C}$-linear $\Leftrightarrow L(iv) = iL(v) \quad \forall v\in\mathbb{C}^n$.

L is $\mathbb{C}$-antilinear $\Leftrightarrow L(iv) = -iL(v)$ — " —

c) Show that every $\mathbb{R}$ linear $L:\mathbb{C}^n\to\mathbb{C}^m$ splits uniquely in a $\mathbb{C}$-linear and $\mathbb{C}$-antilinear part

$$L = L_{\mathbb{C}} + L_{\bar{\mathbb{C}}}$$

$$L_{\mathbb{C}}(v) = \frac{1}{2}\left(L(v) - iL(iv)\right), \quad L_{\bar{\mathbb{C}}} = \frac{1}{2}\left(L(v) + iL(iv)\right).$$

Def. $f:\Omega\to\mathbb{C}$ is called $\mathbb{C}$-differentiabel at a if

$$\lim_{\lambda\to 0}\frac{f(a+\lambda)-f(a)}{\lambda}$$

exists. This is denoted by $f'(a)$.

$\Updownarrow$

$$f(a+\lambda) = f(a) + f'(a)\lambda + o(|\lambda|)$$

f is $\mathbb{C}$-diff. at $a \Leftrightarrow f$ is differentiable at a and df_a is $\mathbb{C}$-linear

<u>Def.</u> Let Ω be an open subset of $\mathbb{C}$. We say that a complex function $f(z)$ defined in Ω is holomorphic if

$f \in C^1(\Omega)$ and f is complex differentiable at all points in Ω, i.e. f satisfies the C-R equations.

- The set of holomorphic functions is denoted by $\mathcal{O}(\Omega)$
- It is not necessary to assume $f \in C^1(\Omega)$. (this follows automatically when f is $\mathbb{C}$-differentiable), but it makes things easier, because we can use Green's theorem in the plane.

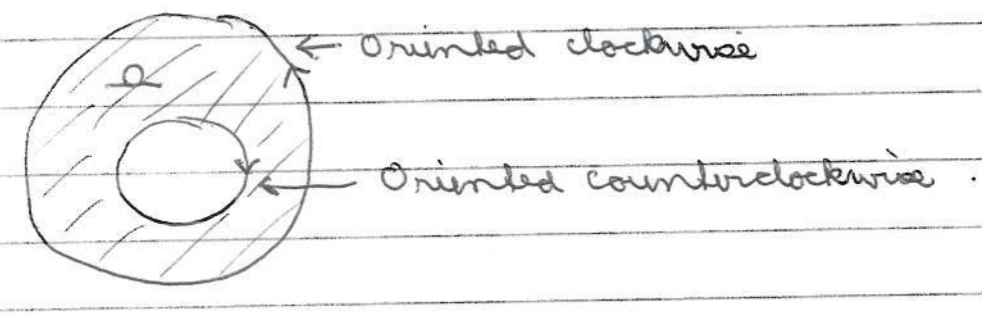
- <u>Green's theorem in the plane</u>

If $\Omega \subset\subset \mathbb{R}^2$ is an open set with piecewise smooth boundary $\partial\Omega$ and M, N are two C^1 functions in $\overline{\Omega} = \Omega \cup \partial\Omega$, then

$$\int_{\partial\Omega} M\,dx + N\,dy = \iint_{\Omega} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Remarks:

1. $\partial\Omega$ is oriented such that Ω lies to the left of $\partial\Omega$.



2. It does not matter if M and N are real or complex valued.

3. $\int_{\partial\Omega} M\,dx + N\,dy$ is computed by parametrizing $\partial\Omega$.

$(x(t), y(t))$, $a \le t \le b$. Then

$$\int_{\partial\Omega} M\,dx + N\,dy = \int_a^b M(x(t),y(t))\,x'(t) + N(x(t),y(t))\,y'(t)\,dt$$

i.e. $dx = x'(t)dt$, $dy = y'(t)dt$.

- If $\gamma \subset \mathbb{C}$ is a curve parametrized by

$$z(t) = x(t) + iy(t), \quad a \le t \le b$$

and f is a complex function on γ, then the complex line integral is defined by

$$\int_\gamma f(z)\,dz = \int_a^b f(x(t)+iy(t))\,z'(t)\,dt = \int_a^b f(x(t)+iy(t))\,(x'(t)+iy'(t))\,dt$$

$$= \int_\gamma f\,dx + i f\,dy \qquad \left(\text{Similar } \int_\gamma f(z)\,d\bar{z}\right)$$

If $\gamma = \partial\Omega$ as in Green's theorem, we get

$$\int_{\partial\Omega} f\,dz = \iint_\Omega \left(i\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy = 2i \iint_\Omega \frac{\partial f}{\partial \bar{z}}\,dx\,dy$$

(Complex form of Green's theorem)

Remarks:

1. If f is holomorphic, we get Cauchy's theorem

$$\int_{\partial\Omega} f\,dz = 0$$

2. If γ is the circle $z = \zeta + re^{i\theta}$, then $dz = ire^{i\theta}d\theta$ and

$$\int_\gamma \frac{f(z)}{z-\zeta}\,dz = \int_0^{2\pi} \frac{f(\zeta+re^{i\theta})}{re^{i\theta}} \cdot ire^{i\theta}\,d\theta = \int_0^{2\pi} i f(\zeta+re^{i\theta})\,d\theta$$

$= 2\pi i \cdot$ average value on circle $\simeq 2\pi i f(\zeta)$

3. Integral of a gradient; If γ is a curve from a to b and f is C^1 on γ, then

$$f(b) - f(a) = \int_\gamma \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = \int_\gamma \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}$$

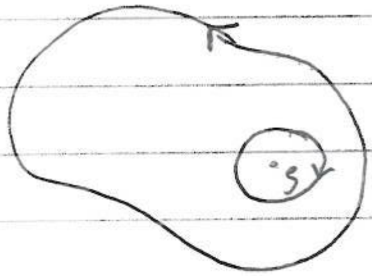
If f is holomorphic, then $f(b) - f(a) = \int_\gamma f'(z)\,dz$

If $|f'(z)| \leq M$ on γ, then $|f(b) - f(a)| \leq M \cdot \ell(\gamma)$.

Cauchy - Stokes formula

Assume f is C^1 in $\overline{\Omega}$, as in Green's theorem and let $\zeta \in \Omega$. For small r, let

$\Omega_r = \Omega \setminus \overline{D}(a,r)$. Then $\partial\Omega_r = \partial\Omega \cup \partial D(a,r)$ where $\partial D(a,r)$ is oriented ~~counter~~clockwise.



Applying the complex form of Green's theorem to $\dfrac{f(z)}{z-\zeta}$ in Ω_r, we get

$$\int_{\partial\Omega} \frac{f(z)}{z-\zeta}\, dz - i \int_0^{2\pi} f(\zeta + re^{i\theta})\, d\theta = 2i \iint_{\Omega_r} \frac{\partial f/\partial \bar z}{z-\zeta}\, dx\, dy$$

$$\downarrow r \to 0 \qquad\qquad \downarrow r \to 0$$

$$2\pi i\, f(\zeta) \qquad\qquad 2i \iint_{\Omega} \frac{\partial f/\partial \bar z}{z-\zeta}\, dx\, dy$$

(In the limit to the right, we have used the fact that $\dfrac{1}{z-\zeta}$ has a finite integral over Ω, i.e. is integrable, see Lemma 2 on page 99 of Narasimhan). This proves:

Theorem If f is C^1 in $\overline{\Omega}$ and $\zeta \in \Omega$ then

$$f(\zeta) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z-\zeta}\, dz - \frac{1}{\pi} \iint_{\Omega} \frac{\partial f/\partial \bar z}{z-\zeta}\, dx\, dy$$

In particular, if f is holomorphic, we get Cauchy's formula

$$f(\zeta) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z-\zeta}\, dz$$

Another particular case is if $f \in C^1(\mathbb{C})$ has compact support, then

$$f(\zeta) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\partial f/\partial \bar z}{z-\zeta}\, dx\, dy \quad \text{for all } \zeta \in \mathbb{C}.$$

3. Some consequences of the integral formulas

The first integral in the previous theorem is defined for all $f \in C(\partial\Omega)$. It is called the Cauchy integral of f. It is actually holomorphic for any curve:

Proposition 3.1 Let $\gamma \subset \mathbb{C}$ be a piecewise smooth (C^1) curve and let $f \in C(\gamma)$. Then the function

$$\tilde{f}(\zeta) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-\zeta}\, dz$$

is holomorphic in $\mathbb{C} \setminus \gamma$. Moreover, $\tilde{f}$ is C^∞ smooth, $\tilde{f}'$ is holomorphic in $\mathbb{C} \setminus \gamma$ and

$$\tilde{f}^{(k)}(z) = \frac{k!}{2\pi i} \int_\gamma \frac{f(z)}{(z-\zeta)^{k+1}}\, dz$$

Def. We say that a sequence of functions $\{f_n\}_{n=1}^\infty$ on Ω converges uniformly on compacts in Ω if there is a function f such that for any compact set $K \subset \Omega$ and $\varepsilon > 0$ there is an integer N $(= N(K, \varepsilon))$ such that

$$|f_n(z) - f(z)| < \varepsilon \quad \text{for all } n \geq N \text{ and } z \in K.$$

Proposition 3.2. Let $f_n \in \mathcal{O}(\Omega)$ and assume that $f_n \to f$ uniformly on compacts in Ω. Then $f \in \mathcal{O}(\Omega)$ and $f_n^{(k)} \to f^{(k)}$ uniformly on compacts in Ω for any $k \in \mathbb{N}$.

Proof: Enough to prove on closed discs $\overline{D}(a,r) \subset \Omega$. This follows since f is given by an integral formula in $D(a,r)$ as in the previous proposition.

Definition 3.3 We say that a function f on Ω is analytic if f is given by a power series in all discs in Ω, i.e. if $D(a,r) \subset \Omega$, then
$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \quad \text{for all } z \in D(a,r)$$

Proposition 3.4 If f is analytic in Ω then $f \in \mathcal{O}(\Omega)$.

Proof: Enough to prove that f is holomorphic in some disc $D(a,t)$ for all $a \in \Omega$. For simplicity of notation, assume $a = 0$ and that $D_r = \{|z| < r\} \subset \Omega$. If $0 < t < s < r$, then there exists $M > 0$ such that $|c_j s^j| < M$ for all $j \in \mathbb{N}$. Then for all $z \in \overline{D}_t$ we have
$$\left|\sum_{j=0}^{\infty} c_j z^j\right| \le \sum_{j=0}^{\infty} |c_j s^j| \left(\frac{t}{s}\right)^j \le M \sum_{j=0}^{\infty} \left(\frac{t}{s}\right)^j$$

The geometric series on the right converges. This shows that f is the limit of a sequence of polynomials on $\overline{D}_t$, hence f is holomorphic in D_t by proposition 3.2

Proposition 3.5 (Cauchy estimates) If $f \in \mathcal{O}(D_r) \cap C(\overline{D}_r)$ then
$$|f^{(k)}(0)| \le \frac{k!\, \|f\|_{\partial D_r}}{r^k}$$

PROOF. By (3.2) we have that

$$\begin{aligned}|f^{(k)}(0)| &\le \frac{k!}{2\pi}\Big|\int_{b\mathbb{D}_r}\frac{f(z)}{z^{k+1}}dz\Big|\\ &= \frac{k!}{2\pi}\Big|\int_0^{2\pi}\frac{f(re^{it})}{(re^{it})^{k+1}}ire^{it}dt\Big|\\ &\le \frac{k!\cdot\|f\|_{b\mathbb{D}_r}}{r^k}.\end{aligned}$$

□

COROLLARY 3.6. *(Simple Maximum principle for a disk) Let $f \in \mathcal{O}(\mathbb{D}_r)\cap C(\overline{\mathbb{D}}_r)$. Then $|f(0)| \le \|f\|_{b\mathbb{D}_r}$.*

THEOREM 3.7. *(Montel) Let $\Omega \subset \mathbb{C}$ be an open set, and $\mathcal{F}$ be a family of holomorphic functions on Ω with the property that for each compact set $K \subset \Omega$ there exists a constant $C_K > 0$ such that $\|f\|_K \le C_K$ for all $f \in \mathcal{F}$. Then for any sequence $\{f_j\}_{j\in\mathbb{N}} \subset \mathcal{F}$ there exists a subsequence $\{f_{n(j)}\}$ such that $f_{n(j)} \to f \in \mathcal{O}(\Omega)$ uniformly on compact subsets of Ω.*

PROOF. Let $A \subset \Omega$ be a dense sequence of points, and let $\{f_j\} \subset \mathcal{F}$ be a sequence such that $f_j(a) \to \tilde{a} \in C$ for all $a \in A$. We claim that the sequence $\{f_j\}$ converges to a holomorphic function f uniformly on compact subsets of Ω. Choose an exhaustion of Ω by compact sets $K_j \subset K^{\circ}_{j+1}$. For any j we have that $\|f_i\|_{K_j} \le M_j$ for all i. By the Cauchy estimates there is a constant N_j such that $\|f_i'\|_{K_j} < N_j$ for all i.

