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1. Del

Simple aspects of complex functions

Version 0.62 — Monday, September 26, 2016 9:16:18 AM
Still preliminary version prone to errors. At least for some time it will not be changed (except may be for correction of stupid errors)
Changes:
A lot of minor and stupid errors corrected
Thank's to all that have contributed by finding errors!

(1.1) A *a domain* in the complex plane Ω is an open non-empty and connected subset of $\mathbb{C}$. Recall that a subset A of $\mathbb{C}$ (or any topological space for that matter) is said to be *connected* if it is not the union of two disjoint open sets. Equivalently one may require that A not be the union of two disjoint closed sets. The set A is *pathwise connected* if any two of its points can be joined by a continuous path, clearly a pathwise connected set is connected, but for general topological spaces the converse dos not hold; but luckily, it holds true for open subsets of the complex plane; so an open subset Ω of $\mathbb{C}$ is connected if and only if it is pathwise connected.

(1.2) The union of two connected sets is connected provided the two sets are not disjoint. Hence any point in A is contained in a maximal connected set. These maximal sets called a connected component of A, and they form a partition of A—they are pairwise disjoint and their union equals the whole space. Connected components are always closed subsets, but not necessarily open. An everyday example being the rationals $\mathbb{Q}$ with the topology inherited from the reals. As every non-empty open interval contains real numbers, the connected components of $\mathbb{Q}$ are just all the points. One says that $\mathbb{Q}$ is *totally disconnected*.

The *path-component* of a point z consists of all the point in the set A that can be joined to z by a continuous path. The different path-components form, just like the connected components, a partition of the space.

OPPGAVE 1.1.

a) Show that a path-wise topological space is connected.

b) A space is called *locally pathwise connected* if every point admits a neigbourhood basis consisting of open and path-wise connected sets; equivalently for every point p and every open set U containing p, one may find an open and path-wise connected set contained in U and containing p. Show that if a space is locally pathwise connected, it is connected if and only if it is pathwise connected.

✷

Domains can be very complicated and their geometric complexity and subtleties form now and again significant parts of the theory— or at least, are the reasons behind long and tortuous proofs of statements seeming obvious in simple situations one often has in mind—like slightly and nicely deformed disk with a whole or two. So a few example are in place:

EKSEMPEL 1.1. If Z is any closed subset of the real axis not being the whole axis. Then clearly $\mathbb{C} \setminus Z$ is open and connected (one can pass from the upper to the lower half plane by sneaking through $\mathbb{R} \setminus \mathbb{Z}$) Two specific examples of interesting closed sets Z can be $\{\, 1/n \mid n \in \mathbb{N} \,\} \cup 0$ and the Cantor set $\mathfrak{c}$. ❄

EKSEMPEL 1.2. For each rational number p/q in reduced form, let $L_{p/q}$ be the (closed) line segment of length $1/q$ emanating from the origin forming the angle $2\pi p/q$ with the positive real axis; *i.e.*, the points of $L_{p/q}$ are of the form $te^{2\pi pi/q}$ with $0 \leq t \leq 1/q$. Let $L = \bigcup_{p/q} L_{p/q}$. Then L is closed. This is not completely obvious (so prove it!). It hinges on the fact that only finitely many of the segments $L_{p/q}$ appear in the vicinity of a point z different from the origin. The complement U of L is therefore open, and it is connected (the ray from the origin through a point in U has just the origin in common with L, and z can be connected to points outside the unit disk, and as L is contained in the closed unit disk, this suffices) so it is a domain. The set U is not simply connected but has the homotopy type of a circle. ❄

EKSEMPEL 1.3. This example is a variant of the previous example; the origin and the point at infinity are just exchanged via $z \to 1/z$. Here it comes: Let $L_{p/q}$ consist of the points $te^{2\pi ip/q}$ with t real and $|t| > q$, and let U be the complement of $\bigcup_{p/q} L_{p/q}$. On shows that U is open as in the previous example. The line segment joining the origin to a point z in U is contained in U, and this shows that U is connected; in fact, it even shows that U is contractible. ❄

OPPGAVE 1.2. Let U be the complement of the product $\mathfrak{c} \times \mathfrak{c}$ in the open unit square $(0,1) \times (0,1)$. Show that U is a domain. ✷

1.1 Derivatives and the Cauchy-Riemann equations

In this section Ω will be a domain and f will be a complex valued function defined in Ω. The function f has two components, the real-valued functions $u(z) = \operatorname{Re} f(z)$, called *the real part* of f, and $v(z) = \operatorname{Im} f(z)$, *the imaginary part* of f. With this notation one writes $f = u + iv$.

The complex variable z is of course of the form $z = x + iy$ with x and y real, so any function $f(z)$ may as well be regarded as a function of the the two real variables x and y. All results about real functions of (some regularity class) from Ω to $\mathbb{R}^2$ apply to complex functions—but imposing the condition of holomorphy (that is, differentiability in the complex sense) on a function f makes it very special indeed, its properties will by far be stronger than those of general C^∞-function (or even real analytic functions).

(1.1) We adopt the convention of indicating partial derivatives by the use of subscripts, like *e.g.*, u_x, u_y. Taking a partial derivative is of course a differential operator and as such it will now and again be denoted by ∂_* with $*$ an appropriate subscript;*e.g.*, u_x will be denoted $\partial_x u$ and u_y by $\partial_y u$.

Clearly one has $f_x = u_x + iv_x$ and $f_y = u_y + iv_y$, or in terms of differential operators $\partial_x = \partial_x u + i\partial_x v$ and $\partial_y f = \partial_y u + i\partial_y v$. It turns out to be very convenient to use the differential operators ∂_z and $\partial_{\bar{z}}$ defined as

$$\partial_z = (\partial_x - i\partial_y)/2 \qquad \partial_{\bar{z}} = (\partial_x + i\partial_y)/2.$$

One verifies easily that $\partial_z\partial_{\bar{z}} = \partial_{\bar{z}}\partial_z$ at least when applied to functions for which ∂_x and ∂_y commute;*e.g.*, function being C^1. Another important formula, valid whenever ∂_x and ∂_y commute, is

$$4\partial_z\partial_{\bar{z}} = \Delta$$

where Δ is the Laplacian operator $\Delta = \partial_x^2 + \partial_y^2$; indeed, one finds

$$(\partial_x - i\partial_y)(\partial_x + i\partial_y) = \partial_x^2 + i\partial_x\partial_y - i\partial_y\partial_x - i^2\partial_y^2 = \partial_x^2 + \partial_y^2.$$

EKSEMPEL 1.4. As a simple illustration let us compute $\partial_z z$ and $\partial_{\bar{z}} z$. One finds $\partial_z z = (\partial_x(x+iy) - i\partial_y(x+iy))/2 = (1 - i\cdot i)/2 = z$ and similarly $\partial_{\bar{z}} z = (\partial_x(x+iy) + i\partial_y(x+iy))/2 = (1 + i\cdot i)/2 = 0$. ❋

OPPGAVE 1.3. Show that ∂_z and $\partial_{\bar{z}}$ satisfy Leibnitz' rule for products. ✷

1.1.1 The constituting definition — differentiability

The concept of holomorphy, that we are about to introduce, is constituting for the course, everything we shall do will hover about holomorphic functions, so the definitions in this paragraph are therefore the most important ones.

The notion we shall introduce is that of a *differentiable function in in the complex sense*, or *$\mathbb{C}$-differentiable* for short, and their derivatives. As f is a function of two real variables as well, there is also the notion of f being differentiable as such. In that case we shall call f differentiable in the real sense, or $\mathbb{R}$-differentiable—the long annotated names are there to distinguish the two notions. Function being $\mathbb{R}$-differentiable but not $\mathbb{C}$-differentiable are however rear creature in our story, so we shall pretty soon drop the annotations in the complex case, just keeping them the in the real case.

(1.1) To tell when a complex differentiable[1] function is differentiable at a point $a \in \mathbb{C}$ and to define its derivative there, we mimic the good old definition of the derivative of a real-valued function. One forms the complex differential quotient associated to two nearby points, and tries to take the limit as the two points coalesce:

ComplexDiff

Definisjon 1.1 *Let a be a point in Ω. We say that f is differentiable at a if the following limit exists:*

$$\lim_{h \to 0} (f(a+h) - f(a))/h. \tag{1.1}$$

Diff1

If so is the case, the limit is denoted by $f'(a)$ and is called the derivative of f at a. If f is differentiable at all points in Ω one says that f is holomorphic in Ω. A function holomorphic in the entire complex plane (i.e., if $\Omega = \mathbb{C}$) is said to be entire.

An equivalent way of formulating this definition is to say that there exists a complex number $f'(a)$ such that for z in a vicinity of a one has

$$f(z) = f(a) + f'(a)(z-a) + \epsilon(z), \tag{1.2}$$

Diff2

where the function $\epsilon(z)$ is such that $|\epsilon(z)/(z-a)| \to 0$ as $z \to a$.

(1.2) The usual elementary rules for computing derivatives that one learned once upon a time during calculus courses, are still valid in this context, and the proofs are *mutatitis mutandis* the same.

Taking derivatives is a complex linear operation: For complex constants α and β the linear combination $\alpha f + \beta g$ is differentiable at a when both f and g are, and it holds true that $(\alpha f + \beta g)'(a) = \alpha f'(a) + \beta g'(a)$.

Leibnitz' rule for a product still holds: If f and g are differentiable at a, the product fg is as well, and one has $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$. Similarly for a fraction: Assume f and g differentiable at a and that $g(a) \neq 0$, then the fraction f/g is differentiable and $(f/g)'(a) = (g(a)f'(a) - g'(a)f(a))/g(a)^2$.

The third important principle is the chain rule. If f is differentiable at a and g at $f(a)$, then the composition $g \circ f$ is differentiable at a with derivative given as $(g \circ f)'(a) = g'(f(a))f'(a)$.

[1] The annotation in the complex case did not survive particularly long!

(1.3) An obvious consequence of the elementary rules is that a polynomial $P(z)$ is holomorphic in the entire complex plane. Almost the same applies to rational functions. They are quotients P/Q between two polynomials P and Q and are holomorphic where they are defined; that is at at least[2] in the points where the denominator Q does not vanish.

1.2 The Cauchy-Riemann equations

Any function from Ω to $\mathbb{C}$ is also a function of two real variables taking values in $\mathbb{R}^2$ with component functions being the real part u and the complex part v of f. For such functions the derivative at the point $z = \alpha + i\beta$ is an $\mathbb{R}$-linear map $D_a f \colon \mathbb{R}^2 \to \mathbb{R}^2$, that is a map $D_a f \colon \mathbb{C} \to \mathbb{C}$ being linear over the reals.

The derivative, if it exists, satisfies a condition very much like condition (1.2) in the complex case, namely for z close to a one has

$$f(z) = f(a) + D_a f(z-a) + \epsilon(z), \tag{1.3}$$

where $\epsilon(z)$ is a function with $|\epsilon(z)/(z-a)|$ tending to zero when z tends to a. The difference from the condition (1.2) lies in the second term to the right: For f to be $\mathbb{C}$-differentiable, the map real linear $D_a f \colon \mathbb{C} \to \mathbb{C}$ must be multiplication by a complex number! DiffReell

(1.1) Casting a glance on the two definitions (1.1) and (1.2) it seems clear that a $\mathbb{C}$-differentiable function is $\mathbb{R}$-differentiable as well. The Cauchy-Riemann equations are a pair of differential equations that guarantee that a $\mathbb{R}$-differentiable function is $\mathbb{C}$-differentiable, and they are in essence contained in the last sentence of the previous paragraph—that $D_a f$ be multiplication by a complex number. To give the equations a concrete form however, we must exhibit the matrices of the derivative-maps in the two cases, in both cases relative to the semi-canonical basis for $\mathbb{C}$ as a real vector space—*i.e.*, the basis the numbers 1 and i constitute[3].

Multiplication by at complex number $c = \alpha + i\beta$ send 1 to $\alpha + i\beta$ and i to $-\beta + i\alpha$, hence its matrix is

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}. \tag{1.4}$$

JacComplex

In the calculus courses (surely, calculus of several variables) we learned that the matrix of the derivative-map $D_a f$ in the semi-canonical basis is just the *Jacobian matrix*:

$$\begin{pmatrix} u_x(a) & v_x(a) \\ u_y(a) & v_y(a) \end{pmatrix}. \tag{1.5}$$

[2] Why "at least"!

[3] Why "semi-canonical"?

JackReal Comparing the two matrices, one sees that a function f, being differentiable in the real sense, is $\mathbb{C}$-differentiable if and only if the derivatives of its two component functions satisfy the relations

$$u_x(a) = v_y(a) \qquad u_y(a) = -v_x(a).$$

These are the famous Cauchy-Riemann equations. Remembering that $\partial_x f = \partial_x u + i\partial_x v$ and $\partial_y f = \partial_y u + i\partial_y v$, one observes they being equivalent to the single equation

$$\partial_x f(a) = -i\partial_y f(a), \tag{1.6}$$

CR3 and, of course, this common values equals $f'(a)$.

(1.2) So far we have considered differentiability in a point, but being $\mathbb{C}$-differentiable *e.g.*, in solely one isolated point, has no serious implications. If, for example, both partials of f vanishes there, the Cauchy-Riemann equations are trivially satisfied, and the only implication is that both the real and the imaginary part of f has a stationary point. The full weightiness of being differentiable[4] comes into play only when the function is differentiable[5] everywhere in a domain, that is, it is holomorphic. So, when summing up, we formulate the Cauchy-Riemann equations in that context:

Setning 1.1 *Let Ω be a domain in $\mathbb{C}$ and let $f = u + iv$ be a complex valued function in Ω. Then f is differentiable throughout Ω if and only if it is differentiable in the real sense throughout Ω, and the real and imaginary parts satisfy the Cauchy-Riemann equations*

$$\partial_x u = \partial_y v \qquad \partial_y u = -\partial_x v \tag{1.7}$$

CR1 *in Ω. If f is differentiable in Ω, one has*

$$f' = \partial_x f = -i\partial_y f. \tag{1.8}$$

CR2

(1.3) Recall the differential operators ∂_z and $\partial_{\bar z}$ we defined by

$$\partial_z = (\partial_x - i\partial_y)/2 \qquad \partial_{\bar z} = (\partial_x + i\partial_y)/2.$$

In view of equation (1.8) the Cauchy-Riemann equations when formulated in terms of the operators ∂_z and $\partial_{\bar z}$, translate into the following proposition, the simplicity of the equation appearing is one virtue of the $\partial_{\bar z}$ and ∂_z notation:

CauchyRiemannDBar

[4] in the complex sense
[5] ditto

Setning 1.2 *An $\mathbb{R}$-differentiable function f in the domain Ω is holomorphic in Ω if and only if it satisfies*

$$\partial_{\bar{z}} f = 0,$$

and in that case the derivative of f is given as $f' = \partial_z f$.

Bevis: This is indeed a simple observation. One has $\partial_{\bar{z}} f = (\partial_x f + i\partial_y f)/2$, which vanishes precisely when (1.8) is satisfied. One has $\partial_z f = (\partial_x f - i\partial_y f)/2$ which equals $\partial_x f$ (and $\partial_y f$ as well) whenever $\partial_{\bar{z}} f = 0$, *i.e.*, whenever $\partial_x f = -i\partial_y f$. ❑

1.2.1 Power series

Rational functions are, although they form very important class of functions, very special. A rather more general class of functions are those given by power series—and indeed, as we shall see later on, it comprise all functions holomorphic in a disk.

(1.1) Recall that a power series $f(z) = \sum_{n\geq 0} a_n(z-a)^n$ has a radius of convergence given as $R^{-1} = \limsup \sqrt[n]{|a_n|}$. That is, the series converges absolutely for $|z-a| < R$, and the convergence is uniform on compact sets included in $|z-a| < R$; *e.g.*, closed disks given by $|z-a| \leq \rho < R$. For short we say that the convergence is *normal*.

Indeed, if $|z-a| < \rho < R$, choose ϵ with $0 < \epsilon \leq (R-\rho)/R\rho$. By definition one has $\sqrt[n]{|a_n|} < 1/R + \epsilon$ for $n >> 0$, and this gives

$$\sqrt[n]{|a_n|}\,|z-a| < \rho/R + \rho\epsilon < 1.$$

Thus we may appeal to Weierstrass M-test comparing with the series $\sum_{n\geq 0} M^n$ where $M = \rho/R + \rho\epsilon$.

(1.2) It is a theorem of Abel's that f is holomorphic in the disk of convergence and that the derivative may be found by termwise differentiation:

Teorem 1.1 *Assume that the power series $f(z) = \sum_{n\geq 0} a_n(z-a)^n$ has radius of convergence equal to R. Then f is holomorphic in the disk D centered at a and with radius R, and the derivative is given as* AbelLeddvisDiff

$$f'(z) = \sum_{n\geq 1} n a_n (z-a)^{n-1}. \qquad (1.9)$$

that is, the power series can differentiated term by term. PowSeriesDerivativ

Bevis: We may assume that $a = 0$. Since $\lim_{n\to\infty} \sqrt[n]{n} = 1$, the derived series has the same radius of convergence as the one defining f. Let R be the radius of convergence and denote by D the disk where the convergence takes place; that is, the disk given by $|z-a| < R$ and fix a point $z \in D$.

By the binomial theorem one has $(z+h)^n - z^n = nz^{n-1}h + h^2 R_n(z,h)$. It follows that the series $\sum_{n\geq 1} a_n R_n(z,h)$ converges normally for those h with $z+h \in D$, since both the series for f and the derived series converge normally in D.

Hence the sum $\sum_{n\geq 1} a_n R_n(z,h)$ is continuous and therefore bounded on a closed disk centered at z sufficiently small to be contained in D. We deduce that for h close to zero it holds true that

$$f(z+h) - f(z) = h \sum_{n\geq 1} a_n z^{n-1} + h^2 \sum a_n R_n(z,h),$$

where the term $\sum a_n R_n(z,h)$ is bounded, and the claim follows. ❑

(1.3) Successive applications of Abel's theorem shows that a function $f(z)$ i given by a power series has derivatives of all orders, and by an easy induction argument one finds the series

$$f^{(k)}(z) = \sum_{n\geq k} n(n-1)\ldots(n-k+1)a_n(z-a)^{n-k}$$

for the k-derivative of f. The constant term of this series equals $k!a_k$, so substituting a for z gives $k!a_k = f^{(k)}(a)$. Hence we have the following result, which may informally be stated as if f has a power series expansion, the expansion is the Taylor series of f.

TaylorPowerSeries **Setning 1.3** *A function f given as a power series*

$$f(z) = \sum_{n\geq 0} a_n(z-a)^n$$

converging normally a disk D centered at a, has derivatives of all orders, and it hold true that

$$a_n = \frac{f^{(n)}(a)}{n!}.$$

Oppgave 1.4. Prove the Cauchy-Riemann equations by letting h approach zero through respectively real and purely imaginary values in (1.1) . ✷

Oppgave 1.5. Assume that $f = u + iv$ is holomorphic in the domain Ω. Use the Cauchy-Riemann equations to show that the gradient of u is orthogonal to the gradient of v and conclude that the level sets of the real part of f are orthogonal to the level sets of the imaginary part. ✷

Oppgave 1.6. Assume that V is a complex vector space and that $A\colon V \to V$ is an $\mathbb{R}$-linear map. One says that A is *$\mathbb{C}$-anti-linear* if $A(zv) = \overline{z}A(v)$ for all $z \in \mathbb{C}$ and all $v \in V$. Show that A is $\mathbb{C}$-anti-linear if and only if $A(iv) = -iA(v)$ for all vectors $v \in V$. Show that any A may be decomposed in a unique way as a sum $A = A_+ + A_-$, where A_+ is $\mathbb{C}$-linear and A_- is $\mathbb{C}$-anti-linear. Hint: Let $A_+(v) = (A(v) - A(iv))/2$ and $A_-(v) = (A(v) + A(iv))/2$. ✷

Oppgave 1.7. Assume that V is a one-dimensional complex vector space and that $A\colon V \to V$ is an $\mathbb{R}$-linear map. Show that A is multiplication by a complex number if and only if its $\mathbb{C}$-anti-linear part vanishes; *i.e.,* $A_- = 0$. ✷

Oppgave 1.8. Show that complex conjugation $\overline{z}$ is not $\mathbb{C}$-differentiable at any point. ✷

Oppgave 1.9. Show that for any complex $\mathbb{R}$-differentiable function it holds that $\overline{\partial_{\overline{z}} f} = \partial_z \overline{f}$. ✷

Oppgave 1.10. Show that $\partial_{\overline{z}} \overline{z} = 1$ and that $\partial_z \overline{z} = 0$. ✷

Oppgave 1.11. A function f $\mathbb{R}$-differentiable in the domain Ω is called *anti-holomorphic* if $\partial_z f = 0$ throughout Ω. Show that $f(z)$ is anti-holomorphic if and only if $\overline{f(z)}$ is holomorphic. ✷

1.3 Integration and Cauchy's formula

Recall that a *line integral* is an integral on the form $\int_\gamma pdx + qdy$ where γ is a path in the complex plane and p and q are two functions, real or complex, defined and continuous along the path γ. The path γ is a parametrization of a curve in $\mathbb{C}$, *i.e.,* a function $\gamma\colon [\alpha, \beta] \to \mathbb{C}$ that in our context always will be piecewise C^1; that is, in addition to γ being continuous, there should be a partition of the parameter-interval $[\alpha, \beta]$ such that γ is continuously differentiable on each of the closed subintervals.

Now and then, as a shortcut, we shall specify a curve C instead of a path in the integral; in that case it should be clearly understood from the context which way the curve should be parametrized. A frequently occurring example, is that of a circle C. The implied parametrization will be $\gamma(t) = a + re^{it}$ with the parameter t running from 0 to 2π and a being the center and r the radius of C—the circle is traversed once counterclockwise. Circles appear frequently in the disguise as boundaries of disks D; that is, as ∂D.

1.3.1 Differential forms

The integrand in a line integral, that is the expression $\omega = pdx + qdy$ is called a *differential form*, more precisely one should say a *differential one form*, since, as the name indicates, there are also *two-forms* and even n-forms for any natural number n. We shall make use two-forms, but no n-form with n larger than two will appear.

(1.1) You will find no mystery in the definition of a line integral if the path γ is C^1 and given as $\gamma(t) = x(t) + y(t)i$ with $t \in [\alpha, \beta]$. One simply proceeds in the direction the nose points, replacing x and y in the functions p and q with $x(t)$ and $y(t)$, and replacing dx and dy with $x'(t)dt$ and $y'(t)dt$. This gives a conventional integral over the interval $[\alpha, \beta]$:

$$\int_\gamma \omega = \int_\gamma pdx + qdy = \int_\alpha^\beta p(\gamma(t))x'(t)dt + q(\gamma(t))y'(t)dt.$$

In case γ is just piecewise C^1, one follows this procedure for each of the subintervals where γ is C^1, and at the end sums the appearing integrals.

(1.2) Given a real valued function u in the domain Ω. The differential du of u is the one-form

$$du = \partial_x dx + \partial_y u dy,$$

and forms of tis type are said to be *exact forms*. It is particularly easy to integrate exact forms, they behave just like derivatives (in some sense, they are derivatives). One has

$$\int_\gamma du = u(\gamma(\beta)) - u(\gamma(\alpha)), \tag{1.10}$$

IntExactForm The integral is just the difference between the values of u at the two ends of the path and does not depend on which path one follows, as long as it starts and ends where at the same places as γ. In particular if a path γ is closed, the integral of du round γ vanishes.

The formula 1.10 follows from the fundamental theorem of analysis and the chain rule. The chain rule immediately gives

$$\frac{d}{dt} u(\gamma(t)) = u_x(\gamma(t))x'(t) + u_y(\gamma(t))y'(t),$$

and one finishes off with fundamental theorem.

(1.3) Speaking about two-forms, in our case they are just expressions $pdx \wedge dy$ where p is a function of the appropriate regularity (*e.g.*, continuously differentiable) in the domain Ω where the form lives. The "wedge product" is anti-commutative, *i.e.*, $dx \wedge dy = -dy \wedge dx$, a feature that becomes natural when one defines the integral of w. To do this, let $r(s,t) = u(r,s) + iv(r,s)$ be a parametrization of Ω; *i.e.*, a continuously differentiable homeomorphism from some open set $U \subseteq \mathbb{R}^2$ (of course life could be as simple as U being equal to Ω and r being the identity). With the parametrization in place, one has the Jacobian determinant

$$\frac{\partial(u,v)}{\partial(s,t)} = \det \begin{pmatrix} u_s & u_t \\ v_s & v_t \end{pmatrix},$$

and one defines the integral $\int_\Omega \omega$ as

$$\int_\Omega \omega = \iint_U p(r(s,t)) \frac{\partial(u,v)}{\partial(s,t)} dudv \tag{1.11}$$

DefIntTwoForm Exchanging u and v changes the sign of the Jacobian determinant and by consequence the sign the double integral to the right in (1.11) . So the definition is consistent with $du \wedge dv = -dv \wedge du$, *i.e.*, the wedge product being anti-commutative.

(1.4) A one form $\omega = pdx + qdy$ in Ω with p and q C^1-functions, has a derivative d which is a two-form. It is given by the rules DiffOfOneForm

$$d^2 = 0 \qquad d(u\omega) = du \wedge \omega + ud\omega \tag{1.12}$$

Hence with $\omega = pdx + qdy$ we find

$$\begin{aligned} d\omega &= dp \wedge dx + pd^2x + dq \wedge dy + qd^2y \\ &= (\partial_x pdx + \partial_y pdy) \wedge dx + (\partial_x qdx + \partial_y qdy) \wedge dy \\ &= (\partial_x q - \partial_y p)dx \wedge dy. \end{aligned}$$

1.3.2 Complex integration

(1.1) Now, let $f(z)$ be a complex function defined in the domain Ω whose real part is u and imaginary part is v, so that $f(z) = u(z) + iv(z)$. We want to make sense of integrals of the form

$$\int_\gamma f(z)dz,$$

where the complex differential dz is defined as $dz = dx + idy$. Introducing this into the expression $f(z)dz$, multiplying out and separating the real and imaginary parts, we find

$$\int_\gamma f(z)dz = \int_\gamma (udx - vdy) + i(vdx + udy), \tag{1.13}$$

which is just a combination of two ordinary real integrals. KompleksIntegral

(1.2) It is a fundamental principle (universally valid only interpreted with care[6]) principle "that integrating the derivative of a function gives us the function back", and in our context it remains in force—frankly speaking, any thing else would be unthinkable. A complex function f differentiable in the domain Ω whose derivative is continuous[7] satisfies the equality

$$\int_\gamma f'(z)dz = f(b) - f(a), \tag{1.14}$$

where γ is any path joining the point a to the point b. The chain rule and the Cauchy-Riemann equations give FundFact

$$\begin{aligned} du &= u_x dx + u_y dy = u_x dx - v_x dy \\ dv &= v_x dx + v_y dy = v_x dx + u_x dy \end{aligned}$$

[6] There are increasing real functions having a derivative that vanishes almost everywhere
[7] One of the marvels of complex function theory is, as we soon shall se, that this is always true

combining this with the definition of the integral (1.13) we obtain $f'(z)dz = du + idv$, and the formula follows by the corresponding formula for exact real forms.

For a *closed* path γ with parameter running from α to β one has $\gamma(\alpha) = \gamma(\beta)$, and consequently the integral around γ vanishes. We have

IntegralAvDerivertForsvinner
Setning 1.4 *If f is differentiable in the domain Ω with a continuous derivative, and γ is a closed path in Ω, then*

$$\int_\gamma f' = 0.$$

Cauchy's integral theorem—the corner stone of complex function theory—states that under certain topological condition on the closed path γ and the domain Ω, a similar statement is valid for any holomorphic function—that is, its integral along γ vanishes. We are going to establish this, step by step in progressively more general variants. The start being the case when γ is the circumference of a triangle.

(1.3) As an illustration we cast a glance on the rational functions. Every polynomial $P(z)$ trivially has a primitive (as you should know, the derivative of $z^{n+1}/(n+1)$ equals z^n), and therefore $\int_\gamma P(z)dz = 0$ as long as the path γ is closed. The same is true for any rational function of the type $c(z-a)^{-n}$ where $n \geq 2$ (a primitive being $(z-a)^{1-n}/(1-n)$, as you should know). The only obstruction for a rational function having a primitive is therefore the occurrence of terms of type $c/(z-a)$ in its decomposition in partial fractions. When being free of such terms, the rational function $F(z)$ satisfies

$$\int_\gamma F(z)dz = 0$$

for closed paths γ avoiding the points where F is not defined.

(1.4) The converse of proposition 1.4 above also holds. One has

EksistensAvPrimitive
Setning 1.5 *Let $f(z)$ be continuos in the domain Ω and assume that $\int_\gamma f(z)dz = 0$ whenever γ is a closed path in Ω. Then $f(z)$ has primitive in Ω, in other words, there is a function $F(z)$ defined in Ω with $F'(z) = f(z)$.*

Bevis: We begin with choosing a point z_0 in Ω. Since the integral of f round any closed path vanishes, we may define a function $F(z)$ by declaring

$$F(z) = \int_\gamma f(z)dz,$$

where γ is any path from z_0 to z; Indeed, the integral has the same value whatever path of integration we chose, as long as it connects z_0 to z: If γ_1 and γ_2 are two of the kind, the path $\gamma_1\gamma_2^{-1}$ is closed, and thus we have

$$0 = \int_{\gamma_1\gamma_2^{-1}} f(z)dz = \int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz.$$

We have to verify that F is differentiable and that the equality $F'(z) = f(z)$ holds. The difference $F(z+h) - F(z)$ can be computed by integrating $f(z)$ along any path leading from z til $z+h$. As h is small in modulus, we may assume that $z+h$ lies in a disk centered at z. Then the line segment parametrized as $\gamma(t) = z + th$ with $0 \leq t \leq 1$ is contained in Ω. Now, $dz = ht$ along γ, and we find the following expression for the differential quotient of F:

$$h^{-1}\big(F(z+h) - F(z)\big) = h^{-1}\int_\gamma f(z)dz = \int_0^1 f(z+th)dt$$

It is a well known matter, and trivial to prove, that $\lim_{h\to 0}\int_0^1 f(z+th)d = f(z)$ when f is continuous at the point z, and with that, we are through. ❏

(1.5) Cauchy's approach to the his theorem was via what is now called Green's theorem, which by the way never is mentioned in any of Green's writings. The first time the statement occurs is in a paper by Cauchy from 1846. However Cauchy does not prove it, he promised a proof that never appeared, and the first proof was given by Riemann. For an extensive history of these matters one may consult [?]. The theorem is today stated in calculus courses as

$$\iint_\Omega (\partial_x q - \partial_y p)dxdy = \int_{\partial\Omega} pdx + qdy$$

where $\partial\Omega$ is the border of the domain Ω, and this form is very close to the way Cauchy stated it. In terms differential forms, it it takes the following appealing look:

$$\int_\Omega d\omega = \int_{\partial\Omega} \omega,$$

a formula that obtained by substituting the equality $d\omega = (\partial_x q - \partial_y p)dx \wedge dy$ from paragraph (1.4) in formula in Green's theorem.

There are two fundamental assumptions in Green's theorem. One about the functions involved, they must continuously differentiable (in the real sense) and one on the geometry. The border $\partial\Omega$ must be a curve that has a piecewise parametrized by continuously differentiable functions in a way that Ω lies to the left of $\partial\Omega$. This the current "calculus way" to state Green's theorem, but there are stronger versions around.

The general geometrical assumptions are notoriously fuzzy, and the proof in the general case is involved, but of course in simple concrete situations proof is simple. Just a combination of Fubini's theorem about iterated integration and the fundamental theorem of analysis. We shall not dive into general considerations about Green's theorem, but will only use it in clear cut situations.

(1.6) It is interesting to give Green's theorem a formulation adapted to the specific context of complex function theory; *i.e.,* a formulation in terms of the differential operators ∂_z and $\partial_{\bar z}$: As $d^2z = 0$ and $dz \wedge dz = 0$, one has

$$d(fdz) = (\partial_z f dz + \partial_{\bar z} f d\bar z) \wedge dz = \partial_{\bar z} f d\bar z \wedge dz$$

which gives

$$-\int_{\Omega} \partial_{\overline{z}} f dz \wedge d\overline{z} = \int_{\partial\Omega} f.$$

In view of the equality $d\overline{z} \wedge d_z = 2idx \wedge dy$, one obtains

$$\int_{\partial\Omega} f(z)dz = 2i \iint_{\Omega} \partial_{\overline{z}} f(z)dxdy$$

In view of the $\partial_{\overline{z}}$-formulation of the Cauchy-Riemann equations as in theorem 1.2 on page 10; that is $\partial_{\overline{z}} = 0$ for holomorphic f's the form of Greens theorem in the form above, one obtains a version of the Cauchy's theorem:

Teorem 1.2 *Let f be a function that is holomorphic with continuous derivative in a domain Ω for which Green's theorem is valid; i.e., the border $\partial\Omega$ has a piecewise C^1-parametrization. Then it holds true that*

$$\int_{\partial\Omega} f(z)dz = 0.$$

This is of course a very nice result, but it is not entirely satisfying. In the days of Cauchy a holomorphic function had a continuous derivative by assumption, but nowadays that condition is dropped—as in our definition. The reason one can do this, is that Cauchy's theorem remains valid when the continuity of the derivative is not assumed; a result due to Edouard Jean-Baptiste Goursat, and which is the topic of the next section.

1.3.3 Moore's proof of Goursat's lemma

As announced, this paragraph is about Goursat's lemma the vanishing of integrals of holomorphic functions round triangles, of course without assumptions about continuity of the derivative. Goursat published this in 1884, and the simple and beautiful proof we give—really one of the gems in mathematics—is now standard and was found by Eliakim Hastings Moore in [?] from 1900 , and it is not due to Goursat as claimed in many texts. There is a point of exception occurring in the statement, which makes it easy to deduce Cauchy's formula from the lemma (which by the way we have promoted to a theorem).