Now we fix K_j and show that $\{f_i\}|_{K_j}$ is a Cauchy sequence. Note that by the Mean Value Theorem we have for $z, z' \in K_{j+1}$ that $|f_i(z) - f_i(z')| \le N_{j+1}|z - z'|$. Given any $\epsilon > 0$ we may choose a finite subset $\tilde{A} \subset K_{j+1}$ of A such that for any $z \in K_j$, there exists an $a \in \tilde{A}$ with $|z-a| < \frac{\epsilon}{4N_{j+1}}$. Furthermore, since $\{f_i\}|_{\tilde{A}}$ is Cauchy, we may find $N \in \mathbb{N}$ such that $|f_l(a) - f_m(a)| < \frac{\epsilon}{2}$ for all $m, n \ge N$. So given any $z \in K_j$ we may pick $a \in \tilde{A}$ to see that

$$\begin{aligned}|f_l(z) - f_m(z)| &\le |f_l(z) - f_l(a)| + |f_l(a) - f_m(a)| + |f_m(a) - f_m(z)|\\ &\le 2N_{j+2}|z-a| + \epsilon/2 < \epsilon,\end{aligned}$$

for all $l, m \ge N$, hence $\{f_i\}|_{K_j}$ is a Cauchy sequence. □

THEOREM 3.8. *Let $f \in \mathcal{O}(\Omega)$* and $\overline{D}(a,r) \subset \Omega$. Then

$$f(\zeta) = \sum_{j=0}^{\infty} c_j (\zeta - a)^j \quad \text{in } D(a,r) \tag{3.5}$$

where

$$c_j = \frac{1}{2\pi i}\int_{b\mathbb{D}_r}\frac{f(z)}{(z-a)^{j+1}}dz \tag{3.6}$$

PROOF. Note that $\frac{1}{z-\zeta} = \frac{1}{z(1-\zeta/z)} = 1/z\sum_{j=0}^{\infty}(\frac{\zeta}{z})^j$ as long as $|\zeta| < |z|$, and plug this into Cauchy's Integral Formula. $\square$

PROPOSITION 3.9. *(Identity principle) Let $f \in \mathcal{O}(\Omega)$. If $Z(f) = \{z \in \Omega : f(z) = 0\}$ has non-empty interior, then $f \equiv 0$ on Ω.* (Ω connected

PROOF. For each $a \in \Omega$ we have that $f(z) = \sum_{j=0}^{\infty} c_j(a)(z-a)^j$ on a small enough disk centered at a. By (3.6) we see that $c_j(a)$ is continuous in a for all j. So the set of points $\{a \in \Omega : c_j(a) = 0 \text{ for all } j \in \mathbb{N}\}$ is non-empty, open and closed in Ω. $\square$

PROPOSITION 3.10. *Let $f \in \mathcal{O}(\Omega)$. Then $Z(f)$ is discrete unless f is constantly equal to zero.*

PROOF. We assume that f is not constant. Near a point $a \in \Omega$ with $f(0) = 0$ we have that $f(z) = \sum_{j=k}^{\infty} c_j(z-a)^j, k \geq 1, c_k \neq 0$, so we can write $f(z) = (z-a)^k(c_k + \sum_{j=1}^{\infty} c_{k+j}(z-a)^j)$. $\square$

<u>Def.</u> $\mathcal{O}^*(\Omega) = \{ f \in \mathcal{O}(\Omega) ; f(z) \neq 0 \quad \forall z \in \Omega \}$

<u>Th.</u> $D = D(a, r)$ disc. If $f \in \mathcal{O}(D)$, then f has a holomorphic antiderivative, i.e there is $F \in \mathcal{O}(D)$ such that $F' = f$. If $f \in \mathcal{O}^*(D)$, then f has a holomorphic logarithm and m-th root of any order.

Pf: We know that $f = \sum_{n=0}^{\infty} c_n (z-a)^n$ in D.
Let $F = \sum_{n=0}^{\infty} \frac{c_n}{n+1} (z-a)^{n+1}$.

If $f \in \mathcal{O}^*(D)$, then $\frac{f'}{f} \in \mathcal{O}(D)$ and there is $F \in \mathcal{O}(D)$ such that $F' = \frac{f'}{f}$. Then $g = fe^{-F} \in \mathcal{O}^*(D)$ and $g' = f'e^{-F} + f \cdot e^{-F} \cdot \left(-\frac{f'}{f}\right) = 0$, hence $g = c \neq 0$, a constant. Pick $\alpha \in \mathbb{C}$ such that $e^{\alpha} = c$. Then $f = e^{F+\alpha}$, ~~so~~ $G = F + \alpha$ is a holomorphic logarithm and $e^{\frac{1}{m}G}$ is a holomorphic m-th root for any $m \in \mathbb{N}$.

· Remark: This result is true in any simply connected domain Ω.

· <u>Theorem 3.11</u> If Ω is a domain and $f \in \mathcal{O}(\Omega)$ is nonconstant, then $f(\Omega)$ is open.

Pf: Pick $a \in \Omega$. We have to show that $f(\Omega)$ contains a nbh of $f(a)$. We may assume $a = 0 = f(a)$. Ω contains a disc $D = D(0, r)$ and f is not constant in D. If $f(D)$ does not contain a nbh of 0, there exist $a_j \to 0$ such that $f(z) \neq a_j$ in D, i.e $g_j = \frac{1}{f - a_j} \in \mathcal{O}(D)$. If $r' < r$ is such that

$f(z) \neq 0$ for all z with $|z| = r$, then $|g_j|$ is uniformly bounded on this circle, but $|g_j(0)| = 1/|a_j| \to \infty$ as $j \to \infty$. This contradicts the maximum principle on a disc.

<u>Cor 3.12</u> (Maximum principle) If Ω is a domain, $f \in \mathcal{O}(\Omega)$ and $a \in \Omega$ such that $|f(z)| \leq |f(a)|$ for all $z \in \Omega$, then f is constant.

Pf: Follows from Open Mapping Th.

<u>Prop 3.13</u> (Hurwitz theorem). If Ω is a domain, $f_j \in \mathcal{O}^*(\Omega)$ and $f_j \to f$ uniformly on compacts then either $f \in \mathcal{O}^*(\Omega)$ or $f \equiv 0$ in Ω.

Pf: If $f(a) = 0$ and $f \not\equiv 0$, pick $r > 0$ such that $f(z) \neq 0$ when $|z-a| = r$. Then $|f(z)| \geq \delta > 0$ when $|z-a| = r$, hence $|f_j(z)| \geq \frac{1}{2}\delta$ when $|z-a| = r$ for sufficiently large j. Therefore $g_j = \frac{1}{f_j} \in \mathcal{O}(\Omega)$ and $|g_j(z)| \leq \frac{2}{\delta}$ when $|z-a| = r$. But this is impossible, since $g_j(a) = \frac{1}{f_j(a)} \to \infty$ when $j \to \infty$. ∎

- Punctured disc around a : $D^*(a,r) = \{z \in \mathbb{C} \mid 0 < |z-a| < r\}$

- If $a \in \Omega$ and $f \in \mathcal{O}(\Omega \setminus \{a\})$, we say that f has a <u>pole of order $k \in \mathbb{N}$</u> at a if in some punctured disc around a we have

$$f(z) = \frac{g(z)}{(z-a)^k}$$

where $g(z) \neq 0$ in $D^*(a,r)$. We then have

$$f(z) = c_{-k}(z-a)^{-k} + c_{-k+1}(z-a)^{-k+1} + \cdots \qquad = \sum_{n=-k}^{\infty} c_n (z-a)^n$$

in $D^*(a,r)$.

- The <u>residue of f at a</u> is defined by

$$\operatorname{res}_a f = c_{-1}$$

In $D^*(a,r)$ we then have

$$f(z) = \frac{c_{-1}}{z-a} + \frac{d}{dz}\Big(\sum_{\substack{n=-k \\ n \neq -1}}^{\infty} \frac{c_n}{n+1}(z-a)^{n+1} \Big)$$

Hence for $r' < r$ we have

$$\int_{|z-a|=r'} f(z)\,dz = 2\pi i \, c_{-1} = 2\pi i \operatorname{res}_a f.$$

- <u>Prop 3.14</u> If $\Omega \subset\subset \mathbb{C}$ has piecewise smooth C^1 boundary, $f \in \mathcal{O}(\Omega) \cap C^1(\bar{\Omega})$, except for poles $a_1, \dots, a_N \in \Omega$ then

$$\frac{1}{2\pi i} \int_{\partial\Omega} f\,dz = \sum_{i=1}^{N} \operatorname{res}_{a_i} f.$$

(This is called the residue theorem).

Proof: Let $D_1, \cdots, D_N$ be disjoint small discs around $a_1, \cdots, a_N$ and put $\Omega' = \Omega \setminus \bigcup_{j=1}^{N} \overline{D_j}$. Then Cauchy's theorem gives

$$0 = \frac{1}{2\pi i}\int_{\partial\Omega'} f\,dz = \frac{1}{2\pi i}\int_{\partial\Omega} f\,dz - \sum_{j=1}^{N} \frac{1}{2\pi i}\int_{\partial D_j} f\,dz =$$

$$\frac{1}{2\pi i}\int_{\partial\Omega} f\,dz - \sum_{j=1}^{N} \operatorname{res}_{a_j} f$$

- Def. $f \in \mathcal{O}(\Omega \setminus \{a\})$ has order k at a if

$$f(z) = (z-a)^k g(z)$$

where $g \in \mathcal{O}(\Omega)$ and $g(a) \neq 0$.

$k > 0$: Zero of order k

$k < 0$: Pole of order $-k$.

It follows that

$$\frac{f'}{f} = \frac{k}{z-a} + \frac{g'}{g} \quad \text{near } a,$$

hence $\operatorname{res}_a \frac{f'}{f} = k = \operatorname{ord}_a f$

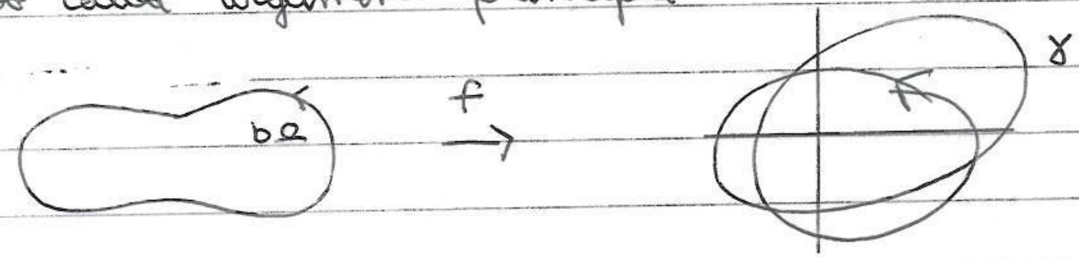
- Corollary If $\Omega \subset\subset \mathbb{C}$ as above, $f \in \mathcal{O}(\Omega) \cap C^1(\overline{\Omega})$ with $f(z) \neq 0$ on $\partial\Omega$, then

$$\int_{\partial\Omega} \frac{f'}{f}\,dz = 2\pi i \sum_{a \in \Omega} \operatorname{ord}_a f$$

If f only has simple zeroes and poles, this is

$\#$zeroes $-$ $\#$poles.

Also called argument principle



$$\int_{b\Omega} \frac{f'}{f}\, dz = \int_{\gamma} \frac{1}{z}\, dz = 2\pi i \cdot \text{winding number}.$$

This is still true if f has poles in Ω.

Theorem 3.15 (Rouché's theorem) $\Omega \subset\subset \mathbb{C}$ as above, $f, g \in \mathcal{O}(\Omega) \cap C^1(\bar{\Omega})$ such that $|f(z) - g(z)| < |f(z)|$ for all $z \in b\Omega$. Then f and g have the same number of zeroes in Ω, i.e.

$$\sum_{z \in \Omega} \mathrm{ord}_z f = \sum_{z \in \Omega} \mathrm{ord}_z g.$$

Pf: Clearly f has no zeroes on $b\Omega$ and $\left| 1 - \frac{g(z)}{f(z)} \right| < 1$ on $b\Omega$, so $F = \frac{g}{f}$ takes values in the disk $D(1,1)$ on $b\Omega$ and therefore has a holomorphic logarithm near $b\Omega$. We have

$$(\log F)' = \frac{F'}{F} = \frac{\frac{g'f - f'g}{f^2}}{\frac{f}{g}} = \frac{g'}{g} - \frac{f'}{f}$$

Hence

$$0 = \int_{b\Omega} (\log F)' dz = \int_{b\Omega} \frac{g'}{g} - \int_{b\Omega} \frac{f'}{f} = \sum_{z \in \Omega} \mathrm{ord}_z g - \sum_{z \in \Omega} \mathrm{ord}_z f.$$

<u>Prop 3.17</u> If Ω is a domain, $f_j \in \mathcal{O}(\Omega)$ are injective for all j and $f_j \to f$ uniformly on compacts, then either f is injective or f is constant.

Proof: Assume $a, b \in \Omega$ and $f(b) = f(a)$. Let $g_j(z) = f_j(z) -$
Then $g_j \in \mathcal{O}^*(\Omega \setminus \{a\})$ and $g_j \to f - f(a)$ uniformly on compacts
Then either $f - f(a)$ is constant, which must be zero, so $f \equiv f(a)$ or $f - f(a)$ is without zeroes, which contradicts the fact that $f(b) = f(a)$.

<u>Prop 3.18+19</u> If $f \in \mathcal{O}(\Omega)$ is injective, then $f'(z) \neq 0$ for all $z \in \Omega$ and f has a holomorphic inverse $f^{-1} \in \mathcal{O}(f(\Omega))$.

Proof: We may assume $z = 0$ and $f(z) = 0$. We shall show that f has a zero of order 1 at 0. We have that $f(z) = z^k g(z)$ with $g \in \mathcal{O}(\Omega)$, $g(0) \neq 0$, $k \in \mathbb{N}$.
In a disc $\mathbb{D}_r$, g has a holomorphic k-th root, i.e. there is $h \in \mathcal{O}(\mathbb{D}_r)$ with $g(z) = h(z)^k$ and $h(0) \neq 0$.
We get $f(z) = (z\, h(z))^k$. The function $z\, h(z)$ is nonconstant, hence open. But then f takes values in a small disc at least k times i $\mathbb{D}_r$. Hence $k = 1$.

By the inverse mapping theorem f has a C^∞ smooth inverse $f^{-1}: f(\Omega) \to \Omega$. The derivative df^{-1} is the inverse of df, hence it is complex linear and f^{-1} is holomorphic.

- $A(r, s) = \{\zeta \in \mathbb{C} \mid r < |\zeta| < s\}$, $0 \leq r < s \leq \infty$

- Prop 3.20 (Laurent expansion) If $f \in \mathcal{O}(A(r,s))$ then f has a unique Laurent series expansion in $A(r,s)$

$$f(\zeta) = \sum_{j=-\infty}^{\infty} c_j \zeta^j$$

where $c_j = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{f(z)}{z^{j+1}} dz$, any $\rho \in (r, s)$. The series $\sum_{j \geq 0} c_j \zeta^j$ converges for $|\zeta| < s$ and the series $\sum_{j<0} c_j \zeta^j$ converges for $|\zeta| > r$.

Proof: The Cauchy theorem gives that $\int_{|z|=\rho} \frac{f(z)}{z^{j+1}} dz$ is independent of $\rho \in (r, s)$. Let $\zeta \in A(r,s)$ and pick r', s' such that

$$r < r' < |\zeta| < s' < s.$$

By the Cauchy-Stokes formula, we have

$$f(\zeta) = \frac{1}{2\pi i} \int_{|z|=s'} \frac{f(z)}{z-\zeta} dz - \frac{1}{2\pi i} \int_{|z|=r'} \frac{f(z)}{z-\zeta} dz$$

$$= \frac{1}{2\pi i} \int_{|z|=s'} \frac{f(z)}{z} \frac{1}{1-\frac{\zeta}{z}} dz + \frac{1}{2\pi i} \int_{|z|=r'} \frac{f(z)}{\zeta} \cdot \frac{1}{1-\frac{z}{\zeta}}$$

$$= \quad \mathrm{I} + \mathrm{II}.$$

$$I = \frac{1}{2\pi i} \int_{|z|=s'} \frac{f(z)}{z} \cdot \sum_{j=0}^{\infty} \left(\frac{\zeta}{z}\right)^j dz = \sum_{j=0}^{\infty} \left(\frac{1}{2\pi i} \int_{|z|=s'} \frac{f(z)}{z^{j+1}} \right) \zeta^j$$

$$II = \frac{1}{2\pi i} \int_{|z|=r'} \frac{f(z)}{\zeta} \cdot \sum_{j=0}^{\infty} \left(\frac{z}{\zeta}\right)^j = \sum_{j=0}^{\infty} \left(\frac{1}{2\pi i} \int_{|z|=r'} f(z) z^j \right) \zeta^{-(j+1)}$$

$$\overset{j'=-(j+1)}{=} \sum_{j'<0} \left(\frac{1}{2\pi i} \int_{|z|=r'} f(z) z^{-(j'+1)} \right) \zeta^{j'}$$

- <u>Exercise</u> If $r=0$, $A(r,s)$ is the punctured disc $D_s^* = \{\zeta \mid 0 < |\zeta| < s\}$. f has a singularity at 0. There are three types

  (1) Removable singularity: $a_m = 0$ for $m<0$ $\Leftrightarrow$ f is bounded in D_s^*

  (2) Pole of order k: $a_{-k} \neq 0$, $a_m = 0$ for $m < -k$ $\Leftrightarrow$ $|f| \to \infty$ when $z \to 0$

  (3) Essential singularity: $a_m \neq 0$ for infinitely many $m < 0$

  $\Updownarrow$

  $f(D_t^*)$ is dense in $\mathbb{C}$ for all $0 < t \le s$.