GoursatLemma **Teorem 1.3** *Let Ω be a domain containing the triangle Δ and let $p \in \Omega$ be a point. Let f be a function continuous in Ω and differentiable through out $\Omega \setminus \{p\}$. Then*

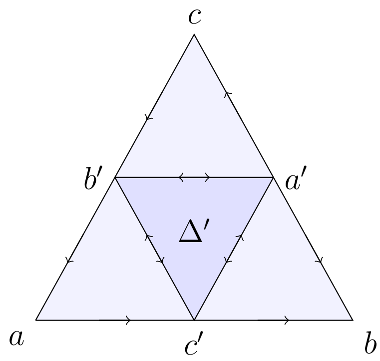
$$\int_{\partial\Delta} f(z)dz = 0.$$

BEVIS: In the first, and essential part, of the proof the special point p is assumed to lie outside the triangle Δ.

We shall describe a process that when fed with a triangle Δ, returns a new triangle Δ' contained in Δ and having the the following two properties:

1. $\left|\int_{\partial\Delta} f(z)dz\right| \leq 4\left|\int_{\partial\Delta'} f(z)dz\right|$
2. Both the diameter and the perimeter of Δ' is half of those of Δ.

Let the corners of Δ be a, b and c; and denote by c' the midpoint of the edge of Δ from a to b, by b' the midpoint of the edge from a to c and by a' the midpoint of the edge from b to c. These six points serve as corners of four new triangles that subdivide Δ; say Δ^i with $1 \leq i \leq 4$. As the new corners are the midpoints of the old edges, the perimeter of each of the triangles Δ^i is half that of Δ, and similarly for the diameters, they all equal half the diameter of Δ. So any of the four Δ^i-s satisfies the second requirement above.



Figur 1.1: *A triangles $\Delta = abc$ and the $\Delta' = a'b'c'$*

So to the first requirement. In the sum to the right in (1.15) below, the integrals of f along edges sheared by two of the four triangle cancel, and hence the equality in (1.15) is valid:

$$\int_{\partial\Delta} f(z)dz = \sum_i \int_{\partial\Delta^i} f(z)dz, \tag{1.15}$$

IntCancels

$$\left|\int_{\partial\Delta} f(z)dz\right| \leq \sum_i \left|\int_{\partial\Delta^i} f(z)dz\right|.$$

Among the four triangles Δ^i-s we pick the one for which $\left|\int_{\partial\Delta^i} f(z)dz\right|$ is maximal as the new triangle Δ', the output of the process. One obviously has $\left|\int_{\partial\Delta} f(z)dz\right| \leq 4\left|\int_{\partial\Delta'} f(z)dz\right|$, and the second requirement above is fulfilled as well.

Iterating this process one constructs a sequence of triangles Δ_n all contained in Ω having the three properties below (where as usual $\lambda(A)$ stands for the perimeter of a figure A and $d(A)$ for the diameter)

- ☐ $\Delta_{n+1} \subseteq \Delta_n$;
- ☐ $\left|\int_{\partial\Delta} f(z)dz\right| \leq 4^n \left|\int_{\partial\Delta_n} f(z)dz\right|$;
- ☐ $\lambda(\Delta_n) < 2^{-n}\lambda(\Delta)$;

□ $d(\Delta_n) < 2^{-n}d(\Delta)$.

The triangles form a descending sequence of compact sets with diameters shrinking to zero; their intersection is therefore one point, say a. By assumption f is differentiable at a, and we may write

$$f(z) = f(a) + f'(a)(z-a) + \epsilon(z-a)$$

where $|\epsilon(z-a)/(z-a)|$ tends towards zero when z approaches a; so if $\eta > 0$ is a given number, $|\epsilon(z-a)| < \eta\,|z-a|$ for z sufficiently close to a; that is for $z \in \Delta_n$ for $n >> 0$. As the integrals of both the constant $f'(a)$ and of $f'(a)(z-a)$ around any closed path vanish, one finds

$$\int_{\partial\Delta_n} f(z)dz = \int_{\partial\Delta_n} \epsilon(z-a)dz,$$

and hence

$$4^{-n}\left|\int_{\partial\Delta} f(z)dz\right| \leq \left|\int_{\partial\Delta_n} f(z)dz\right| = \left|\int_{\partial\Delta_n} \epsilon(z-a)dz\right| \leq$$
$$\leq \int_{\partial\Delta_n} \eta\,|z-a|\,|dz| \leq \eta \cdot 2^{-n}d(\Delta)\cdot 2^{-n}\lambda(\Delta),$$

Things are now so beautifully constructed that factor 4^{-n} cancels, and the inequality becomes

$$\left|\int_{\partial\Delta} f(z)dz\right| < \eta d(\Delta)\lambda(\Delta)$$

The positive number η being arbitrary, we conclude that $\int_{\partial\Delta} f(z)dz = 0$.

Finally, if the point p is among the corners of Δ, we may subdivide Δ in two triangles Δ' and Δ'', one of them, say Δ', containing the special point p and having perimeter as small we want. As the point p lies outside Δ'', the integral of f round $\partial\Delta''$ vanishes by what we have already done; hence $\int_{\partial\Delta} f(z)dz = \int_{\partial\Delta'} f(z)dz$. This integral can be maid arbitrarily small since f is bounded in Δ and the perimeter of Δ' can maid arbitrarily small.

At the very end, we get away with the case of p lying inside Δ, but not being a corner, by decomposing Δ into three (or two if p lies on an edge of Δ) new triangles, each having p as one corner and two of the corners of Δ as the other two. ❏

FiniteP **OPPGAVE 1.12.** Let Ω be a domain and f a continuous function in Ω. Assume that for a finite set P of points in Ω, the function f is differentiable in $\Omega \setminus P$. Prove that $\int_{\partial\Delta} f(z)dz = 0$ for all triangles Δ in Ω. HINT: Induction and decomposition. ✷

OPPGAVE 1.13. Let Ω be a domain and f continuous and holomorphic in $\Omega \setminus P$ as in exercise 1.12. Show that the conclusion of 1.12 holds even if one only assumes that P is a closed subset without accumulation points in Ω. HINT: Triangles are compact. ✷

1.3.4 Cauchy's theorem in star-shaped domains

StarShaped

To continue the development of the Cauchy's theorem and expand its validity, we now pass to arbitrary closed paths in a *star-shaped domains* domain. Recall that the domain Ω is star-shaped if there is one point c, called the apex, such that for any z i Ω the line segment joining c to z is entirely contained in Ω. The point c is not necessarily unique, many domains have several apices.

Of course all convex domains are star-shaped, and this includes circular disks, the by far most frequently occurring domains in the course. The idea is to show that differentiable functions have primitives just by integrating them along line segments emanating from a fixed point. This is very close to the fact that continuous functions whose integral round any closed path vanishes, has a primitive (proposition 1.5 on 16), in star-shaped domains the vanishing of integrals round triangles suffices.

(1.1) So assume that f is continuous throughout a star-shaped domain Ω with apex c and assume that f is differentiable everywhere in Ω except possibly at one point p.

For any two points a and b belonging Ω, we denote by $L(a,b)$ the line segment joining a to b, and we assume tacitly that it is parametrized in the standard way; that is as $(1-t)a+tb$ with the parameter t running from 0 to 1. The domain Ω being star-shaped with apex c by assumption the segment $L(c,a)$ is entirely contained in Ω. Now, we define a function F in Ω by integrating f along $L(c,z)$, that is we set

$$F(z)=\int_{L(c,z)} f(z)dz. \tag{1.16}$$

The claim is that F is continuos throughout Ω and differentiable except at p with derivative equal to f; in other words, the function F is what one usually calls a *primitive* for f: Primitive

Setning 1.6 *Let Ω be a star-shaped domain and let p be a point in Ω. A continuous function f in Ω which is differentiable away from p, has a primitive in $\Omega\setminus\{p\}$.* CauchyConcex

Bevis: The task is to prove that $F(z)$ as defined by equation (1.16) is differentiable and that the derivative equals f. The proof is very close to the proof of proposition 1.5 (in fact, it is *mutatis mutandis* the same).

The obvious line of attack is to study the difference $F(a+h)-F(a)$ where a is an arbitrary point in Ω different from p and h is complex number with a small modulus. We fix disk centered at a contained in Ω. If $a+h$ lies in D, the line segment $L(a,a+h)$ lies in Ω as well.

We find

$$F(a+h)-F(a)=\int_{L(c,a+h)} f(z)dz-\int_{L(c,a)} f(z)dz=\int_{L(a,a+h)} f(z)dz, \tag{1.17}$$

the last and crucial equality holds true since the integral of f around the triangle with TriangleFormula corners c, a and $a+h$ vanishes by Goursat's lemma (theorem 1.3 on page 18).

The path $L(a, a+h)$ is parametrized as $a+th$ with the parameter t running from 0 to 1. Hence $dz = hdt$ along $L(a, a+h)$, and we find

$$\int_{L(a,a+h)} f(z)dz = h \int_0^1 f(a+th)dt.$$

The function f being continuous at a implies that given $\epsilon > 0$ there is $\delta > 0$ such that

$$|f(a+h) - f(a)| < \epsilon$$

whenever $|h| < \delta$. Hence

$$F(a+h) = F(a) + hf(a) + h \int_0^1 (f(a+th) - f(a))dt$$

where

$$\left| \int_0^1 (f(a+th) - f(a))dt \right| < \int_0^1 |f(a+th) - f(a)| \, dt < \epsilon,$$

once $|h| < \delta$. ❑

(1.2) Combining the theorem with the fact that the integral of a derivative round a loop vanishes, one obtains as an immediate corollary Cauchy's formula for star-shaped domains, namely:

CauchyInteFortmel

Korollar 1.1 *If f is a function continuous in the star-shaped domain Ω and holomorphic in $\Omega \setminus \{p\}$, then $\int_\gamma f(z)dz = 0$ for all closed paths γ in Ω.*

1.3.5 Cauchy's formula in a star-shaped domain

StrShap

By far the most impressive tool in the toolbox of complex function theory is Cauchy's formula, expressing the value of f at a point as the integral round a loop circling the point. Taking a step in the direction towards the general case, we proceed to establish this formula for star-shaped domains. This includes Cauchy's formula for disks. Albeit a modest version, it has rather strong implications for the local behavior of holomorphic functions. A crucial feature in the proof is the exceptional point p allowed in corollary 1.1 above—and this is the sole reason for including the exceptional point in the hypothesis of 1.1.

(1.1) The setting is as follows: The domain Ω is star-shaped and a is any point Ω. Furthermore γ a closed path in Ω not passing through a and f is function holomorphic throughout Ω.

The auxiliary function

$$g(z) = \begin{cases} \dfrac{f(z) - f(a)}{z - a} & \text{when } z \neq a \\ f'(a) & \text{when } z = a \end{cases} \tag{1.18}$$

is continuous at a since f is differentiable there, and in $\Omega \setminus \{a\}$ it is obviously holomorphic. Hence g fulfills the hypothesis in Cauchy's theorem (corollary 1.1 on page 22) and the integral of f round closed paths vanish. As a is not lying on the path γ it holds true that

$$\int_\gamma \frac{f(z)-f(a)}{z-a}dz = 0,$$

from which one easily deduces

$$\int_\gamma f(z)(z-a)^{-1}dz = f(a)\int_\gamma (z-a)^{-1}dz. \tag{1.19}$$

The integral $\int_\gamma (z-a)^{-1}dz$ is, as we shall see later on, an integral multiple of $2\pi i$, and we defines the integer $\mathrm{n}(\gamma, a)$ by setting CauchY00

$$\mathrm{n}(\gamma, a) = \frac{1}{2\pi i}\int_\gamma (z-a)^{-1}dz.$$

It is called the *winding number* of g round a. We have thus establish the following version of Cauchy's formula for star-shaped domains:

Teorem 1.4 *Let f be holomorphic in the star-shaped domain Ω and a a point in Ω. For any closed path γ, one has* CauchyStarShaped

$$\frac{1}{2\pi i}\int_\gamma f(z)(z-a)dz = \mathrm{n}(\gamma, a)f(a).$$

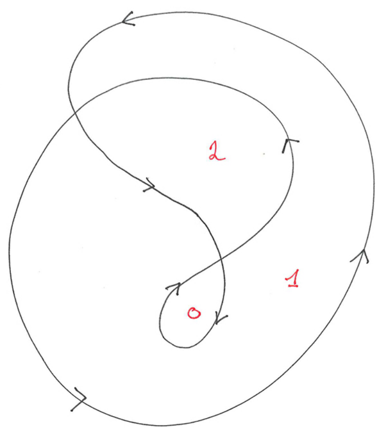
Of course this formula comes to its full force only when the winding number $\mathrm{n}(\gamma, a)$ is known. Hence it is worth while using some time and energy in studying the winding number and establish some of its general properties. We do that in the next paragraph.

(1.2) The following lemma is just a rephrasing in the lingo of function theory of a small lifting lemma from topology saying that any continuous map from an interval to the circle $\mathbb{S}^1$ lifts to universal cover $\mathbb{R}$ of $\mathbb{S}^1$. It is simple but crucial in our context, so we offer a proof.

Lemma 1.1 *Any path $\gamma(t)$ satisfying $|\gamma(t)| = 1$ for all values t of the parameter, may be brought on form $\gamma(t) = e^{i\phi(t)}$.*

If you wonder what kind of path γ is, it si just a movement on the unit circle. The function ϕ is a logarithm of $\gamma(t)$. So along portions of the path where the complex logarithm is defined, it is trivial that $\phi(t)$ exists. The function ϕ is also only unique up to additive constants of the form $2n\pi i$ with $n \in \mathbb{Z}$.

Bevis: For simplicity, we assume that parameter interval of γ is the unit interval $[0,1]$. Let τ be the supremum of the numbers s such that $\phi(t)$ exists for over $[0,s]$. In a neighbourhood U of $\gamma(\tau)$ the complex logarithm $\log w$ is well defined. We choose one of the branches and let $\psi(t) = \log \gamma(t)$ for $t \in \gamma^{-1}(U)$. One of the connected components of the inverse image $\gamma^{-1}(U)$ is an open interval J containing τ, and over $[0,\tau) \cap J$ the two functions ϕ and ψ differ only by an additive constant. Hence by adjusting ψ we can make them agree, and ϕ can be extended beyond τ, contradicting the definition of τ. ❏



The lemma allows paths to be parametrized with polar coordinates centered at points not on the path. The radius vector is just $r(t) = |\gamma(t) - a|$, and the angular coordinate is given as in the lemma; it is one of the functions $\phi(t)$ with $e^{i\phi(t)} = (\gamma(t) - a)\,|\gamma(t) - a|^{-1}$. Thus one has

$$\gamma(t) = a + r(t)e^{i\phi(t)}.$$

With this parametrization one finds $\gamma'(t) = r'(t)e^{i\phi(t)} + ir(t)e^{i\phi(t)}\phi'(t)$, and upon integration we arrive at

$$\int_\gamma (z-a)^{-1}dz = \int_\alpha^\beta (r'(t)r(t)^{-1} + i\phi'(t))dt =$$
$$= \log r(\beta) - \log r(\alpha) + (\phi(\beta) - \phi(\alpha))i.$$

where log designates the good old and well behaved real logarithm. As the path γ is closed, $r(\beta) = r(\alpha)$ and $e^{i\phi(\beta)} = e^{i\phi(\alpha)}$, the latter equality implying that $\phi(\beta) - \phi(\alpha)$ is an integral multiple of 2π. We have establish

Lemma 1.2 *The winding number* $\mathrm{n}(\gamma, a) = \frac{1}{2\pi i}\int_\gamma (z-a)^{-1}dz$ *is an integer.*

Finally, we examine to which extent $\mathrm{n}(\gamma, a)$ varies with the point a, and we shall prove

WindNumbConnComp **Setning 1.7** *If a and b belong to the same connected component of $\mathbb{C}\setminus\gamma$, then* $\mathrm{n}(\gamma, a) = \mathrm{n}(\gamma, b)$*, and the winding number* $\mathrm{n}(\gamma, a)$ *vanishes for a in the unbounded component.*

Assume that a and b are two different points and let z be any point in the plane. An elementary geometric observation is that the point z lies on the line through a and b if and only if the two vectors $z - a$ and $z - b$ are parallel or anti-parallel; phrased in other words, one is a real multiple of the other. They point in opposite directions if z belongs to the line segment $L(a, b)$ joining a to b, and in the same direction if not. The fractional linear transformation

$$A(z) = \frac{z-a}{z-b}$$

therefore maps the line segment between a and b onto the negative real axis.

Now, the principal branch $\log w$ of the logarithm is well defined and holomorphic in the split plane $\mathbb{C}^-$; that is in the complement of the negative real axis. Since the line segment $L(a, b)$ corresponds to the negative real axis under the map A, we conclude that $\log A(z) = \log(z - a)(z - b)^{-1}$ is well defined and holomorphic in the complement $\mathbb{C} \setminus L(a, b)$.

Lemma 1.3 *Let a and b bee different point in the complex plane and let γ be any closed path in $\mathbb{C}$. If γ does not intersect the line segment from a to b, the winding numbers of γ around a and b are the same, that is* $\mathrm{n}(\gamma, a) = \mathrm{n}(\gamma, b)$*.*

BEVIS: The function $g(z) = \log(z - a)(z - b)^{-1}$ is defines and holomorphic along γ, and its derivative is given as

$$g'(z) = (z - a)^{-1} - (z - b)^{-1}.$$

As the integral of a derivative round a loop vanishes, we obtain

$$0 = \int_\gamma g'(z)dz = \int_\gamma (z-a)^{-1}dz - \int_\gamma (z-b)^{-1}dz$$

and we are happy! ❑

The proof of proposition 1.7 will be complete once we prove that any to points a and b in same component U of $\mathbb{C} \setminus \gamma$ can be connected by a piecewise *linear* path.

Connect a and b by a continuous path δ, and cover d by finitely many disks V_j all lying in U. By Lebesgue's lemma there is a partition $\{t_i\}$ of the parameter interval , such the portions of the path with parameter in the subintervals $[t_{i-1}, t_i]$ is contained in one of the V_j-s. But V_j being convex, the line segments between $\delta(t_{i-1})$ and $\delta(t_i)$ lie in V_j and a fortiori in U. Thus any two points in U can be connected by a piecewise path, and we are done.

As an illustration, be offer the nice curve drawn in figure xxx. It divides the plane into four connected components and the corresponding winding numbers are indicated in red ink. In two of the components the winding number is zero, and in two others they are 1 and 2 respectively.

Lemma 1.4 *If D is a disk and a is any point in D, then* $\mathrm{n}(\partial D, a) = 1$.

Bevis: Winding numbers being constant throughout components (proposition 1.7 on page 25) it suffices to check that winding number of ∂D round the center of the disk equals one, so we take a to be the center of D and parametrize ∂D as $z(t) = a + re^{it}$ with t running from 0 to 2π . One has $dz = ire^{it}dt$ and as $z - a = re^{ir}$ the integral becomes

$$\int_\gamma (z-a)^{-1}dz = i\int_0^{2\pi} dt = 2\pi i,$$

and $\mathrm{n}(\partial D, a) = 1$. ❑

Oppgave 1.14. Let C be the circle centered at a having radius r. Assume that γ is the path $a + re^{int}$ with n and integer and the parameter running from 0 to π—that is, it traverses the circle C n times in the direction indicated by the sign of n—then the winding number is $\mathrm{n}(\gamma, a) = n$. ✷

(1.3) A special case of theorem 1.4 is the Cauchy's formula for a circle:

LocalCauchy

Teorem 1.5 *Let D be a disk centered at a and f a function holomorphic in a domain containing the closure $\overline{D}$. The one has*

$$\frac{1}{2\pi i}\int_{\partial D} f(z)(z-a)^{-1}dz = f(a),$$

where the circumference ∂D is traversed once counterclockwise.

(1.4) In polar coordinates; *i.e.*, $z(t) = a + re^{it}$ this reads

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{it})dt$$

So the value of f at a equals the mean value of f along any circle centered at a on which f is holomorphic.

1.4 Consequences of the local version Cauchy's formula

The Cauchy formula has an impressive series of very strong consequences for holomorphic functions; the most important is that they will be infinitely many times differentiable; *i.e.*, have derivatives of all orders. Other important results are the maximum principle (which also has a global aspects) and the open mapping theorem, and finally Liouvilles theorem. This definitively a global statement saying that a bounded entire function is constant.

1.4.1 Derivatives of all orders and Taylor series

DerivativesOfAllOrders The setting in this section is slightly more general than in the previous section. Basically we introduce a way of getting hand on a lot of holomorphic functions by integration along curves, and we show that these functions are analytic, *i.e.*, they have well behaved Taylor expansions round every point where they are defined, and finally, by Cauchy's formula any f holomorphic in a disk, is obtained in this way.

(1.1) We start out with a path γ and a function ϕ defined on γ. The only hypothesis on ϕ is that it be integrable; that is the function $\phi(\gamma(t))$ must be a measurable function on the parameter interval $[\alpha, \beta]$, and the integral $\int_\gamma |\phi(z)|\,|dz| = \int_\alpha^\beta |f(\gamma(t))\gamma'(t)|\,dt$ must be a finite number. We reserve the letter M for that number. Integrating $\phi(z)(z-w)$ along γ gives us a function $\Phi(z)$ defined at every point z not lying on γ; that is, we have

$$\Phi(z) = \int_\gamma \phi(w)(w-z)^{-1}dw$$

for z not on γ. We shall see that Φ has derivatives of all orders, and we are going to give formula for the Taylor polynomials of Φ round any point a (not on γ) with a very good and practical estimate for the residual term. From this, we extract formulas for the derivatives of Φ analogous to Cauchy's formula and show that Taylor series converges to Φ.

Setning 1.8 *The function $\Phi(z)$ is holomorphic and has derivatives of all orders off the path γ. Its n-th derivative is given as the integral* AuxFuncPhi

$$\Phi^{(n)}(z) = n!\int_\gamma \phi(w)(w-z)^{-n-1}dw.$$

The Taylor series of Φ at any point not on γ converges normally to Φ in the largest disk centered at z not hitting γ.

Bevis: We shall exhibit the Taylor series of Φ round any point a not lying on the curve γ. The tactics are simple and clear: Expand $(w-z)^{-1}$ in finite sum of powers of $(z-a)$ (with a residual term), multiply by $\phi(w)$, integrate along γ and hope that we control the residual term sufficiently well.

We begin carrying out this plan by recalling a formula from the old days in high school when one learned about geometric series, that is

$$\frac{1}{1-u} = 1 + u + \cdots + u^n + \frac{u^{n+1}}{1-u}, \tag{1.20}$$

where u is any complex number. We want to develop $(w-z)^{-1}$ in powers of $(z-a)$, GeomSeries
and to that end we observe that

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{(w-a)}\frac{1}{1-\dfrac{z-a}{w-a}},$$

and in view of (1.20) above, we find by putting $u = (z-a)(w-a)^{-1}$

$$\frac{1}{w-z} = \sum_{k=0}^{n} \frac{(z-a)^k}{(w-a)^{k+1}} + \frac{(z-a)^{n+1}}{(w-z)(w-a)^{n+1}}.$$

Multiplying through by $\phi(w)$ and integrating along the path γ yields

$$\Phi(z) = \sum_{k=0}^{n} (z-a)^k \int_\gamma \phi(w)(w-a)^{-k-1} dw + R_n(z)(z-a)^{n+1}.$$

The factor $R_n(z)$ in the residual term equals

$$R_n(z) = \int_\gamma \phi(w)(w-z)^{-1}(w-a)^{-n-1} dw,$$

an expression that has a for our purpose a good upper bound. Indeed, let $d = \inf_{w\in\gamma} |w-a|$ be the distance from a to the curve γ. It is strictly positive since γ is compact and a does not lie on γ. Pick a positive number η with $\eta < 1$. For any z with $|z-a| < \eta d$ one has $|w-z| \geq |w-a| - |z-a| \geq (1-\eta)d$, and it is easily seen that these estimates give

$$|R_n(z)| < (1-\eta)^{-1} M / d^{-n-2}.$$

Hence

$$\left|R_n(z)(z-a)^{n+1}\right| < (1-\eta)^{-1} d^{-1} M \big(\frac{z-a}{d}\big)^{n+1} < (1-\eta)^{-1} d^{-1} M \eta^{n+1}.$$

The residual term tends uniformly to zero as n tends to infinity because $\eta < 1$, and we have established that $\Phi(z)$ has a power series expansion in any disk centered at a whose closure does not hit γ, and furthermore the n-th coefficient of the power series equals

$$\int_\gamma f(w)(w-z)^{-n-1} dw.$$

F The theorem now follows now from the principle that "every power series is a Taylor series" (proposition 1.3 on page 12). ❑

(1.2) In the theorem we assumed that γ parametrizes a compact curve, but the proof goes through more generally at least for points having a positive distance to γ; of course the main hypothesis is that ϕ be integrable along γ. For example, if γ is the real axis (strictly speaking, the parametrization of the real axis with the identity) and ϕ is any integrable function, the corresponding Φ is holomorphic off the axis.

Oppgave 1.15. Let $X \subseteq \mathbb{C}$ be a measurable subset and let ϕ be an integrable function on X. Define

$$\Phi(z) = \iint_X \phi(w)(w-z)^{-1} dxdy$$

where $dxdy$ is the two-dimensional Lebesgue measure. Show that Φ is holomorphic off X. ✷

nyoppgave1 Oppgave 1.16. Assume that Γ is an "infinite contour", that is a path parametrized over the open interval $I = (0, \infty)$. Let $\phi(t)$ be an integrable function on Γ; that is, ϕ is measurable and the integral $\int_0^\infty |\phi(\Gamma(t))\Gamma'(t)|\, dt$ is finite. Define

$$F(z) = \int_\Gamma \phi(w)(w-z)^{-1} dw,$$

for z not on Γ. Show that this is meaningful; *i.e.*, both the real and the imaginary part of the integral are convergent. Show that $F(z)$ is a holomorphic function off Γ. ✷

(1.3) Our main interest in proposition 1.8 above are the implications it has for holomorphic functions. So let f be a function holomorphic in the domain Ω. For any point $z \in \Omega$ and any disk D contained Ω with center at z, Cauchy's local formula (theorem 1.5 on page 1.5) tells us that

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} f(w)(w-z)^{-1} dw.$$

As usual, the boundary ∂D is traversed once counterclockwise. Hence we are in a good position to apply proposition 1.8 with the path γ being ∂D and the function ϕ being the restriction of f to ∂D—indeed, from Cauchy's formula we deduce that the function Φ then equals f, and 1.8 translates into the fundamental and marvelous

Teorem 1.6 *Assume that f is holomorphic in the domain Ω. Then f has derivatives of all orders, and for the n-th derivative the following formula holds true* CauchyNthDerivat

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D} f(w)(w-z)^{-n-1} dw,$$

where D is any disk centered at z and contained in Ω, and where, as usual, ∂D is traversed once counterclockwise. The Taylor series of f about any point z, converges normally to $f(z)$ in D.

1.4.2 Cauchy's estimates and Liouville's theorem

This paragraph is about *entire* functions; that is, functions being holomorphic in the entire complex plane. For such functions one may apply Cauchy's formula for the higher derivatives from the previous paragraph over any disk in $\mathbb{C}$. Using the disk centered at a point a with radius R one obtains upper bounds for the modulus of the higher derivative. These are famous the Cauchy estimates:

$$\left|f^{(n)}(a)\right| = \frac{n!}{2\pi}\int_{\partial D}\left|f(w)(w-a)^{-n-1}dw\right| < n!\sup_{w\in\partial D}|f(w)|/R^n, \qquad (1.21)$$

CauchyEstimate the perimeter of D being $2\pi R$ and $|z-a|$ being equal R on the circumference ∂D. One of the consequences of these estimates is that entire functions that are not constant must sustain a certain growth as z tends to infinity; they must satisfy growth conditions. The simples case is known as Liouville's theorem, and copes with bounded entire functions

Teorem 1.7 *Assume that f is a bounded entire function. Then f is constant.*

Bevis: Assume that $|f|$ is bounded above by M. For any complex number a, one has the Cauchy estimate for the derivative of f, that is inequality (1.21) with $n = 1$,

$$|f'(a)| \leq M/R,$$

valid for all positive numbers R, as large as one wants. Hence $f'(a) = 0$, and consequently f is constant. ❑

(1.1) The next application of the Cauchy estimates, which we include as an illustration, is a slight generalization of Liouville's theorem. Functions having a *sublinear growth* musts be constant

Setning 1.9 *Assume that $|f(z)| < A|z|^\alpha$ for some number $\alpha < 1$. Then f is constant.*

Bevis: The proof is *mutatis mutandis* the same as for Liouville's theorem. The Cauchy estimate on a disk with radius R and center a gives

$$|f'(a)| < A\sup_{z\in\partial D}|z|^\alpha/R < A(R+|a|)^\alpha/R.$$

The term to the right tends to zero as R tends to infinity since $\alpha < 1$ (by l'Hôpital's rule, for example) , and hence $f'(a) = 0$. Since a was arbitrary, we conclude that f is constant. ❑

(1.2) The third application of Liouville's theorem along this line, it a result saying that entire functions with polynomial growth are polynomials; polynomial growth meaning that $|f|$ is bounded above by $A|z|^n$ for positive constant A and a natural number n. One can even say more, f must be a polynomial whose degree is less than n:

Setning 1.10 *Let f be an entire function and assume that for a natural number n and a positive constant A one has $|f(z)| \leq A|z|^n$ for all z. Then f is a polynomial of degree at most n.*

Bevis: The proof goes by induction on n, the case $n = 0$ being Liouville's theorem. The difference $f - f(0)$ is obviously a polynomial of degree at most n if and only if f is, so replacing f by $f - f(0)$, we may assume that f vanishes at the origin. Then $g(z) = f(z)/z$ is entire and satisfies the inequality $|g| \leq A\,|z|^{n-1}$. By induction g is a polynomial of degree at most $n - 1$, and we are through. ❑

(1.3) For any domain Ω it is important, but often challenging, to determine the group $\mathrm{Aut}(\Omega)$ of holomorphic automorphisms of Ω. It consists of maps $\phi\colon \Omega \to \Omega$ that are *biholomorphic*, that is, they are bijective with the inverse being holomorphic as well. It is a group under composition.

An illustrative example, but also an important result in it self, we shall show that all the automorphisms of the complex plane are the affine functions; *i.e.,* functions of the type $az + b$:

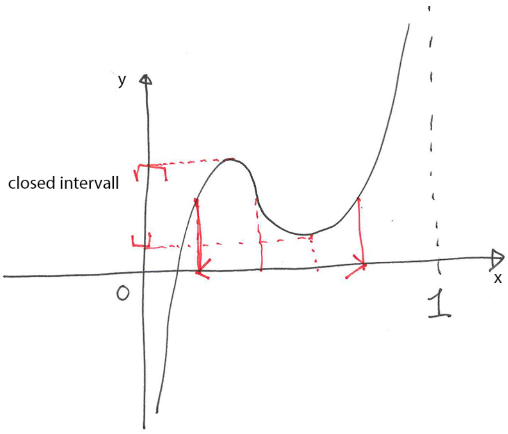
Setning 1.11 *If $\phi\colon \mathbb{C} \to \mathbb{C}$ is biholomorphic, then there are complex constants such that $\phi(z) = az + b$.*

Bevis: After having replaced ϕ by $\phi - \phi(0)$ we can assume that $\phi(0) = 0$, and have to prove that $\phi(z) = az$. The function $\phi(z)/z$ is holomorphic in the entire plane, and will turn out to be bounded. By Liouvilles theorem, it is therefore constant, say equal to a. Hence $f(z) = az$, and we are done.

It remains to see that $\psi(z)$ is bounded. Let $A_R = \{\, z \mid |z| > R \,\}$. Then $\phi(A_R) \cap \phi(\mathbb{C} \setminus A_R) = \emptyset$ and $\phi(\mathbb{C} \setminus A_R)$ is an open neighbourhood of 0. Hence ϕ does not have an essential singularity at infinity, but must have a pole. It must be order one, if not ϕ would not be injective, hence $\phi(z)/z$ is holomorphic at infinity and therefor bounded. ❑

1.4.3 The maximum modulus principle and the open mapping theorem

We start out in a laid back manner and consider a real function f in one variable defined on an open interval I. In general, there is no reason that $f(I)$ should be open, even if f is real analytic—any global maximum or minimum of f kills the openness of $f(I)$. A necessary criterion for f to be an open map (that is $f(U)$ is open for any open U) is that f have no local extrema, and in fact, this is also sufficient. Thus "having local maxima and minima" or "being an open mapping" are close-knit properties of f.



For holomorphic functions f the situation is in one aspect very different. The modulus of an holomorphic function never has local maxima, this is the renowned *maximum modulus principle*. The holomorphic functions are similar to real functions in the aspect that the maximum modulus principle is tightly knit to the functions being open mapping; and since the maximum modulus principle holds, they are indeed open maps.

(1.1) The maximum modulus principle can be approached in several ways, we shall present two. The first, presented in this paragraph, hinges on the Cauchy formula in a disk, and is a clean cut and the reason why the maximums principle holds is quite clear. The other one, which is in a sense simpler just using the second derivative test for maxima, comes at the end of this section.

Teorem 1.8 (The maximum modulus principle) *Let f be a function holomorphic in the domain Ω. Then $|f(z)|$ has no local maximum unless f is constant.*

BEVIS: The crankshaft in this proof is the Cauchy's formula expressed in polar coordinates. If D_r is a disk contained in Ω, centered at a and with r, one has

$$f(a) = \frac{1}{2\pi i}\int_0^{2\pi} f(a + re^{it})dt. \tag{1.22}$$

MeanValue This follows quickly by substituting $z = a + re^{it}$ in Cauchy's formula for a disk (theorem 1.5 on page 26), and the identity may be phrased as the "mean value of f on the circumference equals the value at the center".