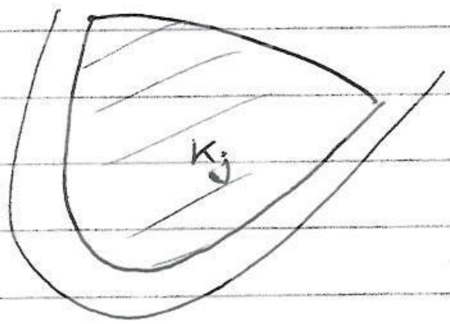
- <u>Liouville's theorem</u> If $f \in \mathcal{O}(\mathbb{C})$ is bounded, then f is constant.

Follows easily from Cauchy estimate of f'.

Partitions of unity

- If $U \subset \mathbb{R}^n$ is open, then there exist an exhaustion $\{K_j\}_{j=1}^{\infty}$ of U by compacts such that
$$K_j \subset K_{j+1}^{\circ} \quad , \quad \bigcup_j K_j = U$$

Proof: If $U = \mathbb{R}^n$ this is trivial. If not, let
$$K_j = \{z \in U ; d(z, \mathbb{R}^n \setminus U) \geq 1/j\} \cap \overline{B}(j)$$



- We say that a family $\mathcal{F}$ of subsets of $\mathbb{R}^n$ is locally finite if every $a \in \mathbb{R}^n$ has a nbh. $B(a,r)$ such that $B(a,r) \cap E \neq \emptyset$ for only a finite number of sets $E \in \mathcal{F}$.
This is equivalent to $K \cap E \neq \emptyset$ for only a finite number of sets $E \in \mathcal{F}$ for any compact K.

- Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a collection of open sets. We say that $\mathcal{V} = \{V_j\}_{j \in J}$ is a refinement of $\mathcal{U}$ if for each V_j there is a U_i with $V_j \subset U_i$ and $\bigcup_{j \in J} V_j = \bigcup_{i \in I} U_i$.

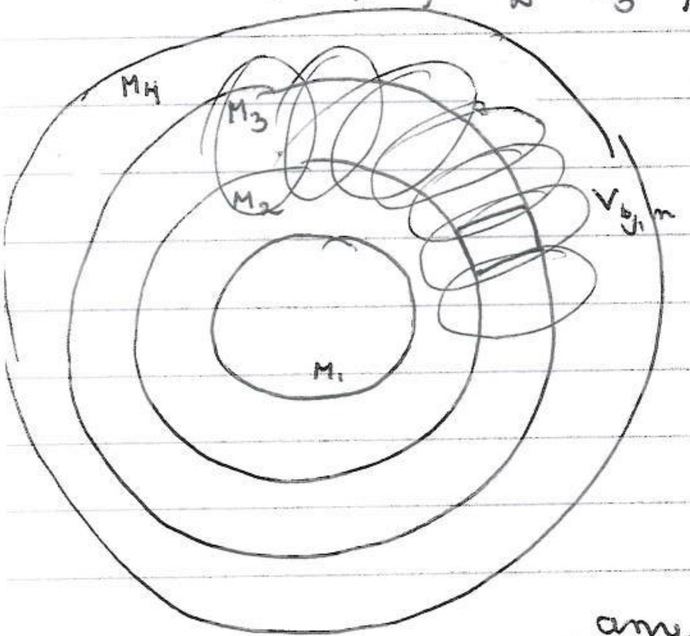
- If $\mathcal{U} = \{U_i\}$ is an open covering of U (i.e. $U = \bigcup U_i$) then there is a locally finite refinement $\mathcal{V} = \{V_j\}$ of $\mathcal{U}$ and compacts $L_j \subset V_j$ such that $\bigcup_{j \in J} L_j = U$.

Proof: Let $\{K_n\}_{n=1}^{\infty}$ be an exhaustion of U. We shall divide U into compact "rings" M_n like this:

- $M_1 = K_1$, $M_{n+1} = K_{n+1} \setminus K_n^{\circ}$, so $\bigcup_{n=1}^{\infty} M_n = U$.

We then define open sets W_n containing M_n which can only intersect the previous and next ring:

- $W_1 = K_2^{\circ}$, $W_2 = K_3^{\circ}$, $W_n = K_{n+1}^{\circ} \setminus K_{n-2}$ for $n \geq 3$



Now $\mathcal{V}_n = \{ V_{i,n} = U_i \cap W_n \}$ is an open cover of M_n and there exist $V_{i_j,n} \in \mathcal{V}_n$, $j = 1, \dots, p_n$ which cover M_n. Then there is some δ $(= \delta(n))$ such that for any $x \in M_n$ there is some i_j such that $B(x, \delta) \subset V_{i_j,n}$. This gives that the compacts

$$L_{i_j,n} = \{ x \in M_n \mid d(x, \mathbb{R}^n \setminus V_{i_j,n}) \geq \delta \} \subset V_{i_j,n}$$

cover M_n. Now, let

$$\mathcal{V} = \{ V_{i_j,n} \mid n \in \mathbb{N} ; j = 1, \dots, i_n \}$$

$\mathcal{V}$ is a refinement of $\mathcal{U}$ and since any compact K is contained in some K_n and therefore will not intersect any $V_{i_j,m}$ when $m > n+1$, it is locally finite. The corresponding $L_{i_j,n}$ cover M_n and hence U.

- If ϕ is a function defined on U, we define

$$\operatorname{supp} \phi = \overline{\{x \,;\, \phi(x) \neq 0\}}^{U}$$

i.e. we are taking the closure in U.

- $C_0^\infty(U) = \{\phi \in C^\infty(U) ;\ \phi$ is real and $\operatorname{supp}\phi$ is a compact subset of $U\}$

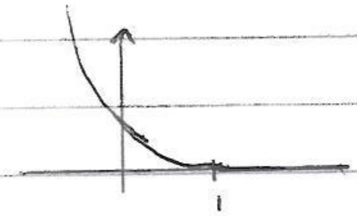
- <u>Def.</u> <u>Partition of unity relative to $\mathcal{U}$</u>

If $\mathcal{U} = \{U_i\}_{i \in I}$ is an open cover of U, then a partition of unity relative to $\mathcal{U}$ is a family $\phi_i \in C^\infty(U)$ such that

  - $\phi_i \geq 0$, $S_i = \operatorname{supp} \phi_i \subset U_i$ (support taken in U)
  - S_i is locally finite
  - $\sum_i \phi_i \equiv 1$ in U.

<u>Lemma 1</u> If U is open, $K \subset U$ is compact, then there is a positive function $\phi \in C_0^\infty(U)$ such that $\phi(x) > 0$ for all $x \in K$.

Proof: The function $\psi(t) = \begin{cases} e^{-1/1-t} & t \leq 1 \\ 0 & t \geq 1 \end{cases}$ is in $C^\infty(\mathbb{R}$



There exist $\delta > 0$ such that $\mathrm{dist}(K, \mathbb{R}^m \setminus U) \geq 2$

There are a finite number of points $a_1, \cdots, a_N \in K$ such that $K \subset \bigcup_{i=1}^N B(a_i, \delta)$. Let

$$\phi(x) = \sum_{i=1}^m \psi\left(\frac{|x-a_i|^2}{\delta^2}\right)$$

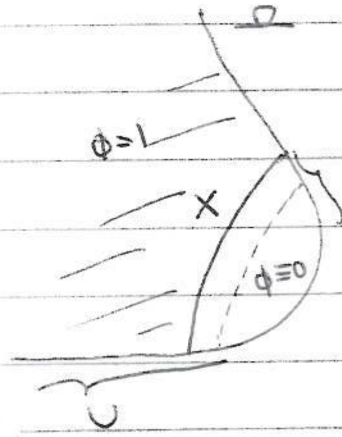
Theorem 1. If $\mathcal{U} = \{U_i\}_{i \in I}$ is an open cover of U, then there is a partition of unity relative to U.

Proof: Let $V = \{V_j\}_{j \in J}$ be a locally finite refinement of $\mathcal{U}$ and $K_j \subset V_j$ compact which cover U. Then there are $\psi_j \in C_0^\infty(V_j) \subset C^\infty(U)$ such that $\psi_j > 0$ in K_j.

Let $\psi = \sum_j \psi_j$. This sum is locally finite, hence $\psi \in C^\infty(U)$ and $\psi > 0$ in U. If we let $\chi_j = \psi_j / \psi$, then χ_j is a partition of unity relative to V_j. For each $j \in J$ pick $\tau(j) \in I$ such that $V_j \subset U_{\tau(j)}$ and for each $i \in I$ define $\phi_i = \sum_{j \in \tau^{-1}(i)} \chi_j \in C^\infty(U)$. Clearly, $\{\mathrm{supp}\, \phi_i\}$ is locally finite.

If $x \in U \setminus U_i$ there is a nbh V of x such that $V \cap \mathrm{supp}\, \chi_j \neq \emptyset$ for only finitely many j. If $j \in \tau^{-1}(i)$, then $\mathrm{supp}\, \chi_j$ is a compact subset of U_i, hence $\phi_i \equiv 0$ in $V \setminus \bigcup_{j \in \tau^{-1}(i)} \mathrm{supp}\, \chi_j$ and $x \notin \mathrm{supp}\, \phi_i$. This proves that $\mathrm{supp}\, \phi_i \subset U_i$.

Theorem 2 (Separation of closed sets) If $\Omega \subset \mathbb{R}^n$ is open, $X \subset \Omega$ closed (relatively), $X \subset U$ open, then there exist $\phi \in C^\infty(\Omega)$, $0 \le \phi \le 1$, $\phi|_X = 1$, $\phi|_{\Omega \setminus U} = 0$



Proof: Let ϕ_U, ϕ_V be a partition of unity relative to the covering $\{U, V = \Omega \setminus X\}$

Must have $\phi_V|_X = 0 \Rightarrow \phi_U = 1$ on X

Also $\phi_U = 0$ in $\Omega \setminus U$.

Patching C^∞ functions on disjoint closed sets

Theorem 3. If $\Omega \subset \mathbb{R}^n$ is open, $X_1, X_2 \subset \Omega$ two disjoint closed sets and $\phi_1, \phi_2 \in C^\infty(\Omega)$. Then there exist $\phi \in C^\infty(\Omega)$ such that $\phi|_{X_1} = \phi_1$, $\phi|_{X_2} = \phi_2$

Proof: Pick $\alpha \in C^\infty(\Omega)$, $0 \le \alpha \le 1$, $\alpha|_{X_1} = 1$, $\alpha|_{X_2} = 0$ and let

$$\phi = \alpha \phi_1 + (1-\alpha)\phi_2.$$

The $\bar{\partial}$ - equation , $\frac{\partial u}{\partial \bar{z}} = \phi$.

Recall Cauchy - Stokes formula in $\Omega \subset \mathbb{C}$ $(z = x+iy, \zeta = \xi + i\eta)$

- $f \in C^1(\bar{\Omega})$, $z \in \Omega \Rightarrow f(z) = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{\pi}\iint_{\Omega}\frac{\partial f/\partial\bar{\zeta}}{\zeta - z}\,d\xi\,d\eta$

- f also holomorphic in Ω : $\quad f(z) = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(\zeta)}{\zeta - z}\,d\zeta$

- $f \in C_0^1(\mathbb{C})$, $z \in \mathbb{C} \Rightarrow f(z) = -\frac{1}{\pi}\iint_{\mathbb{C}}\frac{\partial f/\partial\bar{\zeta}}{\zeta - z}\,d\xi\,d\eta$

Given $\phi \in C_0^1(\mathbb{C})$, we want to find f such that

$$\frac{\partial f}{\partial \bar{z}} = \phi$$

It is natural to try

$$f(z) = -\frac{1}{\pi}\iint_{\mathbb{C}}\frac{\phi(\zeta)}{\zeta - z}\,d\xi\,d\eta = -\frac{1}{\pi}\iint_{\mathbb{C}}\frac{\phi(\zeta + z)}{\zeta}\,d\xi\,d\eta$$

If we can diff. under sign of integration

$$\frac{\partial f}{\partial \bar{z}}(z) = -\frac{1}{\pi}\iint_{\mathbb{C}}\frac{\partial\phi/\partial\bar{z}\,(\zeta+z)}{\zeta}\,d\xi\,d\eta = \phi(z)$$

Differentiation is allowed. Diff. with respect to x, let $h \in \mathbb{R}$

$$\frac{f(z+h)-f(z)}{h} = -\frac{1}{\pi}\iint_{\mathbb{C}} \frac{\frac{1}{h}\left[\phi(\zeta+z+h)-\phi(\zeta+z)\right]}{\zeta}\, d\xi\, d\eta \xrightarrow[h]{\text{dom. conv.}} -\frac{1}{\pi}\iint_{C}\frac{\partial\Phi/\partial x}{\zeta}$$

$\frac{1}{\zeta} \in L^1_{loc}(\mathbb{R}^2)$

Can do the same in y direction.
Hence we have proved

Theorem 2 If $\phi \in C_0^\infty(\mathbb{C})$ and

$$f(z) = -\frac{1}{\pi}\iint_{C} \frac{\phi(\zeta)}{\zeta - z}\, d\xi\, d\eta$$

then $f \in C^\infty(\mathbb{C})$ and $\frac{\partial f}{\partial \bar{z}} = \phi$

- Notice that in general f does not have compact support since for large R

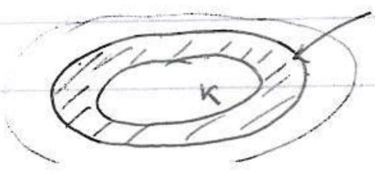
$$0 = \int_{|z|=R} f\, dz = 2i\iint_{|z|\le R}\frac{\partial f}{\partial \bar{z}}\, dx\, dy = 2i\iint_{|z|\le R}\phi\, dx\, dy \Rightarrow \int_{\mathbb{C}}\phi\, dx\, dy = 0.$$

Theorem 3 (Smeared out Cauchy integral formula)
If $K \subset \Omega$ is compact, $f \in \mathcal{O}(\Omega)$ and $\alpha \in C_0^\infty(\Omega)$ is $\equiv 1$ on K, then for $z \in K$

$$f(z) = \frac{-1}{\pi}\iint_{\Omega} f(\zeta)\frac{\partial \alpha}{\partial \bar{\zeta}}\,\frac{1}{\zeta - z}\, d\xi\, d\eta$$

In particular $\iint f(\zeta)\frac{\partial\alpha}{\partial\bar{\zeta}}\, d\xi\, d\eta = 0$

Proof: Apply Cauchy-Stokes to $\phi = \alpha f$.