Aiming for an absurdity, assume that a is a local maximum for the modulus $|f|$, and chose r so small that $|f(a)| \geq |f(z)|$ for all z in D_r. Now, if $|f(a)| = |f(z)|$ for all $z \in D_r$, it follows that f is constant. Hence for at least one r there are points on the circle ∂D_r where $|f|$ assumes values less than $|f(a)|$, and by a well known and elementary property of integrals of continuous functions, we get the following contradictory inequality:

$$|f(a)| \leq \frac{1}{2\pi i}\int_0^{2\pi} \left|f(a + re^{it})\right| dt < \int_0^{2\pi} |f(a)| = |f(a)|$$

❑

The following two offsprings of the maximum modulus theorem are immediate corollaries:

Korollar 1.2 *Let f a function holomorphic in the domain Ω. Then for any point a in Ω it holds true that $|f(a)| < \sup_{z\in\Omega} |f(z)|$ unless f is constant.*

Korollar 1.3 *Let $K \subseteq \Omega$ be compact and f a function holomorphic in Ω. Then f achieves it maximum modulus at the boundary ∂K, and unless f is constant, the maximum is strict.*

(1.2) Knowing there is a maximum principle one is tempted to believe in a minimal principle as well. And indeed, at least for non-vanishing functions, there is one. The proof is obvious: As long as f does not vanish in Ω, the inverse function $1/f$ is holomorphic there, and the maximum modulus principle for $1/f$ yields a minimum modulus principle for f.

Teorem 1.9 (The minimum modulus principle) *Assume that the function f is a non-vanishing and holomorphic in the domain Ω. Then f has no local minimum in Ω unless f is constant.*

(1.3) We have now come to the open mapping theorem, which we deduce from the the minimum modulus principle:

Teorem 1.10 (Open mapping) *Let Ω be a domain and let f be holomorphic in Ω. Then $f(\Omega)$ is an open subset of $\mathbb{C}$ unless f is constant.*

Of course if $U \subseteq \Omega$ is open, it follows that $f(U)$ is open; just apply the theorem with $\Omega = U$. So the theorem is equivalent to f being an open mapping.

Bevis: Let $a \in \Omega$ be a point. We shall show that $f(a)$ is an inner point of $f(\Omega)$.

After replacing f by $f - f(a)$ we may assume that $f(a) = 0$, and since the zeros of f are isolated, there are disks D about a where f has no other zeros than a, and whose boundary is contained in Ω. Our function f does not vanish on boundary ∂D and has a therefore a positive minimum ϵ there.

Now, let w be a point not in $f(\Omega)$ with $|w| < \epsilon/2$. The difference $f(z) - w$ does not vanish in Ω, and on ∂D we have

$$|f(z) - w| \geq |f(z)| - |w| \geq \epsilon - \epsilon/2 = \epsilon.$$

By the minimum modulus principle, $|f(z) - w| \geq \epsilon/2$ throughout D, in particular for $z = a$, which gives the absurd inequality $\epsilon/2 \leq |w| < \epsilon$. ❑

Oppgave 1.17. Prove that the open mapping theorem implies the maximum modulus principle. Hint: Every disk about $f(a)$ contains points whose modulus are larger than $|f(a)|$. ✷

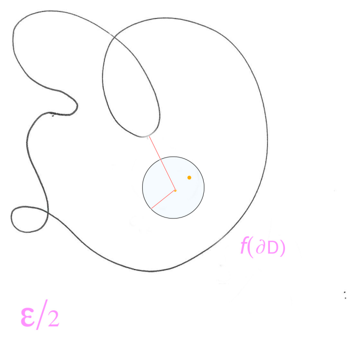


Figure 1.1:

(1.4) There is a simpler approach to the maximum modulus principle then the one we followed above that does not depend on relatively deep results like Cauchy's formula. The principle can be proven just by the good old second derivative test for extrema combined with the Cauchy-Riemann equations. We follow closely the presentation in [?] pages 24–26.

You probably remember from high school, that for a real function ϕ of one variable that is twice continuously differentiable the second derivative is non-positive at a local maximum; *i.e.*, if $a \in I$ is a local maximum for ϕ, then $\phi''(a) \leq 0$.

Now, if u is a twice continuously differentiable function of two variables defined in a domain $\Omega \subseteq \mathbb{C}$ and having a local maximum at $a = (\alpha, \beta)$, the Laplacian $\Delta u = u_{xx} + u_{yy}$ is non-positive at a. Indeed, approaching a along lines parallel to the axes—that is applying the second derivative test to the two functions $u(\alpha, x)$ and $u(x, \beta)$—one sees that the second derivatives satisfy $u_{xx}(a) \leq 0$ and $u_{yy}(a) \leq 0$. With a small trick, this leads to:

Lemma 1.5 *Let the function u be defined and twice continuously differentiable in $\Omega \subseteq \mathbb{C}$ and assume that $\Delta u(z) \geq 0$ throughout Ω. Then for any disk D whose closure is contained in Ω, one has $u(a) \leq \sup_{z \in \partial D} u(z)$ for any $a \in D$. By consequence u has no local maximum in Ω.*

BEVIS: To begin with, assume that $\Delta u(z) > 0$ for all $z \in \Omega$, and let $u_0 = \sup_{z \in \overline{D}} u(z)$. If $u(a) > \sup_{z \in \partial D} u(z)$, the maximum point z_0 does not belong to the boundary ∂D and thus lies in D. But this is impossible as u does not have any local maximum after xxx above. If not, let $\epsilon > 0$ and look at the function $v(z) = u(z) + \epsilon |z|^2$. Then $\Delta v = \Delta u + 4\epsilon > 0$, and we have

$$u(a) < \sup_{z \in \partial D} (u(z) + \epsilon |z|^2) \leq \sup_{z \in \partial D} u(z) + \epsilon \sup_{z \in \partial D} |z|^2,$$

and letting ϵ tend to zero, we are done. ❑

Finally, to arrive at the maximum principle, we observe that if $u(z) = |f(z)|^2$, the Laplacian Δu is given as

$$\Delta u = \partial_z \partial_{\bar{z}} f \overline{f} = \partial_z (f \partial_{\bar{z}} \overline{f}) = \partial_z f \partial_{\bar{z}} \overline{f} = |f'(z)|^2 \geq 0.$$

OPPGAVE 1.18. Show that the Laplacian of the real and of the imaginary part of a holomorphic function vanish identically. ✷

OPPGAVE 1.19. Assume that f does not vanish in a Ω. Show that $u(z) = \log |f(z)|$ is well defined and with its Laplacian vanishing throughout Ω. ✷

OPPGAVE 1.20. Recall that the Hesse-determinant of a function u of two variable is $u_{xx}u_{yy} - u_{xy}^2$. Use the Cauchy-Riemann equations to show that the Hesse-determinant both of the real and of the imaginary part of a holomorphic function is non-positive. ✷

1.4.4 The order of holomorphic functions

The order

A polynomial $P(z)$ has an order of vanishing at any point: The order is zero if P does not vanish at a and equals the multiplicity of the root a in case $P(a) = 0$. The order is characterized by being the largest number n with $(z-a)^n$ dividing P. Holomorphic functions resemble polynomials in this respect, they possess an order at every point where they are defined.

(1.1) Assume that f is a holomorphic function not vanishing identically near a. The Taylor series of f at a converges towards $f(z)$ in a vicinity of a, *i.e.*, one has

$$f(z) = f(a) + f'(a)(z-a) + \cdots + f^{(k)}(a)/k!(z-a)^n + \ldots \tag{1.23}$$

for z near a. Hence if f together with all its derivatives vanish at a, the function f itself vanishes in a neigbourhood of a. So, if this is not the case, there is smallest non-negative integer n for which the n-th derivative $f^{(n)}(a)$ is non-zero. This integer is called the *order* of f at a and is written $\mathrm{ord}_a f$. The n first terms in the Taylor development will all be zero, and the remaining terms will all have $(z-a)^n$ as a factor; hence the Taylor series has the form

$$f(z) = (z-a)^n \big(f^{(n)}(a)/n! + f^{(n+1)}(a)/(n+1)!(z-a) + \ldots\big),$$

where the series converges normally in a disk about a. We have proved

OrderTheorem

Setning 1.12 *Assume that f is holomorphic near a and does not vanish identically in the vicinity of a. Let $n = \mathrm{ord}_a f$ denote the order of f at a. Then we may factor f as*

$$f(z) = (z-a)^n g(z),$$

where g is a holomorphic function near a not vanishing at a. The order of f is the largest non-negative integer for which such a factorization is possible.

OPPGAVE 1.21. Assume that f and g are two functions holomorphic near a.

a) Show that $\operatorname{ord}_a f = 0$ if and only if $f(a) \neq 0$.

b) Show that $\operatorname{ord}_a fg = \operatorname{ord}_a f + \operatorname{ord}_a g$.

c) Show that $\operatorname{ord}_a f + g \geq \min\{\operatorname{ord}_a f, \operatorname{ord}_a g\}$, with equality when the orders of f and g are different. Give examples with strict inequality but with $\operatorname{ord}_a f = \operatorname{ord}_a g$.

✸

(1.2) That holomorphic functions have factorizations like in 1.12 has some strong implications. The first is that the zeros of f must be isolated in Ω, another way expressing this is to say that the zero set $Z = \{ a \in \Omega \mid f(a) = 0 \}$ can not have any accumulation points in Ω. It might very well be infinite, even if Ω is a bounded domain, but its limit points all are situated outside Ω. This is a fundamental property of holomorphic functions, frequently use in sequel. It is called *identity principle*. An example is treated in exercise 1.23 below which is about the function $\sin \pi(z-1)(z+1)^{-1}$ which is holomorphic in the unit disk and has zeros at $(1-n)/(1+n)$ for $n \in N$. There are infinitely many and they accumulate at -1.

IdPrinsipp1

Teorem 1.11 *Let f be holomorphic in Ω. If the zero set Z of f has an accumulation point in Ω, then f vanishes identically.*

BEVIS: Assume that f does not vanish identically, and let $a\imath\Omega$ be any point. Our function f has an order n at a and can be factored as $f(z) = (z-a)^n g(z)$, where g is holomorphic and does not vanish at a. The function g being continuous does not vanish in a vicinity of a, and of course $z - a$ only vanishes at a. We deduce that there is a neigbourhood of a where a is the only zero of $f(z)$, and consequently a is not a accumulation point of the zero set Z. ❏

The may be most frequently used form of the identity principle is the following, which by some authors is called the principle of “solidarity of values”.

IdPrinsipp2

Korollar 1.4 *Assume that f and g are two functions holomorphic in Ω, if they coincide on a set with an accumulation point in Ω, they are equal.*

BEVIS: Apply the identity principle 1.11 to the difference $f - g$. ❏

OPPGAVE 1.22. Let f be holomorphic in Ω and assume that all but finitely many derivatives of f vanish at a point in Ω. Show that f is a polynomial. ✸

NullPSin

OPPGAVE 1.23. Show that $\operatorname{Re}(1-z)(1+z)^{-1} = (1-|z|^2)\,|1+z|^{-2}$ and conclude that the map given by $z \to (1-z)(1+z)^{-1}$ sends the unit disk $\mathbb{D}$ into the right half plane. Let $f(z) = \sin \pi(1-z)(1+z)^{-1}$. Show that f has infinitely many zeros in $\mathbb{D}$ with -1 as an accumulation point. HINT: the zeros are those points in $\mathbb{D}$ such that $(1-z)(1+z)^{-1}$ is an integer. ✸

Oppgave 1.24. Assume that f is holomorphic in the domain Ω. Show that the fibre $f^{-1}(a)$ is a discrete subset of Ω. Conclude that the fibre is at most countable. ✷

Oppgave 1.25. Show that if f is holomorphic in D contained in Ω, and either $\operatorname{Re} f$ or the imaginary part $\operatorname{Im} f$ is constant in a disk $D \subseteq \Omega$, then f is constant. Hint: Use Cauchy Riemann equations and the identity principle. ✷

Oppgave 1.26. Show that if $|f|$ is constant in a disk $D \subseteq \Omega$, then f is constant. Hint: Examine $\log f$. ✷

1.4.5 Isolated singularities

RiemannExtension

For a moment let $R(z) = P(z)/Q(z)$ be a rational function expressed as the quotient of two polynomials. It is of course defined in points where the denominator does not vanish, however, if a is a common zero of the denominator and the enumerator, one may cancel factors of the form $z - a$, and in case the multiplicity of a in numerator happens to be the higher, the rational function $R(z)$ has a well determined value even at a—it has a removable singularity there. Of course this definite value equals the limit $\lim_{z\to a} R(z)$. this not to happen, it is sufficient and necessary that $|R(z)|$ tends to infinity when z tends to a. Similar phenomenon, which we are about to describe, can occur for holomorphic functions.

Let Ω be a domain and $a \in \Omega$ a point. Suppose f is a function that is holomorphic in $\Omega \setminus a$. One sais that f has an *isolated singularity*. The isolated singularities come in three flavours; Firstly f can have a removable singularity (and at the end a is not a singularity at all). This is, as we shall see, equivalent to f being bounded near f. Secondly, the reciprocal $1/f$ can have a removable singularity while f has not; then one sais that f has a *pole* at a, and this occurs if and only if $\lim_{z\to a} |f(z)| = \infty$. In third case, that is if neither of the two first occurs, one says that f has an *essential singularity*.

(1.1) If f is holomorphic in a punctured disk D^* centered at a, one says that f has *a removable singularity* at a if it can be extended to a holomorphic function in D; that is, there is a holomorphic function g defined in D whose restriction to D^* equals f. Clearly this implies that $\lim_{z\to a}(z - a)f(z) = 0$ since f has a finite limit at a, and Riemann proved that also this is sufficient for f to be extendable. Nowadays this is called the *Riemann's extension theorem*:

RiemannExtension

Teorem 1.12 *Assume that f is holomorphic in the punctured disk D^* centered at a. Then f can be extended to a holomorphic function in D if and only if $\lim_{z\to a}(z - a)f(z) = 0$.*

Bevis: If f can be extended, f has a limit at a and hence $\lim_{z\to a}(z - a)f(z) = 0$.

To prove the other implication, one introduces the auxiliary function

$$h(z) = \begin{cases} (z-a)^2 f(z) & \text{when } z \neq a, \\ 0 & \text{when } z = a. \end{cases}$$

Then h is holomorphic in the whole disk D and satisfies $h'(a) = 0$: For $z \neq a$ this is clear, and for $z = a$ one has

$$\big(h(z) - h(a)\big)/(z-a) = (z-a)^2 f(z)/(z-a) = (z-a)f(z),$$

which by assumption tend to zero when z approaches a. It follows that the order of h at a is at least two, and hence $h(z) = (z-a)^2 g(z)$ with g holomorphic near a. Clearly g extends f. ❑

If the function f is bounded near a, one certainly has $\lim_{z\to a}(z-a)f(z) = 0$, and the Riemann's extension theorem shows that f can be extended. Riemann's criterion therefore has the following equivalent formulation:

Teorem 1.13 *Assume that f is holomorphic in the punctured disk D^* centered at a. Then f can be extended to a holomorphic function in D if and only if f is bounded in D^*.*

A familiar example of a function having removable singularity at the origin, is the function $\sin z/z$, and a little more elaborated one is $(2\cos z - 2 - z^2)/z^4$.

(1.2) A function f holomorphic in the punctured disk D^* is said to be *meromorphic* at a if $1/f(z)$ has a removable singularity there; or phrased equivalently: There is a neigbourhood U of a such that in the punctured neigbourhood $U^* = U \setminus \{a\}$ one may write $f(z) = 1/g(z)$ where $g(z)$ is holomorphic in U.

Two different cases can occur. If $g(a) \neq 0$, then $f(z)$ is holomorphic at a and nothing is new. On the other hand, if g vanishes at a, one says that f has a *pole* there, and the order of vanishing of g is called the *order of the pole* or the *pole-order*. One may factor g as

$$g(z) = (z-a)^n h(z),$$

where $n = \mathrm{ord}_a g$ and h is holomorphic near a and $h(a) \neq 0$. Hence

$$f(z) = (z-a)^{-n} h_1(z),$$

where $h_1(z) = 1/h(z)$ is holomorphic and non-vanishing. The *order* of f at a is defined to be $-\mathrm{ord}_a g$, so that at poles the order is negative[8]. For any function meromorphic at a this allows one to write

$$f(z) = (z-a)^{\mathrm{ord}_a f} g(z),$$

where g is holomorphic and non-vanishing at a.

[8]It is slightly contradictory that the order of f is the negative of the pole order, but it is common usage.

(1.3) In this paragraph we study more closely the third case when the singularity of f at a is an *essential* singularity, that is, it is neither removable nor a pole.

By the Riemann extension theorem $1/f$ has a removable singularity if and only if f is bounded near a, which translates into f being bounded away from zero in a neigbourhood of a. This is not the case if f has an essential singularity at a, meaning that for any $\epsilon > 0$ and any $\delta > 0$ there will always be points with $|z-a| < \delta$ with $|f(z)| < \epsilon$. Phrased in a different manner: The function f comes as close to zero as one wants as near a as one wants.