$S = \operatorname{supp}\alpha - K$

Def. Let $K \subset \mathbb{C}$ be compact. Then

$$\mathcal{O}(K) = \{ f \in \mathcal{O}(U_f) \mid U_f \text{ open nbh. of } K \}.$$

Example

$$K = \{ |z| = \tfrac{1}{2} \}$$

$f(z) = z$, $g(z) = \frac{1}{z}$ are both in $\mathcal{O}(K)$.

Runge problem

Let $K \subset \Omega$ be compact and $f \in \mathcal{O}(K)$. Is it possible to approximate f on K by $f_n \in \mathcal{O}(\Omega)$?

Example K as above, f and g also

a) $\Omega = \mathbb{D} = \{ |z| < 1 \}$

$f \in \mathcal{O}(\Omega)$, ~~so~~ no problem. Claim that g cannot be approximated: If $h \in \mathcal{O}(\mathbb{D})$ and $h \sim g$ on K (close) then

$$1 = z g(z) \sim z h(z) \text{ on } K.$$

If $k(z) = z h(z)$ is close to 1 on K, then it also close on $\mathbb{D}_{1/2} = \{ |z| < \frac{1}{2} \}$ by the maximum modulus theorem. But this is not true, since $k(0) = 0$.

b) $\Omega = \mathbb{D}^* = \mathbb{D} \setminus \{0\}$

Then both f and g are in $\mathcal{O}(\Omega)$, ~~so~~ no problem.

The problem in a) is that $\Omega \setminus K$ has a component, $\mathbb{D}_{1/2}$ which is relatively compact in Ω. In b) the corresponding component is $\mathbb{D}_{1/2} \setminus \{0\}$ which is not relatively compact since it goes all the way up to $0 \in \partial \Omega$.

Exercise Let $\Omega \subset \mathbb{C}$ be open, $K \subset \Omega$ compact and U a bounded connected component of $\Omega \setminus K$.
Then the following are equivalent:

1) $\exists\, \delta > 0$ such that $|z - w| \geq \delta$ for all $z \in U$, $w \notin \Omega$.
2) $U \subset\subset \Omega$
3) $\partial U \subset K$
4) U is also a connected component of $\mathbb{C} \setminus K$.

If we negate this, the following are equivalent

1) For all $\delta > 0$ there exist $z \in U$ and $w \notin \Omega$ such that $|z - w| <$
2) $U \not\subset\subset \Omega$
3) $\partial U \cap (\mathbb{C} \setminus K) \neq \emptyset$
4) The connected component U' of $\mathbb{C} \setminus K$ containing U is not contained in Ω, i.e. $U' \cap (\mathbb{C} \setminus \Omega) \neq \emptyset$.

Theorem 1. (Runge) $\Omega \subset \mathbb{C}$ open, $K \subset \Omega$ compact.
The following are equivalent:
(1) $\mathcal{O}(\Omega)|_K$ is dense in $\mathcal{O}(K)$

(2) No connected component of $\Omega \setminus K$ is relatively compact in Ω

(3) $\forall a \in \Omega \setminus K$ there is $f \in \mathcal{O}(\Omega)$ such that $|f(a)| > |f|_K$

Proof:
(1) $\Rightarrow$ (2) If U is a connected component of $\Omega \setminus K$ which is relatively compact in Ω, then $\partial U \subset K$, because otherwise we could attach a disc to $z \in \partial U \setminus K$ to obtain a bigger connected set. If $z_0 \in U$ and $f(z) = \frac{1}{z - z_0} \in \mathcal{O}(K)$, then f cannot be approximated by $f_n \in \mathcal{O}(\Omega)$, because if $\frac{1}{z-z_0} - f_n \to 0$ on K, then $g_n = 1 - (z - z_0) f_n \to 0$ on K, but $g_n(z_0) = 1$, ~~so~~ this violates the maximum modulus theorem since $\partial U \subset K$

(2) $\Rightarrow$ (1) We must prove that every $f \in \mathcal{O}(K)$ can be approximated uniformly on K by $f_n \in \mathcal{O}(\Omega)$.
Pick $f \in \mathcal{O}(W)$ for some open neighbourhood W of K.

Step 1. Approximation of f by rational functions with poles outside K.

Pick $\alpha \in C_0^\infty(W)$ such that $\alpha = 1$ in a nbh W_0 of K
For $z \in K$ we have by Cauchy - Stokes formula

$$f(z) = \frac{1}{\pi} \iint_{\mathbb{C}} f(\zeta) \frac{\partial \alpha}{\partial \bar{\zeta}} \frac{1}{z - \zeta} \, d\xi \, d\eta = \frac{1}{\pi} \iint f(\zeta) \frac{\partial \alpha}{\partial \bar{\zeta}} \frac{1}{z - \zeta} \, d\xi \, d\eta$$

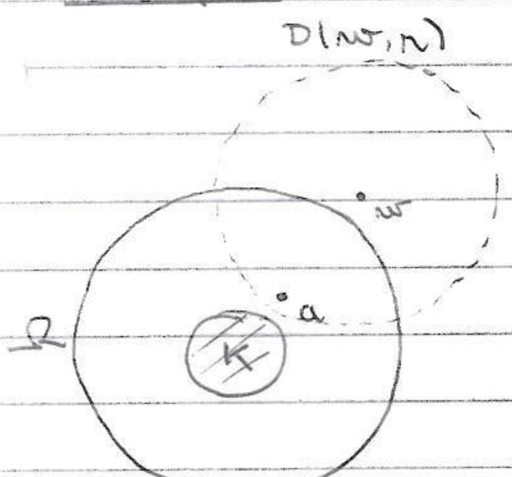
If we subdivide $\mathbb{C}$ by small squares and form the corresponding Riemann sums for the integral,

$$\frac{1}{\pi}\sum_{\nu} f(z_\nu)\frac{\partial\alpha}{\partial\bar{\zeta}}(z_\nu)\,\frac{1}{z-z_\nu}$$

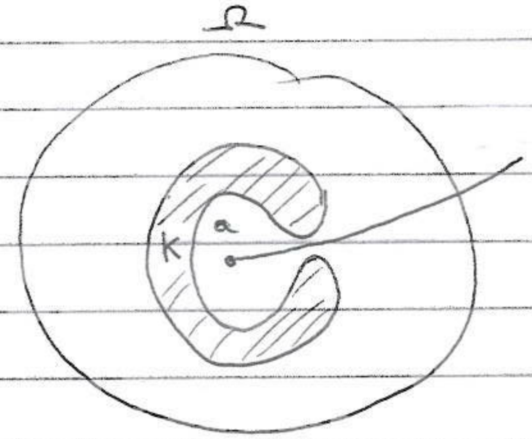
then these Riemann sums will approximate the integrals, uniformly on K, since the integrand is compactly supported, hence uniformly continuous in $\mathbb{C}$. The z_ν's will be close to $L = \operatorname{supp}\alpha \setminus W_0$, hence in $\Omega\setminus K$. It follows that f can be approximated on K by a finite sum $\sum_\nu c_\nu \frac{1}{z-z_\nu}$ with $z_\nu \in \Omega\setminus K$.

Step 2. We now look at terms of the form $\frac{1}{z-a}$ with $a\in\Omega\setminus K$. We shall approximate these by functions which are holomorphic in Ω by "pushing the poles out of Ω".

Examples



$\frac{1}{z-a}$ is holomorphic outside $D(w,r)$ and is given there by a power series in $\frac{1}{z-w}$.



The pole a can be gradually pushed out of Ω.

Therefore, let $a \in \Omega \setminus K$ and let U be the connected component of $\mathbb{C}\setminus K$ containing a. Let
$U_a = \{ w \in U \,;\, \frac{1}{z-a}$ can be approximated on K by polynomials in $\frac{1}{z-w} \}$.

We will show that $U_a = U$. We will show that U_a is both open and closed in $\mathbb{C}\setminus K$.

U_a is open: Suppose $w \in U_a$ and $D(w,r) \cap K = \emptyset$. If P_ε is a polynomial in $\frac{1}{z-w}$ which approximates f on K and $w' \in D(w, r/2)$, then $P_\varepsilon(\frac{1}{z-w})$ is holomorphic outside $\bar{D}(w', r/2)$ and can therefore be developed in a power series in $1/z-w'$ there. A finite sum of this power series will approximate P_ε on the compact $K \subset \mathbb{C}\setminus \bar{D}(w', r/2)$.

U_a is closed in $\mathbb{C}\setminus K$: Assume $w_n \in U_a$ and $w_n \to w \in \mathbb{C}\setminus K$. Then there is a disc $\bar{D}(w,r) \subset \mathbb{C}\setminus K$ and a $w_n \in \bar{D}(w,r)$. $\frac{1}{z-a}$ can be approximated on K by polynomials in $\frac{1}{z-w_n}$. These are holomorphic outside $\bar{D}(w,r)$ and the same argument as above gives that $w \in U_a$.

This proves the claim.
We now prove that $\frac{1}{z-a}$ can be approximated on K by a function which is holomorphic in Ω.

If U_a is bounded, then we claim that $U_a \cap (\mathbb{C}\setminus\Omega) \neq$
Otherwise, $U_a \subset \Omega$ and U_a is a connected component

of $\Omega \setminus K$. But $\partial U_a \subset K$, hence U_a would be relatively compact in Ω, which is impossible. Hence there is some $w \in U_a \setminus \Omega$ and by definition $\frac{1}{z-a}$ can be approximated by a polynomial in $\frac{1}{z-w}$, which is holomorphic in Ω.

If U_a is unbounded, then there is $w \in U_a$ with $|w| > \sup\{|z|;\ z \in K\}$. Let $r = |w|$. In this case a polynomial in $\frac{1}{z-w}$ is holomorphic in the disc $D(0, R)$, hence is given by a power series there, and can be approximated by a polynomial on K.

(3) $\Rightarrow$ (2) is analogous with (1) $\Rightarrow$ (2): If $U \subset\subset \Omega$ is a connected component of $\Omega \setminus K$, then $\partial U \subset K$ and for all $a \in U$ we have by the max. modulus princ

$$|f(a)| \le |f|_{\partial U} \le |f|_K.$$

which contradicts (3).

(2) $\Rightarrow$ (3). If $a \in \Omega \setminus K$, then $L = K \cup \{a\}$ has the same property and by the implication (2) $\Rightarrow$ (1), $\mathcal{O}(\Omega)|_L$ is dense in $\mathcal{O}(L)$. If U and V are disjoint open sets, $K \subset U$, $a \in V$ and ϕ is defined by $\phi = 0$ in U, $\phi = 1$ in V, then $\phi \in \mathcal{O}(L)$, hence there exist $f \in \mathcal{O}(\Omega)$ such that $|f - \phi|_L < \frac{1}{2}$. But then

$$|f|_K < \frac{1}{2} < |f(a)|$$

This completes the proof of the theorem.

- Remark: From the implication (2) ⇒ (1) we that if
    - No connected component of $\Omega \setminus K$ is rel. comp. in Ω
    - $A \subset \mathbb{C}$ is a set which contains at least one point in every bounded component of $\mathbb{C} \setminus \Omega$.
    - $f \in \mathcal{O}(K)$

  then f can be approximated uniformly on K by rational functions with poles in A.

- The polynomials are dense in $\mathcal{O}(\mathbb{C})$. Hence if we let $\Omega = \mathbb{C}$ in Runge's theorem, we get:

<u>Corollary</u> For a compact set $K \subset \mathbb{C}$ the following are equivalent:

(1) Every $f \in \mathcal{O}(K)$ can be approximated by polynomial

(2) $\mathbb{C} \setminus K$ is connected (i.e. K has no holes)

(3) For any $z \notin K$ there is a polynomial P such that $|P(z)| > |P|_K$.

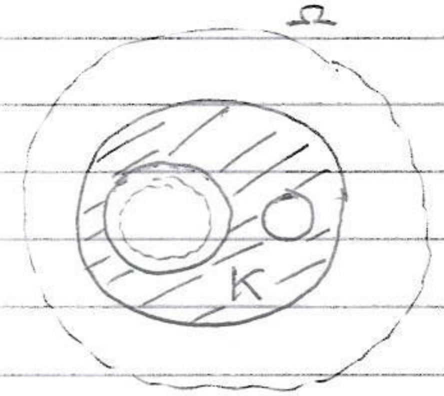
Such K are called <u>polynomially convex</u>.

- <u>Def.</u> Let $K \subset \Omega$ be compact. The <u>holomorphically convex hull</u> of K in Ω is defined by

$$\hat{K}_\Omega = \{ z \in \Omega ; \ |f(z)| \le |f|_K \text{ for all } f \in \mathcal{O}(\Omega) \}.$$

(3) in Runge's theorem states that $\hat{K}_\Omega = K$, in which case we call K <u>holomorphically convex</u> in Ω. We have $\hat{\hat{K}}_\Omega = \hat{K}_\Omega$. We shall see that $\hat{K}_\Omega$ fills in the holes in K which do not contain holes in Ω.

Example.



$\hat{K}_\Omega$ fills in the hole to the right, not the left.

Exercise: - $\hat{K}_\Omega$ does not get closer to $\partial\Omega$ i.e. $d(\hat{K}_\Omega, \partial\Omega) = d(K, \partial\Omega)$.

- $\hat{K}_\Omega$ is compact

Theorem $\hat{K}_\Omega$ is the union of K and all relatively compact components of $\Omega \setminus K$.

Proof: If U is such a component, then $\partial U \subset K$ and therefore $U \subset \hat{K}_\Omega$ by the maximum modulus theorem. This shows that

$$K_1 := K \cup \Big(\bigcup_{U_\alpha \subset\subset \Omega} U_\alpha\Big) \subset \hat{K}_\Omega.$$

Also $\Omega \setminus K_1 = \bigcup_{U_\alpha \not\subset\subset \Omega} U_\alpha$ is open, hence K_1 is closed in Ω

and therefore compact. Also, no components of $\Omega \setminus K_1$ are relatively compact. Runge's theorem gives that any $z \notin K_1$ can be separated from K_1 (and hence K) by a holomorphic function in Ω. This proves that $z \notin \hat{K}_\Omega$, i.e. $\hat{K}_\Omega \subset K_1$.

Lemma If $\Omega \subset \mathbb{C}$ is open, then

$$K_n = \{z \in \Omega;\ d(z, \mathbb{C} \setminus \Omega) \geq \tfrac{1}{n},\ |z| \leq n\}$$

is a holomorphically convex exhaustion of Ω.

Theorem 3 (Classical Runge theorem)
If $\Omega \subset \mathbb{C}$ is open, $A \subset \mathbb{C}$ is a set which contains one point from each bounded component of $\mathbb{C} \setminus \Omega$, then every $f \in \mathcal{O}(\Omega)$ can be approximated uniformly on compacts by rational functions with poles in A.

Proof: Pick $f \in \mathcal{O}(\Omega)$ and a compact set $K \subset \Omega$. Replacing K by $\hat{K}_\Omega$, we may assume that K is holomorphically convex in Ω. The result follows from the remark to Runge's theorem.

Mittag-Lefflers theorem

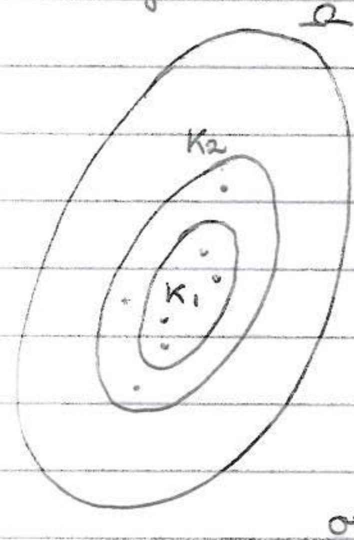
Def. $\mathbb{C}_a^* = \mathbb{C} \setminus \{a\}$, $\mathbb{C}_0^*$ is denoted $\mathbb{C}^*$

If f is holomorphic in a punctured disc around a, we have

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

The negative powers $p_a = \sum_{n=-\infty}^{-1} c_n (z-a)^n$ is called the principal part of f at a. We have $p_a \in \mathcal{O}(\mathbb{C}_a^*)$.

Theorem 1 (Mittag-Leffler) Prescribing principal parts.
If $E \subset \Omega$ is discrete and for every $a \in E$ there is given a principal part $p_a \in \mathcal{O}(\mathbb{C}_a^*)$, then there is $f \in \mathcal{O}(\Omega \setminus E)$ such that $f - p_a$ is holomorphic in a neighbourhood of a for all $a \in E$.



Proof: Let $\{K_n\}$ be a holomorphically convex exhaustion of Ω and put $K_0 = \emptyset$. Let $E_n = E \cap \{K_n \setminus K_{n-1}\}$. E_n is finite. Put

$$g_n = \sum_{a \in E_n}' p_a \in \mathcal{O}(\mathbb{C} \setminus E_n) \supset \mathcal{O}(K_{n-1})$$

Let $f_1 = g_1$. Then $f_1 - p_a$ is holomorphic in a for all $a \in E_1$ and is holomorphic outside K_1. We would like to add g_2, but the problem is convergence. However, since $g_2 \in \mathcal{O}(K_1)$ and K_1 is holomorphically convex, we can find $h_2 \in \mathcal{O}(\Omega)$ such that $|g_2 - h_2|_{K_1} < 2^{-2}$. If we let $f_2 = f_1 + (g_2 - h_2)$, then $f_2 - p_a$ is holomorphic at all $a \in E_1 \cup E_2$. We proceed inductively to find $h_n \in \mathcal{O}(\Omega)$ such that $|g_n - h_n|_{K_{n-1}} < 2^{-n}$. It follows that

$$f = \lim f_n = g_1 + \sum_{n=2}^{\infty} (g_n - h_n) \text{ solves the problem}$$

■

- If every $p_a \in \mathcal{M}(\mathbb{C})$, i.e. only has a pole at a, then $f \in \mathcal{M}(\Omega)$
- Enough to assume $p_a \in \mathcal{O}(D^*(a,r))$ for some $r > 0$.

- Equivalent formulation:

<u>Theorem 1'</u> If $E \subset \Omega$ is discrete, $\Omega = \bigcup_{j \in J} U_j$ and $g_j \in \mathcal{O}(U_j \setminus E)$ such that $g_j - g_k \in \mathcal{O}(U_j \cap U_k)$ for all j, k, then there is $g \in \mathcal{O}(\Omega \setminus E)$ such that $g - g_j \in \mathcal{O}(U_j)$ $\forall j$

$(1') \Rightarrow (1)$ Put $E = \{z_j\}$, $U_j = (\Omega \setminus E) \cup \{z_j\}$ and $g_j = p_{z_j}$

$(1) \Rightarrow (1')$ For $a \in E$ pick $j(a)$ such that $a \in U_{j(a)}$ and let $p_a =$ the principal part of $g_{j(a)}$ at a. This is independent of the choice of $j(a)$. If $g \in \mathcal{O}(\Omega \setminus E)$ such that $g - p_a$ is holomorphic at a for all $a \in E$, then $g - g_j \in \mathcal{O}(U_j)$

In theorem 1', suppose we can find the "holomorphic correction terms", $f_j = g - g_j \in \mathcal{O}(U_j)$ directly. How can we be sure that they patch together to a global g?
We must have

$$f_i + g_i = f_j + g_j \quad \text{in} \quad (U_i \cap U_j) \setminus E$$
$$f_i - f_j = g_j - g_i \quad \text{in} \quad U_i \cap U_j$$

Let $f_{ij} = g_j - g_i \in \mathcal{O}(U_i \cap U_j)$. The existence of f_i follows from:

<u>Theorem 4</u> If $\{U_j\}_{j=1}^{\infty}$ is an open covering of Ω and $f_{ij} \in \mathcal{O}(U_i \cap U_j)$ satisfy the <u>cocycle condition</u>

$$f_{ij} + f_{jk} + f_{ki} = 0 \quad \text{in} \quad U_i \cap U_j \cap U_k$$

for all indices i, j, k. Then there exist $f_j \in \mathcal{O}(U_j)$ such that $f_{ij} = f_i - f_j$ in $U_i \cap U_j$ for all i, j.

- Notice that the cocycle condition implies that $f_{ii} = 0$ and $f_{ji} = -f_{ij}$ for all i, j.
- The argument above shows that Theorem 4 $\Rightarrow$ Theorem 1'
- We shall now prove Theorem 4. ~~We first prove a solution theorem for the $\bar{\partial}$-equation.~~
-

Step 1. We first prove that there are smooth solutions to the problem, i.e. there are $\phi_i \in C^\infty(U_i)$ such that $f_{ij} = \phi_i - \phi_j$ in $U_i \cap U_j$. For this, it is sufficient that $f_{ij} \in C^\infty(U_i \cap U_j)$.

Proof: Let α_i be a partition of unity relative to $\mathcal{U} = \{U_i\}$ and define in U_i:

$$\phi_i = \sum_k \alpha_k f_{ik}$$

This is in $C^\infty(U_i)$, since $\operatorname{supp} \alpha_k \subset U_k$ and the sum is locally finite. In $U_i \cap U_j$ we have

$$\phi_i - \phi_j = \sum_k \alpha_k (f_{ik} - f_{jk}) = \sum_k \alpha_k f_{ij} = f_{ij}.$$

Step 2. We now correct the ϕ_i to make a holomorphic solution. Notice that since $\phi_i - \phi_j$ differ by a holomorphic function on $U_i \cap U_j$, the function

$$\psi(z) = \frac{\partial \phi_i}{\partial \bar{z}} \quad \text{for } z \in U_i$$

is globally defined in Ω. If we can find $u \in C^\infty(\Omega)$ such that

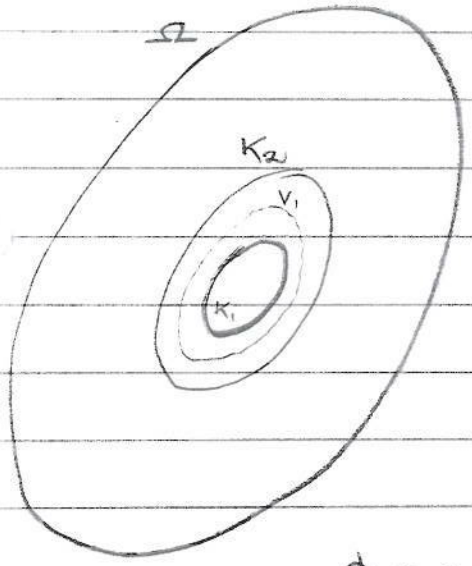
$$\frac{\partial u}{\partial \bar{z}} = \psi$$

then $f_i = \phi_i - u \in \mathcal{O}(U_i)$ and solves the problem. Hence Theorem 4 follows from the following result:

Theorem 2 (Solution of $\bar{\partial}$-equation) If $\psi \in C^\infty(\Omega)$, then there exist $u \in C^\infty(\Omega)$ such that $\frac{\partial u}{\partial \bar{z}} = \psi$.

Proof: Notice that we can solve the equation in a nbh of any compact set $K \subset \Omega$. Just chop off ψ with a smooth function. The solution is in $C^\infty(\mathbb{C})$.

We shall now build the solution inductively as in Mittag-Lefflers theorem. Let $\{K_n\}_{n=1}^\infty$ be a holomorphically convex exhaustion of Ω.



First, solve

$\frac{\partial u_1}{\partial \bar z} = \psi$ in an open nbh V_1 of K_1,

$u_1 \in C^\infty(\mathbb{C})$. We now want to correct u_1 so the equation is satisfied in an open nbh V_2 of K_2. Let $\phi = \psi - \frac{\partial u_1}{\partial \bar z}$. Then $\phi \in C^\infty(\Omega)$ and $\phi = 0$ in V_1. Now solve $\frac{\partial v_2}{\partial \bar z} = \phi$ in V_2,

$v_2 \in C^\infty(\mathbb{C}) \cap \mathcal{O}(V_1)$. $u_1 + v_2$ solves the problem in V_2, but we want the process to converge, so we pick $f_2 \in \mathcal{O}(\Omega)$ such that $|v_2 - f_2|_{K_1} < 2^{-2}$ and let $u_2 = v_2 - f_2$.

Now, proceed to find $u_3, \dots, u_n \in C^\infty(\mathbb{C})$ and open nbh. V_j of K_j, $j = 3, \dots, n$, such that

- $u_j \in \mathcal{O}(V_{j-1})$, $|u_j|_{K_{j-1}} < 2^{-j}$
- $\frac{\partial u_1}{\partial \bar z} + \dots + \frac{\partial u_n}{\partial \bar z_n} = \psi$ in V_n

Then $u = \sum_{n=1}^\infty u_n$ is the required solution. ▮

The winding number

Let γ be a closed piecewise C^1 curve in $\mathbb{C}$. Then for $z \in \mathbb{C} \setminus \gamma$,

$$\operatorname{Ind}(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z}$$

is called the winding number of γ around z. Clearly $\operatorname{Ind}(\gamma, z) \in \mathcal{O}(\mathbb{C} \setminus \gamma)$.

Lemma $\operatorname{Ind}(\gamma, z) \in \mathbb{Z}$.

Pf: Assume γ is parametrized over $[0,1]$, so $\gamma(0) = \gamma(1)$.

Then

$$\frac{d}{dt} \frac{e^{\int_0^t \frac{\gamma'(s)}{\gamma(s) - z} ds}}{\gamma(t) - z} = \frac{e^{\int} \cdot \frac{\gamma'(t)}{\gamma(t) - z} \cdot (\gamma(t) - z) - e^{\int} \cdot \gamma'(t)}{(\gamma(t) - z)^2} = 0.$$

Hence is constant, which must be $\frac{1}{\gamma(0) - z}$. Then

$$e^{\int_0^1 \frac{\gamma'(s)}{\gamma(s) - z} ds} = \frac{\gamma(1) - z}{\gamma(0) - z} = 1, \text{ Hence } \int_0^1 \frac{\gamma'(s)\, ds}{\gamma(s) - z} = 2\pi i \cdot m$$

$m \in \mathbb{Z}$.

- $\operatorname{Ind}(\gamma, z)$ is constant in each connected comp. of $\mathbb{C} \setminus \gamma$; 0 in the unbounded
- Ω is simply connected if any closed curve is homotopic to a constant curve.

TFAE:

(1) Ω is simply connected

(2) Any two curves between two points a and b are homotopic

(3) For any closed curve $\gamma \subset \Omega$ and $z \notin \Omega$, $\operatorname{Ind}(\gamma, z) = 0$.

Two more formulations:

(4) $\mathbb{C} \setminus \Omega$ has no compact components

(5) $\mathbb{P}^1 \setminus \Omega$ is connected

Lemma Suppose $g \in \mathcal{O}^*(\Omega)$. TFAE

(1) g has a holomorphic logarithm in Ω ($e^f = g$)

(2) g'/g has a holomorphic primitive in Ω

(3) $\int_\gamma g'/g \, dz = 0$ for all closed curves in Ω

Proof:

(1) $\Rightarrow$ (2) If $e^f = g$, then $g'/g = f'$

(2) $\Rightarrow$ (1) If $g'/g = f'$, let $h = e^{-f} g$. Then $h' = e^{-f}(g' - f'g) = 0$ hence $h \equiv c$, so $g = ce^f = e^{f+a}$

Eq. of (2) and (3) well known from calculus class.

- Ω simply connected, then g has a holomorphic logarithm because (3) holds.

- Lemma If z_0 and z_1 are in the same component of $\mathbb{C} \setminus K$, then $g(z) = \frac{z - z_0}{z - z_1}$ has a hol. logarithm in a nbh of K. If z_0 is in the unbounded component of $\mathbb{C} \setminus K$, $g(z) = z - z_0$ has a hol. logarithm.

Pf: Pick a nbh Ω of K such that z_0, z_1 are in the same component of $\mathbb{C} \setminus \Omega$. Then

$$\frac{g'(z)}{g(z)} = \frac{1}{z - z_0} - \frac{1}{z - z_1}$$

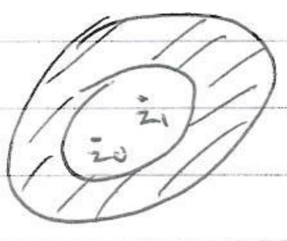
Hence if $\gamma \subset \Omega$ is a closed curve, then

$$\int_\gamma \frac{g'(z)}{g(z)} dz = \int_\gamma \frac{dz}{z - z_0} - \frac{dz}{z - z_1} = \mathrm{Ind}(\gamma, z_0) - \mathrm{Ind}(\gamma, z_1) = 0.$$

For z_0 in the unbounded component, $\frac{g'(z)}{g(z)} = \frac{1}{z - z_0}$, so

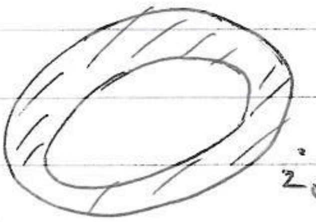
$$\int_\gamma \frac{g'(z)}{g(z)} dz = \int_\gamma \frac{dz}{z - z_0} = \mathrm{Ind}(\gamma, z_0) = 0.$$

Pushing zeroes



$$f(z) = \log \frac{z - z_0}{z - z_1} \in \mathcal{O}(K)$$

$\Rightarrow z - z_0 = e^{f(z)}(z - z_1)$. Now, approximate f on K by $\tilde{f}(z) \in \mathcal{O}(\mathbb{C} \setminus \{z_1\})$, so
$z - z_0 \sim e^{\tilde{f}(z)}(z - z_1)$ on K.