But even more is true. If α is *any* complex number, the difference $f-\alpha$ is meromorphic at a if and only if f is. This is trivial if f is holomorphic, and as the sequence

$$|f| - |\alpha| \leq |f-\alpha| \leq |f| + |\alpha|$$

of inequalities shows, the difference $f-\alpha$ has a pole if and only if f has. So the end of the story is that f has an essential singularity if and only if $f-\alpha$ has. In the light of what we just did above, we have proven the following theorem, the *Casorati-Weierstrass theorem*

Teorem 1.14 *Assume that f has an essential singularity at a and let α be any complex number. Given $\epsilon > 0$ and $\delta > 0$, then there exists points z with $|z-a| < \delta$ and $|f(z) - \alpha| < \epsilon$.*

EKSEMPEL 1.5. The archetype of an essential singularity is the singularity of $e^{1/z}$ at the origin. To get an idea of the behavior of $e^{1/t}$ we take a look at the function along the line where $\operatorname{Im} z = \operatorname{Re} z = t/2$, *i.e.*, where $z = (t+it)/2$. As $1/(1+i) = (1-i)/2$, and we find

$$e^{2/t(1+i)} = e^{1/t}(\cos 1/t - i \sin 1/t).$$

The ever accelerating oscillation of the trigonometric functions $\sin 1/t$ and $\cos 1/t$ as t approaches zero is a familiar phenomenon, and combined with the growth of $e^{1/t}$ illustrates eminently the Casorati-Weierstrass theorem. ❄

OPPGAVE 1.27. Show that $f(z) = \sin \pi(1-z)/(1+z)$ has an essential singularity at $z=-1$. Show that for any real a with $|a| < 1$ there is a sequence $\{x_n\}$ of real numbers converging to -1 such that $f(x_n) = a$. HINT: Study the linear fractional transform $(1-z)/(1+z)$. ✸

OPPGAVE 1.28. Let $g(z) = \exp -(1+z)/(1-z)$. Show that g has an essential singularity at $z=1$. Show that $|g(z)|$ is constant when z approaches 1 through circles that are tangent to the unit circle at 1, and that any real constant can appear in this way. Show that g tends to zero when z approaches 1 along a line making an obtuse angle with A the real axis. HINT: Study the fractional linear transformation $(1+z)/(1-z)$. ✸

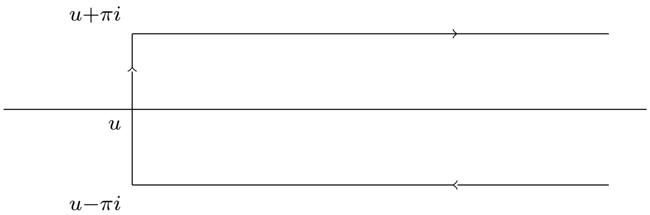
1.4.6 An instructive example

MLEx

The theme of this paragraph, organized through exercises, is an entire function $F(z)$ with peculiar properties constructed by Gösta Mittag-Leffler and presented by him at the International Congress for Mathematicians in Heidelberg in 1905. When z tends to infinity, but stays away from a sector of the type $S_\alpha = \{\, z \mid -\alpha < \operatorname{Im} y < \alpha, \operatorname{Re} z > 0 \,\}$, the function $F(z)$ tends to zero. In addition $\lim_{x\to\infty} F(x) = 0$ where it is understood that x is real. In particular the limit of $F(z)$ is zero when z goes to ∞ along any ray emanating from the origin.

The construction is based on an "infinite contour" $\Sigma(u)$ where u is a positive real number. The path is depicted below in figure 1.2. It has three parts: $\Sigma_1(u)$ is the segment from $x + \pi i$ to infinity, $\Sigma_2(u)$ the segment from ∞ to $x - \pi i$ and $\Sigma_0(u)$ the segment from $x - \pi i$ to $x + \pi i$. All three are parametrized in the simplest way by linear functions.

As a matter of language we say that a point z lies *inside* $\Sigma(u)$ if $\operatorname{Re} z > u$ and $-\pi < \operatorname{Im} z < \pi$; and of course, it lies *outside* $\Sigma(u)$ if it lies neither inside nor on $\Sigma(u)$.



MLVei Figur 1.2: *The path* $\Sigma(u)$.

We begin working with an entire function $f(z)$ merely assuming it be integrable along Σ; that is the integrals $\int_{\Sigma_i(u)} |f(z)|\,|dz|$ are finite for $i = 1, 2$. In the end we specialize f, as Mittag-Leffler did, to be the function

$$f(z) = e^{e^z} e^{-e^{e^z}}.$$

Oppgave 1.29. Show that the integral

$$\int_{\Sigma(u)} f(w)(w-z)^{-1} dw.$$

is independent of u as long as z lies outside $\Sigma(u)$. ✷

Given an arbitrary complex number z and define a function

$$F(z) = \frac{1}{2\pi i} \int_{\Sigma(u)} f(w)(w-z)^{-1} dw. \tag{1.24}$$

MittagLefflerFu where u is any real number such that z lies outside the contour $\Sigma(u)$. After the previous exercise this is a meaningful definition.

Oppgave 1.30. Show that $F(w)$ is an entire function of w. Hint: See exercise 1.16 on page 29. ✷

Oppgave 1.31. Let $z = x + iy$ be any point not on the contour $\Sigma(u)$.

$$|w - z| \geq \begin{cases} |y - \pi| & \text{if } y \neq 0, \\ |x - u| & \text{if } y = 0. \end{cases}$$

Fix the number u and let $z = re^{i\phi}$. Show that

$$\lim_{r\to\infty} \int_{\Sigma(u)} f(w)(w - z)^{-1} dw = 0.$$

Show that $F(z)$ tends to zero when z tends to infinity along any ray emanating from the origin but being different from the positive real axis. Hint: For $|z|$ sufficiently large z stays outside of $\Sigma(u)$ and formula (1.24) is valid. ✷

Now we study what happens on the positive real axis, so assume that $z = x$ is real and positive. Fix a real and positive constant u_0 less than x, and let u be greater than x, and introduce the rectangular path R as illustrated in figure 1.3.

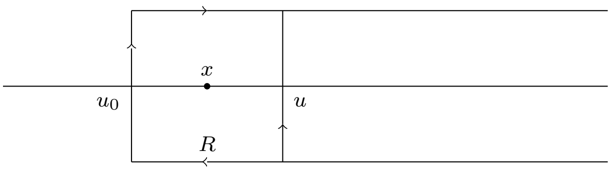
Oppgave 1.32. Show that as chains $\Sigma u_0 = R + \Sigma(u)$, and show that we have

$$F(x) - f(x) = \frac{1}{2\pi i} \int_{\Sigma(u_0)} f(w)(w - z)^{-1} dw.$$

Use this to show that

$$\lim_{x\to\infty} |F(x) - f(x)| = 0.$$

✷



Figur 1.3: *The rectangle R .*

In the last part of this exercise session, we specialize f to be the function $f(z) = e^{e^z} e^{-e^{e^z}}$.

Oppgave 1.33. Show that the integrals

$$\int_{\Sigma(u)} e^{e^w} e^{-e^{e^w}} dw$$

converge absolutely. Show that

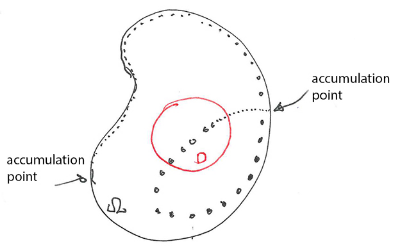
$$\lim_{x\to\infty} e^{e^x} e^{-e^{e^x}} = 0$$

and conclude that the associated function $F(z)$ tends to zero along any ray emanating from the origin. ✷

Oppgave 1.34. Show that F is not identically zero. ✷

1.4.7 The argument principle

It is classical that the multiplicities of the different roots of a polynomial add up to its degree. One can not hope for statements about holomorphic functions as strong as this. Already, there is no notion of degree for a holomorphic function in general. The order at a point is a sort of local degree; the degree of a polynomial is however a global invariant, and there is counterpart for holomorphic functions. And the number of zeros can very well be infinite, a simple example is $\sin \pi z$, which vanishes at all integers.



(1.1) However, there is a counting mechanism for the zeros, which goes under the name of the *argument principle*, which now and then is extremely useful.

So let f be holomorphic in Ω, and let D be any disk whose closure is contained in Ω. Then, as the zeros are isolated i Ω, there is at most finitely many of then in D.

Let $a_1, \ldots, a_r$ de the those of the zeros of f that are contained in the disk D, and denote by $n_1,, \ldots, n_r$ their multiplicities, *i.e.*, $n_i = \operatorname{ord}_{a_i} f$. By repeated application of proposition 1.12, one may write

$$f(z) = \prod_i (z - a_i)^{n_i} g(z),$$

where the index i runs from 1 to r and where g is holomorphic an non-vanishing in D. Taking the logarithmic derivative gives

$$d \log f = \sum_{1 \leq i \leq r}^{r} n_i (z - a_i)^{-1} + d \log g.$$

(Recall that we write $d \log f$ for f'/f). The integral of $d \log f$ around the circumference ∂D, becomes

$$\frac{1}{2\pi i} \int_{\partial D} d \log f = \sum_{1 \leq i \leq r} n_i \, \mathrm{n}(\partial D, a_i) + \frac{1}{2\pi i} \int_{\partial D} d \log g.$$

Now, as g does not vanish in the disk D, it has a logarithm there, and hence $\int_\gamma d \log g = 0$. Consequently the integral $\int_\gamma d \log f$ satisfies

$$\frac{1}{2\pi} i \int_{\partial D} d \log f = \sum_i n_i \, \mathrm{n}(\partial D, a_i) = \sum_i n_i. \tag{1.25}$$

ArgPrin1 where the last equality holds since the winding nu,bers involved all equal one ∂D being traversed once counterclockwise and the a_i's all lying within ∂D. With the right interpretation the formula counts the total number of zeros of f contained in the disk D.

OPPGAVE 1.35. Denote by Z the set of zeros of f in Ω and for each $a \in Z$, let $n(a) = \operatorname{ord}_a f$. Show that

$$\int_{\partial D} d\log f = \sum_{a\in Z} n(a)\,\mathrm{n}(\partial D, a).$$

HINT: The sum is finite, even if it doesn't look like. ✱

(1.2) If a is any complex numbers, the zeros of the difference $f - a$ constitute the fibre $f^{-1}(a)$. Hence the technic in the last paragraph can as well be used to count points in fibres. Every point b in a fibre will contribute to the totality with a multiplicity equal to the multiplicity of the zero b of $f - a$. Denoting this multiplicity by $n(b)$ we have the formula

$$\frac{1}{2\pi i}\int_{\partial D} d\log(f-a) = \sum_{b\in f^{-1}(a)\cap D} n(b).$$

where of course $d\log(f-a) = f'(z)(f(z)-a)^{-1}dz$.

If γ is a parametrization of ∂D, the composition $f\circ\gamma$ is parametrizes a path Γ in $\mathbb{C}$; *i.e.*, we have $\Gamma = f\circ\gamma$. The winding number $\mathrm{n}(\Gamma, a)$ is given by an integral, and substituting $w = f(z)$ this integral changes in the following way:

$$\mathrm{n}(\Gamma, a) = \frac{1}{2\pi i}\int_{\Gamma}(w-a)^{-1}dw = \frac{1}{2\pi i}\int_{\gamma} f'(z)(f(z)-a)^{-1}dz,$$

hence

$$\mathrm{n}(\Gamma, a) = \sum_{b\in f^{-1}(a)\cap D} n(b).$$

We sum up in the following proposition:

Setning 1.13 *Let f holomorphic in Ω and let D be a disk whose closure lies in Ω. Let a be any complex number. The number of points in the fibre $f^{-1}(a)$ lying within the disk D is finite, and counted with multiplicities, equals the winding number $\mathrm{n}(\Gamma, a)$ where Γ is the image of the boundary circle ∂D under f, traversed once counterclockwise.* NumberInFibreIsW

The winding number of a closed path is, as we saw, constant within each connected component of the complement of the path. Applying this to the image Γ of ∂D under f, we conclude that the number of points in $f^{-1}(a)\cap D$ and in $f^{-1}(b)\cap D$—counted appropriately—are the same as long as a and b belongs to the same connected component of $\mathbb{C}\setminus\Gamma$.

In particular, if A is a disk about a contained in the image of ϕ and not intersecting Γ, the two sets $f^{-1}(a)\cap D$ and $f^{-1}\cap D$ have equally many members. This leads to

SameNuberInFibre

Setning 1.14 *Assume that f is a holomorphic map and that a is a solution of $f(z) = f(a)$ of multiplicity n. Then there is a disk D about a such that for b sufficiently close to $f(a)$, all solutions of $f(z) = b$ in D are simple and their number equals n.*

The theorem says that there are disks D and B about a and $f(a)$ respectively such that B lies in the image $f(D)$, and such that the fibers $f^{-1}(b) \cap D$ all are simple—that is every point occurs with multiplicity one—except the central fibre $f^{-1}(a) \cap D$ which reduces to just one point with multiplicity n.

Bevis: As the derivative f' is holomorphic, its zeros are isolated and there is a disk D about a where it does not vanish in other points than a. Making D smaller, if necessary, it will also avoid the points in the fiber $f^{-1}(f(a))$ other than a.

The image $f(D)$ is open, and we chose a disk B containing $f(a)$ and lying in a connected component of the complement $\mathbb{C} \setminus \partial A$. As f' has no zeros in D, except at a, all fibers $f^{-1}(b) \cap D$ for $b \in B$, except $f^{-1}f(a) \cap D$, are simple, and by proposition 1.13 above they all have n points, as fibers over points from the same component of $\mathbb{C} \setminus f(\partial A)$. ❏

(1.3) The case $n = 1$ in 1.14 is a very important special case. Then the statement is that a functions f with $f'(a) \neq 0$ is injective in a disk containing a. This is also a consequence of the inverse function theorem, $f'(a)$ being the jacobian map at a of f; but there is a stronger statement that the inverse map f^{-1} is holomorphic. One has

LocalConform

Setning 1.15 *Let f be holomorphic in Ω and let $a \in \Omega$ be a point with $f'(a) \neq 0$. There is a disk D about a on which f is biholomorphic. That is f is injective and the inverse map $f^{-1}\colon f(D) \to D$ is holomorphic, moreover the its derivative at $f(a)$ equals $1/f'(a)$.*

Bevis: The inverse map f^{-1} is continuous since f is open, and the usual argument for the existence of the derivative of f^{-1} we know from calculus goes trough, letting $w = f(z)$ and $b = f(a)$ we have

$$(f^{-1}(w) - f^{-1}(b))/(w - b) = (z - a)/(f(z) - f(a)) \tag{1.26}$$

DiffOfInvers

and as w tends to b continuity of f^{-1} implies that z tends to a, and the right side of (1.26) tends to $1/f'(z)$. ❏

Another way of proving this, is to appeal to the inverse function theorem. It says that f^{-1} is C^∞ near $f(a)$ and that its jacobian map at a point $f(z)$ equals the inverse of that of f. But of course, the inverse of multiplication by a complex number c is multiplication by c^{-1}, and we conclude by the Cauchy-Riemann equations.

(1.4) A biholomorphic map is frequently called *conformal*, a term coming from cartography and alluding to the fact that a holomorphic function with a non-vanishing derivative at a point a infinitesimally preserves the angel and orientation between vectors at a. This is due to the jacobian map being multiplication by $f'(a)$, so if $f'(a) = re^{i\phi}$ all angels are shifted by ϕ, so the difference between the two is conserved. The proposition 1.15 may be phrased as if f is holomorphic near a with non-vanishing derivative at a, then f is biholomorphic near a.

Oppgave 1.36. Let γ and γ' be two paths that pass by a both with a non-vanishing tangents at a. Let ψ be the angle between the two tangents. Let f be holomorphic near a with $f'(a) \neq 0$. Show that the paths $f \circ \gamma$ and $f \circ \gamma'$ both have non-vanishing tangents at $f(a)$ and that the angle between them equals ψ. ✷

(1.5) Now, consider the case that $f(a) = 0$ and that $f'(a)$ vanishes, say with multiplicity n, Then f may be factored near a as

$$f(z) = z^n g(z)$$

where $g(z)$ is holomorphic and non-vanishing in a neigbourhood of a. It follows that g has an n-th root in a disk about a; say $g = h^n$. We may thus write

$$f(z) = (zh(z))^n = \rho(z)^n$$

where $\rho(z) = zh(z)$. Now $\rho' = h(z) + zh'(z)$ does not vanish at a since g does not, and hence ρ is biholomorphic near ρ. We therefore have

Setning 1.16 *Assume that $f(a)$ has a zero of multiplicity n at a. Then there is a biholomorphic map ρ defined in a neigbourhood U of a such that*

$$f(z) = \rho(z)^n$$

for $z \in U$. In particular, f is locally injective at a if and only if $f'(a) \neq 0$.

Bevis: Only the last sentence is not proven. We have seen that if $n = 1$, then f is locally conformal and, in particular, it is locally injective. So assume that $n > 1$ and we must establish that f is *not* injective. The map ρ is open so the image $\rho(U)$ contains a disk A about the origin. If $\rho(z) \in A$ and η is an n-th root of unity, $\eta\rho(z)$ lies in A as well. Now, A being contained in $\rho(U)$ one has $\eta\rho(z) = \rho(z')$ for some $z' \in U$, and z' is different from z since ρ is injective. It follows that $f(z') = (\eta f(z))^n = f(z)$. ❑

Oppgave 1.37. Show that

$$f^{-1}(b) = \int_{\partial D} z f'(z)(f(z) - b)^{-1} dz$$

✷

1.4.8 The general argument principle

At the end of this section, we give generalization of the formula (1.25) on page 42. extending it to meromorphic functions. In this case one is forced to take both poles and zeros into account—it is their difference in number (with multiplicities) that is a categorical quantity, or said in clear text, a quantity that can be computed formally. This difference is just the sum

$$\sum_{a \in D} \mathrm{ord}_a f$$

where D is a disk whose closure lies within the domain Ω. We can only count the zeros and poles if they are finite in number, and $\overline{D}$ lying in Ω ensures this. Poles and zeros are isolated—that is the points a where $\mathrm{ord}_a f$ does not vanish—hence in the compact disk $\overline{D}$ there can only finitely many of them.