$f(z) = \log(z - z_0) \in \mathcal{O}(K)$

$z - z_0 = e^{f(z)}$. Approximate f on K by $\tilde{f} \in \mathcal{O}(\mathbb{C})$ so $z - z_0 \approx e^{\tilde{f}(z)}$ on K. Thus we have approximated $z - z_0$ on K by a zero free entire function.

Theorem If $K \subset \Omega$ is holomorphically convex, i.e. $\hat{K}_{\mathcal{O}(\Omega)} = K$. Then $\mathcal{O}^*(\Omega)|_K$ is dense in $\mathcal{O}^*(K)$.

Proof: Let $f \in \mathcal{O}^*(K)$ and let $\varepsilon > 0$, $\varepsilon < \min\{|f(z)|;\, z \in K\}$. Then there exists a rational function $R(z) = \frac{P(z)}{Q(z)} \in \mathcal{O}(\Omega)$

such that $|f - R|_K < \frac{1}{2}\varepsilon$. P has no zeroes on K. Let $a_1, \dots, a_k$ be the zeroes of P in the bounded components of $\mathbb{C} \setminus K$, $a_{k+1}, \dots, a_m$ the zeroes of P in the unbounded component of $\mathbb{C} \setminus K$ and pick b_j, $j = 1, \dots, k$, $b_j \notin \Omega$, in the same component as a_j. We may assume
$$P(z) = \prod_{j=1}^{m} (z - a_j)^{m_j}$$

Then
$$g(z) = \sum_{j=1}^{k} m_j \log\left(\frac{z - a_j}{z - b_j}\right) + \sum_{j=k+1}^{m} m_j \log(z - a_j) \in \mathcal{O}(K)$$

and
$$e^{g(z)} = \frac{P(z)}{\prod_{j=1}^{k}(z - b_j)^{m_j}} = \frac{P(z)}{P_0(z)}.$$

We have $\min_{z\in K} |Q(z)| = \delta > 0$. Let $M = \max_{z\in K} |P_0(z)|$, $N = \max_{z\in K} |e^{g(z)}|$ and let $\mu > 0$ be given. If $h \in \mathcal{O}(\Omega)$, $|h-g|_K < \log(1+\mu)$ then $|e^{h-g} - 1|_K < \mu$. Hence for $z \in K$,

$$\left|R(z) - \frac{P_0(z)e^{h(z)}}{Q(z)}\right| = \left|\frac{P_0(z)e^{g(z)}}{Q(z)} - \frac{P_0(z)e^{h(z)}}{Q(z)}\right| \leq \frac{M}{\delta}\left|e^{g(z)} - e^{h(z)}\right|$$

$$\leq \frac{M}{\delta}|e^{g(z)}|\,|1-e^{h(z)-g(z)}| \leq \frac{MN}{\delta}\cdot\mu < \frac{1}{2}\varepsilon$$ when μ is suff. small. Therefore $R_0(z) = \frac{P_0(z)}{Q(z)}e^{h(z)} \in \mathcal{O}^*(\Omega)$ is the required approximation.

<u>Weierstrass theorem</u> Shall prove result on prescription of zeroes and poles. For this we need to study infinite products.

Let $\{a_n\} \in \mathbb{C}$. We say that $\prod_{n=1}^{\infty} a_n$ is <u>convergent</u> if $p_N = \prod_{n=1}^{N} a_n$ is a convergent sequence, and we set

$$\prod_{n=1}^{\infty} a_n = \lim_{N\to\infty} p_N$$

If this limit is nonzero, it is clearly necessary that $\lim_{n\to\infty} a_n = \cancel{0}\ \underline{1}$. We shall consider products

$$\prod_{n=1}^{\infty} (1+u_n) \quad \text{with } u_n \to 0.$$

Sloppy calculation

$$\log \prod^{N} (1+u_n) = \sum_{1} \log(1+u_n) \approx \sum u_n$$

Hence it follows that the convergence of $\prod(1+u_n)$ is related to the convergence of the series $\sum_1 u_n$. Correct calc:

$$|p_N| \le \prod^{N} (1+|u_n|)$$

$$\log|p_N| \le \sum^{N} \log(1+|u_n|) \le \sum^{N} |u_n| \qquad (\log(1+x) \le x)$$

$$|p_N| \le e^{\sum |u_n|}$$

Hence $\{p_N\}$ is bounded if $\sum^{\infty} |u_n| < \infty$

$p_N - 1$ is a polynomial in $u_1, \dots, u_N$, without constant terms. This gives

$$|p_N - 1| \le \prod^{N} (1+|u_n|) - 1 \le e^{\sum |u_n|} - 1$$

Lemma 1 If $\{u_n(z)\}$ are bounded functions on a set E such that $\sum |u_n(z)|$ converges uniformly on E, then

$$f(z) = \prod^{\infty} (1 + u_n(z))$$

converges uniformly on E and $f(z_0) = 0$ iff $u_n(z_0) = -1$ for some n

Proof: It follows ~~that~~ from $|p_N(z)| \leq e^{\sum |u_m(z)|}$ that $\{p_N(z)\}$ is uniformly bounded on E, i.e. $|p_N(z)| \leq C \;\; \forall z \in E$, N

For $M > N$ we have

$$|p_M(z) - p_N(z)| = |p_N(z)| \, \Big| \prod_{N+1}^{M} (1 + u_m(z)) - 1 \Big|$$

$$\leq C \left(e^{\sum_{N+1}^{M} |u_m(z)|} - 1 \right) \xrightarrow[N,M \to \infty]{} 0$$

which proves that $\{p_N(z)\}$ converges uniformly on E.

The inequality also shows that

$$|p_M(z)| \geq |p_N(z)| \, (1 - \epsilon) \quad \text{for } N \text{ suff. large}, M > N$$

Hence, the infinite product has a zero at z_0 iff some finite p_N does. ∎

Theorem If Ω is connected, $f_n \in \mathcal{O}(\Omega)$, no f_n is identically equal to zero and $\sum |1 - f_n(z)|$ converges u.o.c. in Ω, then $f(z) = \prod^{\infty} f_n(z)$ converges u.o.c. and

$$\operatorname{ord}_a(f) = \sum_{n=1}^{\infty} \operatorname{ord}_a(f_n).$$

Theorem 2 Weierstrass theorem

If $E \subset \Omega$ is discrete and for every $a \in E$ there is given an integer $k_a \in \mathbb{Z}$, then there is a holomorphic function $f \in \mathcal{O}^*(\Omega \setminus E)$ such that $(z-a)^{-k_a} f(z)$ is holomorphic and nonzero in a nbh. of a for all $a \in E$.

Proof: Let $\{K_n\}$ be a holomorphically convex exhaustion of Ω and let $E_n = E \cap (K_n \setminus K_{n-1})$, $K_0 = \emptyset$. Let $g_n = \prod_{a \in E_n} (z-a)^{k_a}$. The g_1 has the required property for $a \in E_1$. We would like to multiply by g_2, but the problem is convergence. Notice however that $g_2 \in \mathcal{O}^*(K_1)$, hence there is $h_2 \in \mathcal{O}^*(\Omega)$ such that $|g_2 h_2 - 1|_{K_1} < 2^{-2}$ and $g_1 \cdot (g_2 h_2)$ has the required property for $a \in E_1 \cup E_2$.
Inductively we can find $h_n \in \mathcal{O}^*(\Omega)$ such that $|g_n h_n - 1|_{K_{n-1}} < 2^{-n}$. This implies that

$$f = g_1 \cdot \prod_{n=2}^{\infty} g_n h_n$$

has the required properties.

Exercise The analogous version of Theorem 4 for Weierstrass theorem is the following:

If $\{U_j\}_{j=1}^{\infty}$ is an open covering of Ω and $f_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ satisfy the cocycle condition $f_{ij} f_{jk} f_{ki} = 1$ in $U_i \cap U_j \cap U_k$ then there exist $f_i \in \mathcal{O}^*(U_i)$ such that $f_{ij} = f_i / f_j$ in $U_i \cap U_j$ for all i, j.

Show that this implies Weierstrass theorem.

Theorem 4 (Interpolation in a discrete set)
If $E \subset \Omega$ is discrete and for every $a \in E$ is given $\phi_a \in \mathcal{O}(D^*(a, r_a))$ and $k_a \geq 0$. Then there is $f \in \mathcal{O}(\Omega \setminus E)$ such that $f - \phi_a$ is holomorphic at a and $\mathrm{ord}_a (f - \phi_a) > k_a$ for all $a \in E$.

Proof: By Weierstrass theorem there is $g \in \mathcal{O}(\Omega)$ such that $Z(g) = E$ and $\mathrm{ord}_a g = k_a + 1$ for all $a \in E$. Then $\phi_a / g \in \mathcal{O}(D^*(a, r_a))$ for all $a \in E$ and by Mittag-Leffler there is $h \in \mathcal{O}(D \setminus E)$ such that

$$h - \phi_a/g = O(1) \quad \text{as } z \to a \quad \forall a \in E.$$

Then $h = \phi_a/g + O(1)$ and $f = hg = \phi_a + O(|z-a|^{k+1})$ as $z \to a$. ∎

Notice h can have zeros outside E.

If each ϕ_a is meromorphic then we can find such f without other zeros:

Theorem 5 If $E \subset \Omega$ is discrete and for every $a \in E$ there is given $\phi_a \in \mathcal{O}(D^*(a, r_a))$ such that $\mathrm{ord}_a \phi_a > -\infty$. Then there is $f \in \mathcal{M}(\Omega) \cap \mathcal{O}^*(\Omega \setminus E)$ such that $\mathrm{ord}_a (f - \phi_a) > k_a$ for all $a \in E$.

Proof:
- $E_0 = \{a ; \phi_a \not\equiv 0\}$
- $m_a = \mathrm{ord}_a \phi_a$, $a \in E_0$

- By Weierstrass, we can find $g \in \mathcal{M}(\Omega)$ such that

$$\mathrm{ord}_a g = \begin{cases} m_a & \text{for } a \in E_0 \\ > k_b & \text{for } b \in E \setminus E_0 \end{cases}$$

$g \in \mathcal{O}^*(\Omega \setminus E)$

- If $h \in \mathcal{O}(\Omega)$ and $f = g e^{h(z)}$ then everything holds except possibly $\operatorname{ord}_a (f - \phi_a) > k_a$ for $a \in E_0$. How can we achieve this? Notice that ϕ_a / g is holomorphic and nonzero near a, so there is $h_a \in \mathcal{O}(D^*(a, r_a))$ such that $e^{h_a} = \phi_a / g$. Then

$$\operatorname{ord}_a (g e^{h} - \phi_a) = \operatorname{ord}_a g \left(e^{h} - \frac{\phi_a}{g} \right) = \operatorname{ord}_a g \left(e^{h} - e^{h_a} \right)$$

$$= \operatorname{ord}_a g e^{h_a} \left(e^{h - h_a} - 1 \right) = m_a + \operatorname{ord}_a (h - h_a)$$

By Theorem 4 there is $h \in \mathcal{O}(\Omega)$ such that $\operatorname{ord}_a (h - h_a) > m_a + k_a$. This completes the proof.

Automorphism of the disc

Def: An automorphism of an open set $\Omega \subset \mathbb{C}$ is a biholomorphic map of Ω onto itself, i.e. a holomorphic map $f: \Omega \to \Omega$ which has a holomorphic inverse.
Denoted by $\mathrm{Aut}(\Omega)$. This is a group.
$D = D(0,1) = \{|z| < 1\}$ the unit disc; $T = \{\lambda;\ |\lambda| = 1\}$

Theorem 1. Schwarz lemma.
If $f \in \mathcal{O}(D)$, $|f(z)| \le 1$ for all $z \in D$ and $f(0) = 0$, then

$$|f'(0)| \le 1 \quad \text{and} \quad |f(z)| \le |z|$$

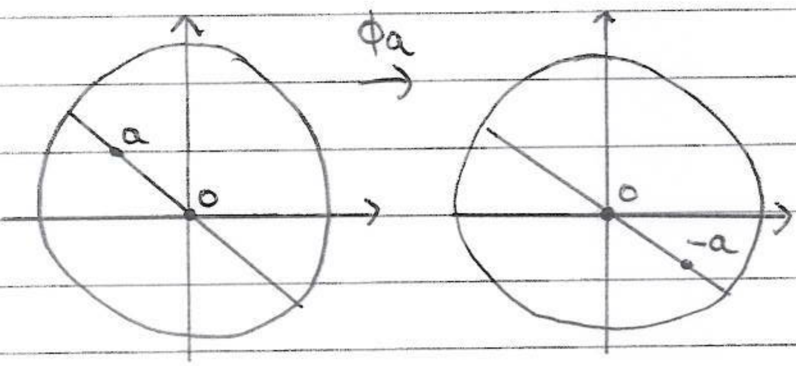
Equality holds for some $z \in D \Leftrightarrow f(z) = \lambda z$, $|\lambda| = 1$.

Proof: Let $g(z) = \dfrac{f(z)}{z}$, $g(0) = f'(0)$. Then $g \in \mathcal{O}(D)$ and $\overline{\lim\limits_{z \to \zeta \in T}} |g(z)| \le 1$, hence the maximum modulus theorem implies that either $|g(z)| < 1$ for all $z \in D$ or $g(z) \equiv \lambda \in T$.

In the first case $|f(z)| < |z|$ and $|f'(0)| < 1$, in the second case $f(z) = \lambda z$.

$\blacksquare$

For $a \in D$, let $\phi_a(z) = \dfrac{z - a}{1 - \bar{a} z}$ $\qquad \phi_a(a) = 0$, $\phi_a(0) = -a$



If $|z| = 1$, then

$$|\phi_a(z)| = \left| \frac{z - a}{(1 - \bar{a} z)\bar{z}} \right|$$

$$= \left| \frac{z - a}{\bar{z} - \bar{a}} \right| = 1.$$

Hence $\phi_a: D \to D$. Easy to see that $\phi_a^{-1} = \phi_{-a}$.
ϕ_a is an automorphism

<u>Theorem 2</u> Every automorphism of D is of the form
$\psi(z) = \lambda \phi_a(z)$ for some $\lambda \in T$.

Proof: If $\psi(0) = 0$, then $(\psi^{-1})'(0) \cdot \psi'(0) = 1$. Since $\psi, \psi^{-1} \in \mathrm{Aut}(D)$ and are 0 at 0, their derivatives at zero must be ≤ 1 in absolute value. $<$ is impossible, so $|\psi'(0)| = 1$ and $\psi = \lambda z$ by the Schwarz lemma.
In general, if $\psi(a) = 0$, consider $\phi = \psi \circ \phi_{-a}$. Then $\phi \in \mathrm{Aut}(D)$, $\phi(0) = 0$, so $\phi(z) = \lambda z$ hence $\psi(z) = \lambda \phi_a(z)$.

<u>Hurwitz theorem</u> If Ω is connected $f_n \in \Theta(\Omega)$ without zeroes and $f_n \to f \not\equiv 0$ uniformly on compacts, then f is without zeroes.

Proof: Let $a \in \Omega$ and pick $r > 0$ such that f has no zeroes on $\gamma = \{z;\ |z-a| = r\}$. Then $f_n'/f_n \to f'/f$ uniformly on γ, so

$$\#\text{ zeroes of } f \text{ in } D(a,r) = \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \lim \frac{1}{2\pi i}\int_\gamma \frac{f_n'}{f_n}\,dz = 0$$

<u>Corollary</u> If Ω is connected, $f_n \in \Theta(\Omega)$ are injective and $f_n \to f \not\equiv c$ uniformly on compacts, then f is injective.

Proof: If $f(a) = f(b) = w$ and $D(a,r) \cap D(b,r) = \emptyset$, then by Hurwitz theorem $f_n(z) - w$ must have a zero in both $D(a,r)$ and $D(b,r)$ for sufficiently large n, hence f_n is not injective.

Riemann mapping theorem

Theorem 1. If $\Omega \neq \mathbb{C}$ is simply connected (and connected), the Ω is biholomorphic to D.

- We shall see that this follows from the fact that every $f \in \mathcal{O}(\Omega)$, f without zeroes, has a holomorphic square root. This is true in a simply connected domain since f has a holomorphic logarithm. If $g = e^{\frac{1}{2}\log f}$, then $g^2 = f$.

- $f : \Omega \to \mathbb{C}$ is biholomorphic onto its image $\Leftrightarrow$ f is injective

- The square root property is invariant under biholomorphism

- If $f : \Omega \to \Omega'$ is biholomorphic and has a holomorphic square root, then $\sqrt{f}$ is also biholomorphic.
  Also; if $w \in \mathrm{Im}(\sqrt{f})$, then $-w \notin \mathrm{Im}(\sqrt{f})$.

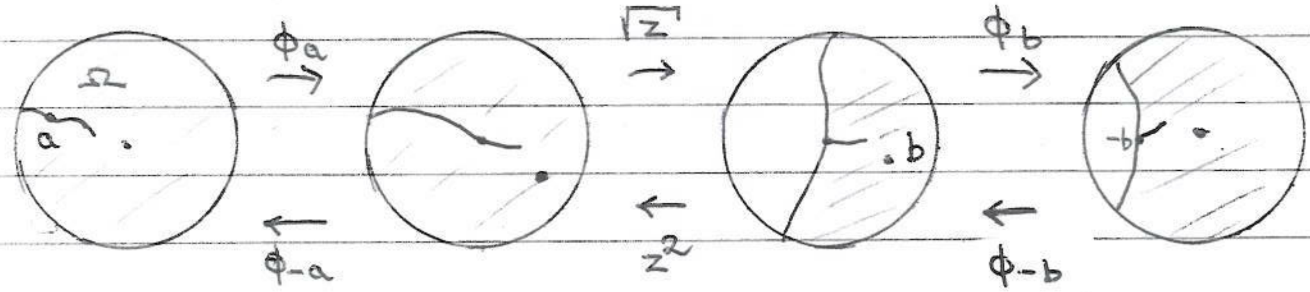
Proposition (Koebe) If $0 \in \Omega \subset D$, $\Omega \neq D$ is connected and has the square root property, then there is a $\mathcal{K} \in \mathcal{O}(\Omega)$ such that

(i) $\mathcal{K}(0) = 0$, $\mathcal{K}(\Omega) \subset D$

(ii) $\mathcal{K}$ is injective

(iii) $|\mathcal{K}(z)| > |z|$ for all $z \in \Omega$, $z \neq 0$.

Proof: Pick $a \in D \setminus \Omega$



Let $\mathcal{H} = \phi_b \circ \sqrt{z} \circ \phi_a$. Then (i) and (ii) holds.
$\mathcal{H}^{-1}$ is defined in all of D and is 2-1 (except at $-b$), therefore $|\mathcal{H}^{-1}(w)| < |w|$ for all $w \neq 0$, so $|\mathcal{H}(z)| > |z|$ for all $z \neq 0$. ∎

Proof Theorem 1. We know that Ω has the square root property.

Step 1. To map Ω biholomorphically onto a bounded domain
Pick $a \in \mathbb{C} \setminus \Omega$ and $g \in \mathcal{O}(\Omega)$ such that $g^2(z) = z - a$.
If $D(w, r) \subset g(\Omega)$ (which is open), then $D(-w, r) \cap g(\Omega) = \emptyset$
and
$$\psi(z) = \frac{1}{g(z) + w}$$
is biholomorphic in Ω and $|\psi(r)| < \frac{1}{r}$.

For small ϵ, $h(z) = \epsilon(\psi(z) - \psi(z_0))$, is biholomorphic onto $0 \in \Omega_0 \subset D$. Ω_0 has the square root property.

Step 2. We shall produce a biholomorphic map $\Omega_0 \to D$ which is "maximal". Let

$$\mathcal{F} = \{ f : \Omega_0 \to D;\ f \text{ is hol, injective and } f(0) = 0 \}$$

Let $z_0 \in \Omega_0$, $z_0 \neq 0$ and put

$$\alpha = \sup_{f \in \mathcal{F}} |f(z_0)| \in (0,1] .$$

and pick $f_n \in \mathcal{F}$ such that $\lim_{n\to\infty} |f_n(z_0)| = \alpha$. By Montel's theorem there is a convergent subsequence, i.e. we may assume $f_n \to f$ u.o.c. Since $f(0)=0$ and $|f(z_0)| = \alpha > 0$, f is not constant. By corollary of Hurwitz theorem, f is injective, so f is a biholomorphism $f: \Omega_0 \to \Omega_1 = f(\Omega_0) \subset D$. We cannot have $\Omega_1 \neq D$, because by Koebe's theorem there is a $\mathcal{X} : \Omega_1 \to D$ injective such that $|\mathcal{X}(f(z_0))| > |f(z_0)| = \alpha$, contradicting the definition of α. ∎

It is instructive to read Theorem 1 of section 7.3.

Schwarz - Pick and Ahlfors lemma.

$$\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$$

$$\varphi_a'(z) = \frac{1-|a|^2}{(1-\bar{a}z)^2}$$

$$\varphi_a'(0) = 1-|a|^2$$

$$\varphi_a'(a) = \frac{1}{1-|a|^2}$$

If $f : D \to D$ is holomorphic, $z \in D$, let

$$g = \varphi_{f(z)} \circ f \circ \varphi_{-z}$$

Then $g(0) = 0$ and

$$g'(0) = \varphi_{f(z)}'(f(z)) \cdot f'(z) \cdot \varphi_{-z}'(0)$$

$$= \frac{1}{1-|f(z)|^2} \cdot f'(z) \cdot (1-|z|^2)$$

We get

Theorem 1.1. If $f : D \to D$ is holomorphic, then

$$\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2}$$

Equality at one point implies that f is an automorphis

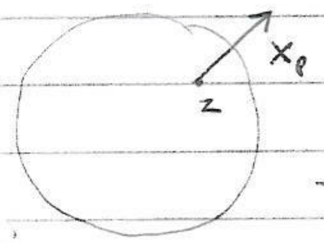
Pf: The last statement follows from $g(w) = \lambda w$, so

$$f(w) = \varphi_{-f(z)}(\lambda \varphi_z(w)) \Rightarrow f = \varphi_{-f(z)} \circ (\lambda \varphi_z)$$

This formulation is equivalent to the Schwarz lemma. Pick gave an invariant definition of this: Consider the (Kähler) metric

$$ds_h^2 = \frac{dz\,d\bar{z}}{(1-|z|^2)^2}$$

on D, i.e for a tangent vector $X \in T_p D$, $p \in D$



$$ds_h^2(X) = \frac{|X|^2}{(1-|z|^2)^2}$$

Then

$$f^*(ds_h^2) = \frac{|f'(z)|^2}{(1-|f(z)|^2)^2}\, dz\,d\bar{z} \le \frac{dz\,d\bar{z}}{(1-|z|^2)^2} = ds_h^2 \quad \text{i.e.}$$

$$f^*(ds_h^2) \le ds_h^2$$

with equality at one point iff f is an automorphism.

- We can define length of curves $\gamma : [a,b] \to D$ using the metric ds_h:

$$L(\gamma) = \int_a^b ds_h(\gamma(t), \gamma'(t))\, dt$$

It follows that holomorphic functions decrease the length of curves:

$$L(f \circ \gamma) \le L(\gamma)$$

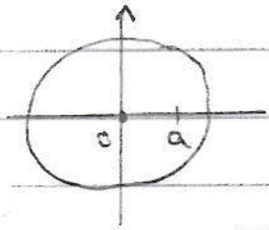
and automorphisms preserve length.

- This defines a distance on D by

$$\rho_h(z_1, z_2) = \inf L(\gamma) \quad , \gamma \text{ curve from } z_1 \text{ to } z_2 .$$

Holomorphic functions are distance decreasing and automorphisms preserve distances. It follows that

$$\rho_h(z_1, z_2) = \rho_h(0, |\varphi_{z_1}(z_2)|)$$



$$\rho_h(0, a) = \int_0^a \frac{dt}{1-t^2} = \frac{1}{2} \log \frac{1+a}{1-a}$$

so

$$\rho_h(z_1, z_2) = \frac{1}{2} \log \frac{1+|\varphi_{z_1}(z_2)|}{1-|\varphi_{z_1}(z_2)|}$$

Theorem 1.2. If $f : D \to D$ is holomorphic, then

1. $f^*(d\sigma_h) \leq d\sigma_h$

2. $\rho_h(f(z), f(w)) \leq \rho_h(z, w)$

Equality in one point in 1 or one pair $z \neq w$ in 2 implies f is auto

$d\sigma_h$ is called the Poincaré metric

ρ_h Poincaré distance.

- The curvature of a metric $h\, dz\, d\bar{z}$ is defined by

$$\mathcal{K} = -\frac{2}{h} \frac{\partial^2}{\partial z \partial \bar{z}} \log h = -\frac{1}{2h} \Delta(\log h)$$

For $h = \dfrac{1}{(1-|z|^2)^2}$ we get

$$\mathcal{K} = -2(1-|z|^2)^2 \frac{\partial}{\partial z \partial \bar{z}} \log (1-|z|^2)^{-2}$$

$$= 4(1-|z|^2)^2 \frac{\partial}{\partial z \partial \bar{z}} \log(1 - z\bar{z}) = 4(1-|z|^2)^2 \frac{\partial}{\partial z} \frac{-z}{1-z\bar{z}}$$

$$= 4(1-|z|^2)^2 \cdot \frac{-1(1-z\bar{z}) - (-z)\cdot(-\bar{z})}{(1-z\bar{z})^2} = 4(1-|z|^2)^2 \cdot \frac{-1}{(1-z\bar{z})^2} = -4$$

$d\sigma_r =$

- If $h\,dz\,d\bar{z}$ is metric on Ω and $f : U \to \Omega$ satisfies $f'(z) \neq 0$ everywhere, then

$$f^*(d\sigma_r^2) = |f'(z)|^2 h(f(z))\,dz\,d\bar{z}$$

and

$$\mathcal{K}_{f^*(d\sigma_r)}(z) = \mathcal{K}_{d\sigma_r}(f(z)). \qquad \left(\frac{\partial^2}{\partial z \partial \bar{z}}(\log h(f(z)) = \left(\frac{\partial}{\partial z \partial \bar{z}} \log h\right)(f(z)) \cdot |f'(z)|^2\right.$$

Thus curvature is a conformal invariant.

- The metric $d\sigma_a^2 = \dfrac{4a^2}{A} \dfrac{dz\,d\bar{z}}{(a^2 - |z|^2)^2}$ on $D_a = \{|z| < a\}$ $\left(\frac{\partial^2}{\partial z \partial \bar{z}} \log \frac{1}{(a^2-|z|^2)^2} = \frac{2a^2}{(\quad)^2}\right)$

has curvature $-A$. Theorem 1.2. generalizes to

Theorem 1.3. Ahlfors lemma.

If M is a Riemann surface with metric $d\sigma_M^2$ with curvature $\leq -B$ $(B > 0)$ and $f : D_a \to M$ is holomorphic, then

$$f^*(d\sigma_M^2) \leq \frac{A}{B}\, d\sigma_a^2$$

Proof:

Define $u \geq 0$ on D_a by $f^*(d\sigma_M^2) = u\, d\sigma_a^2 = u(z) \dfrac{4a^2\,dz\,d\bar{z}}{A(a^2 - |z|^2)^2}$

and for $r \leq a$, u_r is defined by $f^*(d\sigma_M^2) = u_r\, d\sigma_r^2$ on D_r

so $u = u_a$ and

$$u_r(z) = u(z) \frac{a^2 (r^2 - |z|^2)}{r^2 (a^2 - |z|^2)}$$

so $u_r \to u$ when $r \to a$. It is therefore sufficient to prove that $u_r(z) \leq \dfrac{A}{B}$ for $z \in D_r$.

By the formula above, $u_r(z) = 0$ when $|z| = r$. If $u_r(z) \equiv 0$ we are done. Otherwise, u_r has a maximum at some $z_0 \in D_r$. Then f defines local coordinates around z_0, i.e. there is a nbh U of z_0 with $f'(z) \neq 0$ for $z \in U$ and we can compute the curvature of $d\sigma_M^2$ by computing it in U.

We have
$$f^*(d\sigma_M^2) = u_r\, d\sigma_r^2 = u_r(z)\frac{4r^2\, dz d\bar z}{A(r^2-|z|^2)^2} =: h(z)\, dz d\bar z$$

~~so~~

$$\mathcal{K} = -\frac{2}{h}\frac{\partial^2}{\partial z\partial\bar z}\log h = -\frac{2}{h}\frac{\partial^2}{\partial z\partial \bar z}\left(\log u_r + \log\frac{4r^2}{A(r^2-|z|^2)^2}\right)$$

$$= -\frac{2}{h}\left(\frac{\partial^2}{\partial z\partial\bar z}\log u_r + \frac{2r^2}{(r^2-|z|^2)^2}\right)$$

$$= -\frac{2}{h}\frac{\partial^2}{\partial z\partial\bar z}\log u_r - \frac{A}{u_r} \leq -B$$

$$\frac{2}{h}\frac{\partial^2}{\partial z\partial\bar z}\log u_r \geq B - \frac{A}{u_r}$$, but $\frac{\partial^2}{\partial z\partial\bar z}\log u_r(z_0) = \frac{1}{4}\Delta\log u_r(z_0) \leq 0$ since z_0 is a maximum.