A second generalization is the introduction of a closed path γ in D. Loosely speaking, we count the difference of the number of poles and the number of zeros of f "lying with in γ". The precise meaning is the sum

$$\sum_{a \in D} \mathrm{n}(\gamma, a) \mathrm{ord}_a f.$$

where now D is any disk with encompassing γ and with $\overline{D} \subseteq \Omega$—and it essential that f has neither poles nor zeros lying on the path γ. We can safely factor f as a product

$$f(z) = \prod_{a \in D} (z-a)^{\mathrm{ord}_a f} g(z)$$

where $g(z)$ is holomorphic and without zeros in D and, of the course, the product is finite. Taking logarithmic derivatives we get the formula

$$d \log f = \sum_{a \in D} (z-a)^{-1} \mathrm{ord}_a f + d \log g \tag{1.27}$$

ArgIntForm

and integrating along the closed path γ:

$$\frac{1}{2\pi i} \int_\gamma d \log f = \sum n(\gamma, a) \mathrm{ord}_a f,$$

as $\int_\gamma d \log g = 0$, the function g having a logarithm in D.

(1.1) Let us now introduce a second holomorphic function $h(z)$ in Ω, and consider the integral $\int_\gamma g d \log f$. Multiplying (1.27) on page 46 by h gives

$$d \log f(z) = \sum_{a \in D} h(z)(z-a)^{-1} \mathrm{ord}_a f + h(z) d \log g(z).$$

Now, g is holomorphic and without zeros and $d\log g(z)$ is holomorphic as well. Hence $h(z)d\log g(z)$ is holomorphic and consequently its integrals round closed paths vanish by the Cauchy theorem. To integrate the terms in the sum, we appeal to Cauchy's formula which can be applied since h is holomorphic. This gives

$$\frac{1}{2\pi i}\int_\gamma h(z)d\log f(z) = \sum_{a\in D} h(a)\,\mathrm{n}(\gamma,a)\mathrm{ord}_a f, \tag{1.28}$$

which one may interpret as a counting formula for zeros and poles, but this time they are weighted by the function h. We sum up these computations in GeneralArguement

GenerllArgumentP

Teorem 1.15 *Let f be a meromorphic function and h a holomorphic function in Ω. Then for any disk D with $\overline{D}\subseteq\Omega$ and any closed path γ in D, one has the equality*

$$\frac{1}{2\pi i}\int_\gamma h d\log f = \sum_{a\in\Omega} h(a)\,\mathrm{n}(\gamma,a)\mathrm{ord}_a f.$$

There is a still more general version of this theorem. Working with paths being null-homotopic, and this is the most natural hypothesis, one can get rid of the disk D, but for the moment we do not know that integrals of holomorphic functions only depends on the homotopy class of the integration path. Once that is established, the theorem 1.15 in its full force follows easily, but that is for the next section.

1.4.9 The Riemann sphere

The Riemann-sphere $\hat{\mathbb{C}}$ or the extended complex plane is just the one point compactification of the complex plane. We add one point at the infinity, naturally denoted by ∞, so as a set $\hat{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$. The topology is defined a for any one point compactification. The open sets containing ∞ are the sets $K^c\cup\{\infty\}$ where K is any compact subset of $\mathbb{C}$ (and K^c is its complement in $\mathbb{C}$), and the rest of topology, *i.e.*, those open sets not containing the point at infinity, are the open sets in the finite plane $\mathbb{C}$.

One has a coordinate function round ∞ defined by

$$w(P) = \begin{cases} 1/z & P = z \neq \infty \\ 0 & P = \infty. \end{cases}$$

A disk D_R centered at ∞ with radius R, that is $\{\, w \mid |w| < R \,\}$ corresponds to $\{\infty\}\cup\{\, z \mid |z| > R \,\}$, and it intersects the finite plane in $\{\, z \mid |z| > R^{-1} \,\}$.

By using the coordinate w we may extend all theory about the local behavior of a complex function at a finite points, to be valid at infinity as well.

(1.1) Assume that f is a function defined for $|z| > R^{-1}$. We say that f is *holomorphic at ∞* if $f(w^{-1})$ has a removable singularity at $w = 0$. By the Riemann extension theorem this is equivalent to $f(z)$ being bounded as $z\to\infty$, or if you want, to $f(z)$ having a limit when $z\to\infty$. And of course this limit is the value of f at ∞

In the same vain the function f has a pole at infinity if $f(w^{-1})$ has one at the origin. The pole has order n if $f(w^{-1}) = w^{-n}g(w)$ where the function $g(w)$ is holomorphic and non-vanishing at the origin. Substituting $w = z^{-1}$ we see that this becomes $f(z) = z^n g(z^{-1})$ where $g(z^{-1})$ is bonded, but with a non-zero limit when $z \to \infty$. An of course, f has a zero at infinity if $f(z) = z^{-n}g(z^{-1})$ where g has a non-zero limit as z tends to infinity.

Eksempel 1.6. A polynomial of degree n has a pole of order n at infinity. Indeed, we have assuming that polynomial p is monic,

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0 = z^n(1 + a_{n-1}z^{-1} + \cdots + a_0 z^{-n}) = z^n g(z)$$

where $g(z)$ tends to 1 as $z \to \infty$ ❄

(1.2) In the last paragraph we discussed function defined at infinity, we take a closer look at functions taking the value infinity. Saying that f has a pole at a is the same a saying that $\lim_{z\to a} |f(z)| = \infty$. This is equivalent to saying that $f(z)$ tends to ∞ in the Riemann-sphere $\hat{\mathbb{C}}$, so setting $f(a) = \infty$ gives a continuous function into $\hat{\mathbb{C}}$.

Using the coordinate $w = z^{-1}$ at infinity, the behavior of f is described, by the behavior of $1/f(z)$, and it is easily seen that the order of vanishing of f at infinity equals the pole order at a.

Finally, a function f might have a pole at infinity, and its behavior is described by $1/f(z^{-1})$.

1.5 The general homotopy version of Cauchy's theorem

A type problems invariably arising in complex function theory are variants of the following "patching problem": Given a certain number of open subsets $\{U_i\}$ indexed by the set I and covering a domain Ω and for each U_i a function F_i holomorphic in U_i. Assume that any pair F_i and F_j differ by constant on each connected component of the intersection $U_i \cap U_j$—*e.g.*, a situation like this arises when F_i is a primitive for a given function f holomorphic in the union $\Omega = \bigcup_i U_i$.

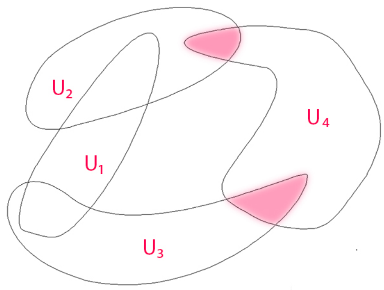
The big question is: When can one change each F_i by a constant such that any pair F_i and F_j agree on the whole $U_i \cap U_j$? Or phrased in precise manner: When can one find complex constants c_i such that for all pairs of indices the equality

$$F_i(z) + c_i = F_j(z) + c_j$$

holds true for all $z \in U_i \cap U_j$? The condition is clearly necessary and sufficient for the existence of a "patch" of the F_i's, meaning a function F defined in the whole of Ω restricting to F_i on each U_i. That is, F satisfy $F|_{U_i} = F_i$ for each i in I.

The question is a non-trivial one; illustrated by the simple situation with just two opens U_1 and U_2, but with the intersection $U_i \cap U_j$ being disconnected. In this situation the answer is positive if and only if F_1 and F_2 differ by the same constant on all the connected components of $U_i \cap U_j$.

In the figure below, for example, F_2 and F_3 on the open sets U_2 and U_3 are easily adjusted to coincide with F_1 on the intersections $U_1 \cap U_2$ and $U_3 \cap U_4$. Of course one can make F_4 mach F_2 on $U_2 \cap U_4$ but at the same time F_4 matches F_3 on $U_3 \cap U_4$, one has extremely lucky or very clever at the choices of F_2 and F_3.



Figur 1.4:

This section offers a variation of this theme. In the bigger picture on has the *cohomology groups* invented precisely for tackling challenges as described this flavour, but those will be for later.

We start out with a short recapitulation of a notion from topology, namely the homotopy of paths, and proceed with the main theme, a general Cauchy type theorem, stating that the integral of a holomorphic function only depends of the homotopy type of the path of integration.

1.5.1 Homotopy

For a moment we take on a topologist glasses and review —in a short and dirty manner — the notion of homotopy between two paths in a domain Ω of the complex plane. Homotopy theory has grown to big theory, nowadays it is a lion's share of algebraic topology, but it originated in complex function theory, and a lot of the results specific for elementary function theory of can be developed in an ad hoc manner without any reference to homotopy. However, let what belongs to the king belong to the king, and more important, pursuing the study of Riemann surfaces one will find that fundamental groups are omnipresent.

For a more thorough treatment one may consult Allan Hatchers book [**?**].

(1.1) For a topologist a path in Ω is a *continuous* path, that is a continuous map $\gamma : [0,1] \to \Omega$. It is convenient in this context to let all parameter intervals be the unit interval $I = [0,1]$. As $[0,1]$ is mapped homeomorphically onto any interval $[\alpha, \beta]$ by the affine function $(1-t)\alpha + t\beta$, this does not impose any serious principal restriction.

Observe that with this definition a constant map $\gamma(t) = a$ is path— a *constant path*. The *reverse path* of γ denoted γ^{-1}, is the path $\phi(1-t)$. If γ_1 and γ_2 are two paths such that the end-point of γ_1 coincides with the starting point of γ_2, one has *the composite path* $\gamma = \gamma_2\gamma_1$ given as

$$\gamma(t) = \begin{cases} \gamma_1(2t) & \text{when } 0 \leq t \leq 1/2 \\ \gamma_2(2t-1) & \text{when } 1/2 < t \leq 1, \end{cases}$$

one first traverses γ_1 and subsequently γ_2.

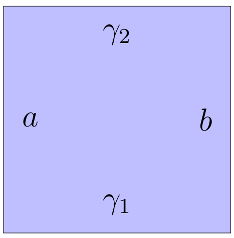
Closed paths, *i.e.,* loops ending where the started, are called loops in topology. And one usually specifies the common end- and start-point and speaks about loops at a point a. Two loops at a can always be composed.

(1.2) The intuitive meaning of two paths being homotopic in the domain Ω is that one can be deformed continuously into the other without leaving Ω. Let γ_0 and γ_1 be the two paths in the domain Ω. They are assumed to continuous and to have a common starting point, say a, and a common end-point b. That is, one has $\gamma_0(0) = \gamma_1(0) = a$ and $\gamma_0(1) = \gamma_1(1) = b$. It is a feature of the notion of homotopy that the starting points and the end-points stay fixed during the deformation.

The precise definition is as follows:

Definisjon 1.2 *Let γ_0 and γ_1 be two continuous paths in the domain Ω both with starting point a and both with end-point b, are* homotopic *if there exists a continuous function $\phi\colon I \times I \to \Omega$ with $\phi(0,t) = \gamma_0(t)$ and $\phi(1,t) = \gamma_2(t)$ and $\phi(s,0) = a$ and $\phi(s,1) = b$.*

In figure below we have depicted $I \times I$ with the behavior of the homotopy ϕ on the boundary indicated.



Figur 1.5: A homotopy

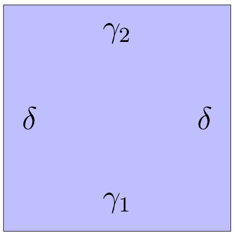
(1.3) It is common to write $\gamma_1 \sim \gamma_2$ if γ_1 and γ_2 are homotopic, and it is not difficult to show that homotopy is an equivalence relation. The algebraic operation of forming the composite of two paths is compatible with homotopy. The composition is associative up to homotopy meaning that $(\gamma_1\gamma_2)\gamma_3 \sim \gamma_1(\gamma_2\gamma_3)$ where of course it is understood that the γ_i's are mutually composabel, and one may show that the homotopy classes of loops at a form a group under composition with the constant path as unit element and, of course, with the inverse path as inverse. It is called the *fundamental group* of Ω at a and it is written $\pi_1(\Omega, a)$.

Eksempel 1.7. If Ω is star-shape, say with a as the central point, then every loop at a is homotopic to the constant loop at a. Indeed, if γ is a loop, the convex combination $\phi(s,t) = (1-s)\gamma(t) + sa$ is a homotopy as required. ❄

Eksempel 1.8. Assume that ϕ is a homotopy between γ_1 and γ_2, and assume that the final point of γ coincides with the common initial point of γ_1 and γ_2. Show that $\gamma_1\gamma \sim \gamma_2\gamma$, and with the appropriate hypothesis on γ, that $\gamma\gamma_1 \sim \gamma\gamma_2$. Conclude that if $\gamma_1' \sim \gamma_2'$, and γ_i''s satisfy the right composability condition, one has $\gamma_1\gamma_1' \sim \gamma_2\gamma_2'$.
Hint: Define a homotopy ψ by $\psi(s,t) = \gamma(2t)$ for $0 \leq t \leq 1/2$ and $\psi(s,t) = \phi(s, 2t-1)$ for $1/2 < t \leq 1$.

❄

(1.4) One can relax the condition on a homotopy and not require that the end-points be fixed. In that case one speaks about *freely homotopic paths*. Although, if the two paths are closed, one requires that the homotopy be a homotopy of closed paths; that is, the deformed paths are all closed. To be precise, one requires that $\phi(s,0) = \phi(s,1)$ for all s. This implies that the two paths $\delta_1(s) = \phi(s,0)$ and $\delta_2(s) = \phi(s,1)$ are the same.



(1.5) Let γ_1 and γ_2 be two piecewise C^1-curves that are composable—the end-point of the first being the start point of the other—and let γ the composite. Clearly γ is also piecewise C^1 and one has

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz.$$

In the same vain, if γ is piecewise C^1, the inverse path γ^{-1} is as well, and one has

$$\int_{\gamma^{-1}} f(z)dz = -\int_\gamma f(z)dz.$$

Integration behaves a little like a group homomorphism, so to speak. It takes composites to sums and inverse to negatives. And in the next section the main result is that integration of holomorphic functions also is compatible with homotopy—that is, the integral only depends on the homotopy class of the path of integration.

1.5.2 Homotopy invariance of the integral I

We come to main concern in this section, the general Cauchy theorem. In the usual setting, we are given a domain Ω and a function f holomorphic in Ω. The main result of the section basically says that the integral of f along a path γ (that must be piecewise C^1 to serve as a path of integration) only depends on the homotopy class of γ, and this means a homotopy that fixes the end points. There is also a version with the homotopy being a free homotopy, but it is only valid for close curves.

From the homotopy invariance we extract the general Cauchy's theorem and with the use of a few results about homotopy groups (that we do not prove) we obtain the general formulation of Cauchy's formula and the counting formula for zeros and poles.

OPPGAVE 1.38. Give an example of two freely homotopic paths and a holomorphic function whose integrals along the to paths differ. ✷

OPPGAVE 1.39. Give an example of two homotopic paths (fixed end-homotopic) and function that is not holomorphic whose integrals along the two paths differ. ✷

(1.1) It is slightly startling that although a homotopy between two piecewise C^1-curves is just required to be continuous (so no integration is allowed along the deformed paths), the integral of f along them remains the same.

If the homotopy is continuously differentiable, however, the independence of the integrals is not difficult to establish. Let $\phi\colon I \times I \to I$ denote the homotopy, and that assume it to be C^∞ in the interior of $I \times I$ and to restrict to piecewise-continuous paths on the boundary $\partial I \times I$.