This gives $u_r(z_0) \leq \frac{A}{B}$. ∎

Which M can have a metric with negative curvature?

1. $\mathbb{C}$ does not have such a metric.
Pf: If $d\sigma_{\mathbb{C}}^2$ is such a metric, let $f: D \to \mathbb{C}$ be defined by $f(z) = az$. Then
$$(f^* d\sigma_{\mathbb{C}}^2)(0) = |a|^2 d\sigma_{\mathbb{C}}^2(0),$$ hence no such inequality can hold. The metric $(1+|z|^2)dz d\bar z$ has curvature $\mathcal{K} = -2/(1+|z|^2)$ and is complete.

2. $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ does not have such a metric, since $f(z) = e^z$ is a covering $\mathbb{C}\to\mathbb{C}^*$, hence if $\mathbb{C}^*$ had a metric with negative curvature, so would $\mathbb{C}$.
The metric $\frac{dz d\bar z}{\log(1+|z|^2)}$ has curvature $\mathcal{K} = \frac{-2}{(1+|z|^2)^2}\left(\frac{|z|^2}{\log(1+|z|^2)} - 1\right) < 0$ and is complete.

3. The upper half plane $\mathbb{C}^+$ has such a metric since it is biholomorphic to D. A biholomorphic map is

$$f(z) = \frac{z-i}{z+i} \qquad f'(z) = \frac{2i}{(z+i)^2}$$

$$f^*\left(\frac{dz\,d\bar{z}}{(1-|z|^2)^2}\right) = \frac{|f'(z)|^2}{(1-|f(z)|^2)^2}\,dz\,d\bar{z} = \frac{4}{|z+i|^2\left(1-\left|\frac{z-i}{z+i}\right|^2\right)^2}\,dz\,d\bar{z}$$

$$= \frac{4}{(|z+i|^2-|z-i|^2)^2}\,dz\,d\bar{z} = \frac{4\,dz\,d\bar{z}}{((x^2+(y+1)^2)-(x^2+(y-1)^2))^2} = \frac{4\,dz\,d\bar{z}}{(4y)^2}$$

$$= \frac{1}{4y^2}\,dz\,d\bar{z}$$

4. The punctured disc D^* has such a metric. We have a covering map $p : \mathbb{C}^+ \to D^*$ given by $p(z) = e^{iz}$. This has local inverses $p^{-1}(w) = \frac{1}{i}\log w$ and

$$(p^{-1})^*\left(\frac{dz\,d\bar{z}}{4y^2}\right) = \frac{|(p^{-1})'(w)|^2\,dw\,d\bar{w}}{4(\operatorname{Im} p^{-1}(w))^2}$$

$$= \frac{dw\,d\bar{w}}{4|w|^2(\log|w|)^2} = \frac{dw\,d\bar{w}}{|w|^2(\log|w|^2)^2} =: d\sigma^2_{D^*}$$

This metric is also complete. If $0 < r < R < 1$, then

$$\rho_{D^*}(r,R) = \int_r^R \frac{dt}{t(-\log t^2)} = -\frac{1}{2}\int_r^R \frac{dt}{t\log t} = -\frac{1}{2}\log(-\log t)\Big|_r^R$$

$$= \frac{1}{2}\left(\log\left(\log\frac{1}{r}\right) - \log\left(\log\frac{1}{R}\right)\right) \to \infty \quad \text{when } r \to 0 \text{ or } R \to 1$$

The circle $\gamma(t) = re^{it}$ has length

$$\ell(\gamma) = \int_0^{2\pi} \frac{r\,dt}{r(-\log r^2)} = \frac{\pi/2}{\log(1/r)} \to 0 \quad \text{when } r \to 0.$$

5. The doubly punctured plane $\mathbb{C} \setminus \{z_0, z_1\}$ has a metric $h(z)\, dz d\bar{z}$ with curvature bounded above by a negative constant.

Remark: It is possible to find such a metric which is also complete, i.e. the distance to the points z_0, z_1 and ∞ is infinite. This metric is much more complicated to construct, however.

Proof: We may assume $z_0 = 0$, $z_1 = 1$. We shall prove that

$$h(z) = \frac{(1+|z|^\alpha)^\beta}{|z|^\gamma} \cdot \frac{(1+|z-1|^\alpha)^\beta}{|z-1|^\gamma}$$

has the required property for suitable α, β and γ.

The expression for the Laplacian of a radial function $f(r)$ is

$$\Delta f(r) = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial f}{\partial r} \qquad \text{(Check this!)}.$$

This gives

$$\Delta\left(\log \frac{(1+r^\alpha)^\beta}{r^\gamma}\right) = \Delta\left(\beta \log(1+r^\alpha) - \gamma \log r\right) = \beta \Delta(\log(1+r$$

$$\frac{\partial}{\partial r} \log(1+r^\alpha) = \frac{\alpha r^{\alpha-1}}{1+r^\alpha}$$

$$\frac{\partial^2}{\partial r^2} \log(1+r^\alpha) = \alpha \left(\frac{(\alpha-1) r^{\alpha-2}(1+r^\alpha) - r^{\alpha-1} \cdot \alpha r^{\alpha-1}}{(1+r^\alpha)^2} \right) = \frac{\alpha r^{\alpha-2}}{(1+r^\alpha)^2} (\alpha - 1 - r$$

Hence

$$\beta \Delta \log(1+r^\alpha) = 2\beta \left(\frac{\alpha r^{\alpha-2}}{(1+r^\alpha)^2} (\alpha - 1 - r^\alpha) + \frac{1}{r} \frac{\alpha r^{\alpha-1}}{1+r^\alpha} \right)$$

$$= \frac{\alpha\beta r^{\alpha-2}}{(1+r^\alpha)^2} \left((\alpha - 1 - r^\alpha) + (1+r^\alpha) \right) = \frac{\alpha^2 \beta r^{\alpha-2}}{(1+r^\alpha)^2}$$

This gives

$$\mathcal{K} = -\frac{1}{2h} \Delta(\log h) = -\frac{\alpha^2\beta}{2} \frac{|z|^\gamma |z-1|^\gamma}{(1+|z|^\alpha)^\beta (1+|z-1|^\alpha)^\beta} \left(\frac{|z|^{\alpha-2}}{(1+|z|^\alpha)^2} + \frac{|z-1|^{\alpha-2}}{(1+|z-1|^\alpha)^2} \right)$$

$$= -\frac{\alpha^2\beta}{2} \left(\frac{|z|^{\gamma+\alpha-2} |z-1|^\gamma}{(1+|z|^\alpha)^{\beta+2} (1+|z-1|^\alpha)^\beta} + \frac{|z-1|^{\gamma+\alpha-2} |z|^\gamma}{(1+|z-1|^\alpha)^{\beta+2} (1+|z|^\alpha)^\beta} \right)$$

Now, if

$$(1) \quad \gamma + \alpha - 2 = 0 \quad , \text{ i.e. } \gamma = 2 - \alpha$$

this is

$$-\frac{\alpha^2\beta}{2} \left(\frac{|z-1|^\gamma}{(1+|z|^\alpha)^{\beta+2} (1+|z-1|^\alpha)^\beta} + \frac{|z|^\gamma}{(1+|z-1|^\alpha)^{\beta+2} (1+|z|^\alpha)^\beta} \right)$$

Hence

$$\lim_{z\to 0} \mathcal{K} = \lim_{z\to 1} \mathcal{K} = -\frac{\alpha^2\beta}{2^{\beta+1}} < 0.$$

As $z \to \infty$, the power of $|z|$ in $\mathcal{K}$ is $\gamma - 2\alpha\beta - 2\alpha$

Hence $\lim_{|z|\to\infty} \mathcal{K} \leq -\alpha^2\beta$ if

$$(2) \quad \gamma - 2\alpha\beta - 2\alpha \geq 0$$

Inserting (1) in (2) gives $2 - \alpha - 2\alpha\beta - 2\alpha \geq 0$

i.e.

$$(3) \quad \alpha \leq \frac{2}{3+2\beta}$$

For instance $\alpha = \frac{1}{3}$, $\beta = 1$, $\gamma = \frac{5}{3}$ solves the problem.

We also get Ahlfors lemma for maps from D^*.
(We have put $A=1$).

Theorem 1.3 b. Ahlfors lemma for D^*
If M is a Riemann surface with metric $d\sigma_M^2$ with curvature $\leq -B$ $(B>0)$ and $f: D^* \to M$ is holomorphic, then

$$f^*(d\sigma_M^2) \leq \frac{1}{B} d\sigma_{D^*}^2$$

Proof: We have $d\sigma_{D^*}^2 = (p^{-1})^* d\sigma_D^2$. $f \circ p : D \to M$ is holomorphic, so by the Ahlfors lemma for D we have

$$(f \circ p)^*(d\sigma_M^2) = p^*(f^*(d\sigma_M^2)) \leq \frac{1}{B} d\sigma_D^2$$

which gives

$$f^*(d\sigma_M^2) = (p^{-1})^*(p^*(f^*(d\sigma_M^2))) \leq (p^{-1})^*\left(\frac{1}{B} d\sigma_D^2\right) = \frac{1}{B} d\sigma_{D^*}^2$$

Theorem 1.4 Suppose $\Omega \subset \mathbb{C}$ has a metric with curvature $\leq -B$. Then

a) There is no nonconstant holomorphic map $f: \mathbb{C} \to \Omega$
b) No holomorphic function $f: D^* \to \Omega$ can have an essential singularity at 0.

Proof:
a) Restricting to a disc of radius a (with $A=1$) the Schwarz lemma gives

$$f^*(d\sigma_\Omega^2) \leq \frac{1}{B} d\sigma_a^2 = \frac{1}{B} \frac{4a^2}{(a^2-|z|^2)^2} dz d\bar{z} \to 0$$

Since $f^*(d\sigma_\Omega^2) = |f'(z)|^2 h(f(z)) dz d\bar{z}$, when $a \to \infty$
this gives $f'(z)=0$, so f is constant.

To prove b) we the following

<u>Lemma</u> If $f \in \mathcal{O}(D^*)$ has an essential singularity at 0 then $f(D^*)$ is dense in $\mathbb{C}$.

Pf: If not, there is $a \in \mathbb{C}$ and $\delta > 0$ such that $|f(z) - a| \geq \delta$ for all $z \in D^*$. But then $g(z) = \dfrac{1}{f(z)-a}$

satisfies $|g(z)| \leq 1/\delta$, hence has a removable singularity at 0. But then $f(z) = \dfrac{1}{g(z)} + a$ either has a pole or a

removable singularity at 0.

To prove b), notice that if $f : D^* \to \Omega$ has an essential singularity at 0, then $f(D^*_r)$ is dense in $\mathbb{C}$ for all $r > 0$ hence there is a sequence $z_n \to 0$ such that $f(z_n) \to p \in \Omega$.
If ρ is the metric defined by $d\sigma_\Omega$, i.e.

$$\rho(z,w) = \inf\left\{ \int_0^1 d\sigma_\Omega(\gamma'(t))\,dt \; ; \; \gamma : [0,1] \to \Omega, \; \gamma(0) = z, \gamma(1) = w \right\}$$

and $\bar{B}(p,r) \subset \Omega$, then $\inf\{\rho(p,z) ; |p-z| = r\} = \delta > 0$.
If $\rho(p, f(z_n)) < \frac{1}{2}\delta$ and γ is a curve of length $\leq \frac{1}{2}\delta$ starting at $f(z_n)$, then $\gamma \subset B(p,r)$, hence $|\gamma(t)| \leq |p| + r = C$ for all t.
We may assume than $r_n = |z_n|$ decrease strictly to zero.
Since $f(z_n) \to p$ there is N such that $\rho(p, f(z_n)) < \frac{1}{2}\delta$ for $n \geq N$.
Let γ_n be the circle $|z| = r_n$. Then

$$L(f \circ \gamma_n) \leq \frac{1}{\sqrt{B}} L(\gamma_n) \leq \frac{\pi}{2\sqrt{B}\log 1/r_n^2} \to 0 \quad \text{when } n \to \infty$$

Hence for large n, $L(f \circ \gamma_n) \leq \frac{1}{2}$. This implies that $|f(z)| \leq C$ for all z with $|z| = r_n$.
This means that $|f(z)| \leq C$ for all z in the annuli $A_n = \{r_{n+1} \leq |z| \leq r_n\}$ and therefore in a punctured disc D_r. Hence f has a removable singularity at 0.

Theorem 1.5

a) Picard's small theorem: A nonconstant entire function cannot omit more than one value.
b) Picard's big theorem: If a holomorphic function has an essential singularity at a, then f takes all complex values except possibly one in any punctured disc around a.

Proof: a) If f omits two values z_0 and z_1, then $f: \mathbb{C} \to \Omega = \mathbb{C} \setminus \{z_0, z_1\}$. Since Ω has a metric with curvature $\leq -B$ this follows from 1.4 a).

b) Follows in the same way from 1.4 b).

∎

We will now use the complete metric on $\mathbb{C} \setminus \{z_0, z_1\}$ mentioned in the remark on page 48.

Theorem 1.6 Schottky's Theorem Given $R_0 > 0$ and $r < 1$, then there is a constant $M (= M(R_0, r))$ such that if $f: D \to \mathbb{C} \setminus \{z_0, z_1\}$ is holomorphic and $|f(0)| \leq R_0$, then $|f(z)| \leq M$ for all z with $|z| \leq r$.

Proof: Let γ be the curve $\gamma(t) = tz$. By Ahlfors lemma
$L(f\circ\gamma) \le \frac{1}{\sqrt{B}} L(\gamma) = \frac{1}{2}\log\frac{1+|z|}{1-|z|} \le \frac{1}{2}\log\frac{1+r}{1-r}$.

It follows that $f(z)$ must be bounded since
$d_\Omega(f(0), w) \to \infty$ as $|w| \to \infty$.

∎

It follows that $f(z)$ must also stay away from z_0 and z_1,
i.e. $|f(z) - z_0| \ge M_0$, $|f(z) - z_1| \ge M_1$.

The same proof can be used to prove bounds on maps $f : D^* \to \mathbb{C} \setminus \{z_0, z_1\}$ on either annular regions or circles. Here is the circle version:

Theorem 1.7. Schottky's Theorem in D^* Given $R_0 > 0$ and $r < 1$ there is a constant M such that if $f : D^* \to \mathbb{C} \setminus \{z_0, z_1\}$ is holomorphic and $f(z) \le R_0$ for some z with $|z| \le r$, then $|f(\zeta)| \le M$ for all ζ with $|\zeta| = |z|$

Pf: We use the curve $\gamma(t) = ze^{it}$, $0 \le t \le 2\pi$ whose length is

$$\frac{\pi}{2\log(1/|z|^2)} \le \frac{\pi}{2\log(1/r^2)}$$

and Ahlfors lemma for D^*.