We cover $\phi(I \times I)$ with finitely many disks ($I \times I$ is compact!). Furthermore we choose a partition $\{t_i\}_{0 \le i \le r}$ of the unit interval I such that if R_{ij} denotes the rectangle $[t_{i-1}, t_i] \times [t_{j-1}, t_j]$, it holds true that each R_{ij} is mapped into one of the covering disks. The restriction of ϕ to the boundary ∂R_{ij} is a closed path lying in the covering disk in which the image of R_{ij} lies, and we denote this path by $\phi(\partial R_{ij})$. The function f is holomorphic in the covering disk, so Cauchy's theorem for disks gives us

$$\int_{\phi(\partial R_{ij})} f(z)dz = 0. \tag{1.29}$$

CCauchy

By a simple and classical cancellation argument, which should be clear from the figure 1.6 below, it follows that

$$\int_{\gamma_1} f(z) - \int_{\gamma_2} f(z) - \int_{\delta_1} f(z)dz + \int_{\delta_2} f(z)dz = \sum_{i,j} \int_{\phi(\partial R_{ij})} f(z)dz = 0$$

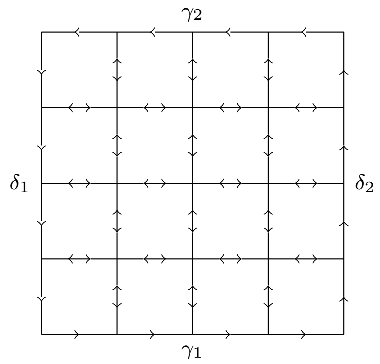
the last equality stemming from (1.29) above. Hence we have

$$\int_{\gamma_1} f(z) - \int_{\gamma_2} f(z) = \int_{\delta_1} f(z)dz - \int_{\delta_2} f(z)dz. \tag{1.30}$$

FinalEqiality

Now assume that the γ_i's are closed paths. If we require that the the deformation of γ_1 into γ_2 should be through closed paths, we must have δ_1 and δ_2 to be the same paths. Then the right side in (1.30) above vanishes, and we can conclude that

$$\int_{\gamma_1} f(z) = \int_{\gamma_2} f(z).$$



Figur 1.6:

Fig10

1.5.3 Homotopy invariance of the integral II

We closely follow the presentation of Reinholdt Remmert (page 169–174 in the book [Rem]), and proof is inspired by the proof of the so called van Kampen theorem in algebraic topology—a important theorem used to compute the fundamental group of unions—one would find in most textbooks in algebraic topology (*e.g.*, in [?]).

(1.1) The proof we present seems long and complicated, but the core is very simple. Most of it consists of rigging (which is the same rigging as we did in the case of a C^∞ homotopy)—one might be tempted to compare it to assembling a full orchestra to play a ten second jingle.

Teorem 1.16 *If γ_1 are γ_2 are two homotopic piecewise C^1-paths in the domain Ω and $f(z)$ is a holomorphic function in Ω, then one has the equality* GenerelCauchy

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

BEVIS: The basic rigging is as follows: Let $\{U_k\}$ be cover of Ω by open disks. Then f has a primitive function over each U_k; that is, there are functions F_k holomorphic in U_k with $F_k' = f$ in U_k, and these functions are unique up to an additive constant.

The inverse images $\phi^{-1}(U_k)$ form an open cover of $I \times I$ and by Lebesgue's lemma there is a partition $0 = t_0 < \cdots < t_r = 1$ of I such that each of the subrectangles

$R_{ij} = [t_{i-1}, t_i] \times [t_{j-1}, t_j]$ are contained in $\phi^{-1}(U_k)$ for at least one k. We rename the R_{ij}'s and call them R_k indexed with k increasing t te left and upwards; that is, R_0 is the bottom left rectangle and R_n, say, the upper right one. The U_k's are renumbered accordingly. (Two U_k' for different k's can be equal).

The point of the proof is to construct a continuous function $\psi \colon I \times I \to \Omega$ with the property

$$\psi(s,t) = F_k(\phi(s,t)) \text{ for } (s,t) \in R_k, \tag{1.31}$$

NokkelLigning where each F_k is a primitive function for f in U_k. which is as close to finding a primitive to $f(\phi(s,t))$ we can come. A crucial fact is that in the intersections $U_i \cap U_j$, which are connected, the functions F_i and F_j differ by a constant both being a primitive for f, and the salient point in the construction of ψ is to change the F_k's by appropriate constants (it might even happen that U_k and $U_{k'}$ are equal for $k \neq k'$ but the two functions F_k and $F_{k'}$ are different).

SmallLemma **Lemma 1.6** *Once we have established the existence of a function ψ satisfying (1.31) the theorem follows.*

Bevis: The map ϕ is just a continuous map, but on the boundary of $I \times I$ it restricts to the two original piecewise C^1-paths; so $\phi(0,t)$ is just the parametrization γ_1. Hence we get

$$\int_{\gamma_1} f(z)dz = \sum_{i=1}^{r} \int_{t_{i-1}}^{t_i} f(\phi(0,t))\phi'(0,t)dt = \sum_{i=1}^{r} \int_{t_{i-1}}^{t_i} F_i'(\phi(0,t))\phi'(0,t)dt =$$
$$= \sum_{i=1}^{r} \psi_i(t_{i-1}) - \psi_i(t_i) = \psi(0,0) - \psi(0,1).$$

In a similar way, one finds

$$\int_{\gamma_2} f(z)dz = \psi(1,0) - \psi(1,1).$$

Now, the homotopy ϕ fixes the end-points, which means that $\psi(s,0)$ and $\psi(s,1)$ are independent of s, in particular it follows that $\psi(0,1) - \psi(1,1) = \psi(0,0) - \psi(0,1)$, and in view of the computations above, that is exactly what we want. ❑

We carry on with the jingle, the construction of the mapping ψ so as to satisfy the condition (1.31) above. The tactics consist in using induction and successive extensions to exhibit, for each m, a function ψ_m on the union $\bigcup_{0\leq k\leq m} R_k$ extending ψ_{m-1} and satisfying (1.31) for $k \leq m$. And at the end of the process, we let ψ be equal to ψ_r — the last of the functions ψ_m.

So we assume that ψ_m is constructed on $\bigcup_{0\leq k\leq m} R_k$ subjected to (1.31) and try to extend it to

$$\bigcup_{0\leq k\leq m+1} R_k = R_{m+1} \cup \bigcup_{0\leq k\leq m} R_k.$$

The most difficult case is when R_{m+1} is located in a corner, as depicted in the figure below. We concentrate on that situation, leaving to the zealous student the easier case when R_m is situated in the bottom row or at the leftmost boundary of $I \times I$ and only intersects one of the previous rectangles in one edge.

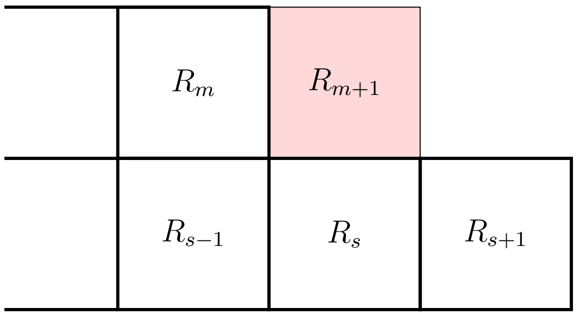
The image of the edge $R_m \cap R_{m+1}$ under ϕ is contained in $U_m \cap U_{m+1}$. After possibly having changed F_{m+1} by a constant, we may assume that F_m and F_{m+1} coincide in $U_m \cap U_{m+1}$ (which is connected), and hence $F_m(\phi(s,t))$ and $F_{m+1}(\phi(s,t))$ are equal along $R_m \cap R_{m+1}$.

By induction, $F_s(\phi(s,t))$ and $F_m(\phi(s,t))$ coincide in the corner $R_m \cap R_s \cap R_{s-1}$, both being equal to $\psi_m(s,t)$ there.

So along the edge $R_m \cap R_{m+1}$ the functions $F_m(\phi(s,t))$ and $F_{m+1}(\phi(s,t))$ agree, and along $R_m \cap R_s$ the functions $F_m(\phi(s,t))$ and $F_{m+1}(\phi(s,t))$ agree, hence $F_s(\phi(s,t))$ and $F_{m+1}(\phi(s,t))$ take the same value in the corner-point!

The salient point is to see that the functions F_s and F_{m+1} agree along the edge $R_{m+1} \cap R_s$, because then they patch up to a continuous function on $R_{m+1} \cup \bigcup_{0 \le k \le m} R_k$.

Luckily, they differ only by a constant in the intersection $U_{m+1} \cap U_s$, and the image of the corner lies there. As F_m and F_s agree in the corner, as do F_m and F_{m+1}, it follows that F_{m+1} and F_s are equal in the corner. Since their difference in $U_{m+1} \cap U_s$ is a constant, it follows that they are equal there, and in particular they coincide along the edge $R_{m+1} \cap R_s$. And that is what we were aiming for!



❏

(1.2) One can relax the condition on a homotopy and not require that the end-points be fixed in which case one speaks about *freely homotopic paths*. Although, if the two paths are closed, one requires that the homotopy be a homotopy of closed paths; that is, the deformed paths are all closed. To be precise, one requires that $\phi(s,0) = \phi(s,1)$ for all s.

In general integrals are obviously not invariant under free homotopy for non-closed paths, but for closed paths it holds true. One has

HomotopyInvarian

Teorem 1.17 *Let γ_1 and γ_2 be two closed piecewise C^1-paths in the domain Ω that are freely homotopic. Let f be holomorphic in Ω. Then*

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

BEVIS: The proof is the virtually same as for theorem 1.16, with only one small exception: The maps $\psi(s,0)$ and $\psi(s,1)$ are no longer constant. However, we know that $\phi(s,0) = \phi(s,1)$ for all s, which is sufficient to save the proof. As a matter of notation we let δ denote this path.

For each s it holds true that $\psi(s,0) = F_{i_s}(\phi(s,0))$ for some index i_s, however this index may change along the path δ. In an analogous manner, $\psi(s,1) = F_{j_s}(\phi(s,1))$ with the index j_s possibly varying with s. Now, $\phi(s,0) = \phi(s,1)$ and the F_k's differ only by constants. Therefore the difference $\psi(s,0) - \psi(s,1)$ is locally constant along δ and hence constant by continuity. It follows that

$$\psi(0,0) - \psi(0,1) = \psi(1,0) - \psi(1,1),$$

and by reference to the proof of lemma 1.6 we are done. ❑

(1.3) As an example, but important example, let us show that any closed path $\gamma(t)$ in the star-shaped domain Ω with apex a is freely homotopic to any circle contained in Ω and centered at a —traversed a certain number of times, in any direction. That γ is freely homotopic to a path of the form re^{it}, the parameter t running from 0 to $2n\pi$ and n being an integer and r sufficiently small so the circle lies in Ω. Express the path $\gamma(t)$ in polar coordinate as

$$\gamma(t) = a + r(t)e^{i\phi(t)},$$

with t running from 0 to $2n\pi$. Define a homotopy Φ by

$$\Phi(s,t) = (1-s)r(t)e^{i(1-s)\phi(t)} + sre^{ist},$$

where t runs from 0 to $2n\pi$—since the segment from a to $\gamma(t)$ is contained in Ω, clearly the segment from $a + re^{it}$ is as well. This shows that two closed curves are freely homotopic in Ω if and only if their winding numbers about a are equal.

vamKampen1

(1.4) The previous example can be generalized using van Kampen's theorem. One may show that if Ω is any domain and Ω' is obtained from Ω by removing a point a (or a closed disk $\overline{D}$) contained in Ω, there is an exact sequence of fundamental groups

$$1 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \pi_1(\Omega') \longrightarrow \pi_1(\Omega) \longrightarrow 1 \qquad (1.32)$$

and where the map α sends the generator 1 of $\mathbb{Z}$ to a circle around a contained in Ω and being traversed once counterclockwise, so a closed path γ lying in Ω' and being null-homotopic in Ω, has a homotopy class in Ω' that is a multiple of $\alpha(1)$; that is, the path is homotopic in Ω' to a small circle round a traversed a certain number of times, in one direction or the other.

1.5.4 General Cauchy theorem

From the invariance of the integral, we immediately obtain the following fundamental theorem:

Teorem 1.18 *Let Ω be a domain and f a function holomorphic in Ω and let γ be a piecewise closed C^1-path in Ω. Assume that γ is null-homotopic. Then*

$$\int_\gamma f(z)dz = 0.$$

Bevis: Let α be "half" the path γ, that is $\alpha(t) = \gamma(t/2)$ for $t \in [0,1]$, and let β be the other half, that is the one given by $\beta(t) = \gamma(t/2 + 1/2)$. Then of course γ is the composite $\beta\alpha$. The composite being null-homotopic implies that $\alpha \sim \beta^{-1}$, and hence by theorem 1.16 one has

$$\int_\alpha f(z)dz = -\int_\beta f(z)dz,$$

but then

$$\int_\gamma f(z)dz = \int_\alpha f(z)dz + \int_\beta f(z)dz = 0.$$

❏

For simply connected domains we get the general Cauchy theorem as an immediate corollary

Korollar 1.5 *Let Ω be a simply connected domain and let f be holomorphic ion Ω. Then for any closed path γ it holds true that*

$$\int_\gamma f(z)dz = 0.$$

(1.1) An in view of the existence criterion for primitives (proposition 1.5 on page 16) we see holomorphic functions in simply connected domains all have primitives:

Korollar 1.6 *If f is a holomorphic function in the simply connected domain Ω, then f has a primitive.*

(1.2) In particular, and of particular interest, this applies to the logarithm. Any holomorphic function f vanishing nowhere in the simply connected Ω has a logarithm; *i.e.*, there is a function, which we denote by $\log f$, and that satisfies the equation

$$\exp \circ \log f = f \tag{1.33}$$

throughout Ω. Indeed, as f is without zeros in Ω, the logarithmic derivative f'/f is holomorphic there and, Ω being simply connected, has a primitive there. We temporarily denote this primitive by L (as long as (1.33) is not verified, it does not deserve to be ExpLog titled $\log f$). A small and trivial computation using standard rules for the derivative, shows that

$$\partial_z f^{-1}(z)\exp(L(z)) = 0.$$

Hence $\exp(Lz) = Af(z)$ for some constant A. Of course it might be that $A \neq 1$, but then we change the primitive L into $L - \log A$, which is another primitive for f'/f.

As usual $\log f$ is unique only up to whole multiples of $2\pi i$.

When the logarithm $\log f$ is defined, the function f also possesses roots of all types. More generally for any complex constant α, the power f^α is defined; it is given as $f^\alpha = \exp(\alpha \log f)$.

1.5.5 The Genral Cauchy formula

Using the remark in example 1.4, we obtain the general form of the formula of Cauchy, valid for null-homotopic paths in any domain Ω:

Teorem 1.19 *Assume that f is a holomorphic function in the domain Ω, and let $a \in \Omega$ be a point. Then for any closed path γ being null-homotopic in Ω, it holds true that*

$$\mathrm{n}(\gamma, a)f(a) = \frac{1}{2\pi i}\int_\gamma f(z)(z-a)^{-1}dz.$$

BEVIS: By the homotopy invariance of the integral (theorem 1.17 on page 55) and the remark in paragraph 1.4, the integral in the theorem equals

$$\frac{1}{2\pi i}\int_{n\partial D} f(z)(z-a)^{-1}dz$$

for a certain integer n. In this integral D denotes a disk whose closure is contained in Ω, and $n\partial D$ indicates the path that is the boundary circle of D traversed n times. ❏

(1.1) There is also a generalization of the argument principle—giving us the ultimate formulation. However, it needs some preparation, the first being a common technic, which as well will be useful later, called *exhausting by compacts*. Recall the notation A° for the set of interior points of a set A.

Exhausting

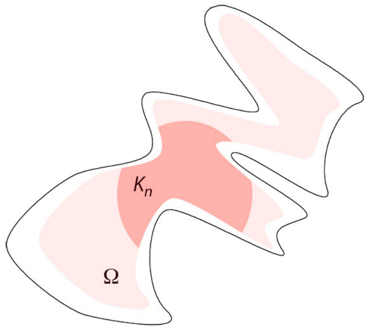
Lemma 1.7 *Assume that Ω is a domain in the complex plane. Then there exists a sequence of compact sets K_n all contained in Ω satisfying the two properties*

- *The sequence is increasing: $K_n \subseteq K_{n+1}$;*
- *Their interiors cover Ω, that is: $\bigcup_n K_n^\circ = \Omega$.*

BEVIS: For each n we put

$$K_n = \{\, z \in \Omega \mid d(z, \partial D) \geq 1/n \,\} \cap \{\, z \mid |z| \leq n \,\}.$$

Then K_n is closed and bounded (the distance function being continuos) and the K_n-s form an increasing sequence. For every point z in Ω one has $d(z, \partial D) > 1/n$ and $|z| < n$ for n sufficiently large, hence the interiors of the K_n cover Ω. ❏



Figur 1.7: One of the compact exhausting sets.

(1.2) The reason we are interested in this process of exhausting by compacts at this stage, is that it guaranties there only being finitely poles and zeros of f having non-vanishing winding number with respect to a given closed path γ in Ω.

Indeed, γ is compact and hence must be contained in some K_n. Points outside K_n belong to the unbounded component of the complement $\mathbb{C}\setminus\gamma$ and the winding numbers of γ round them vanish. But zeros and poles of f are isolated, so in compact sets there is only finitely many. Hence

Lemma 1.8 *Assume that γ is a closed path in the domain Ω and that f is meromorphic in Ω. Then there is only a finite number of points $a \in \Omega$ such that $\mathrm{n}(\gamma, a) \neq 0$.*

(1.3) The second preparation is a formula from homotopy theory analogous to the exact sequence in example 1.4 on page 56, but involving not only one point, and just as is the case with 1.4, it hinges on the van Kampen theorem. We shall not prove it, so if you do not know the van Kampen theorem, you have no choice but trusting us.

Given a finite number $a_1, \ldots, a_r$ of points in the domain Ω and given r little disks D_i, centered at a_i respectively and so little that they are contained in the domain Ω. LetΩ' be Ω with the r given points deleted; *i.e.*, $\Omega' = \Omega \setminus \{a_1, \ldots, a_r\}$.

Denote by c_i the homotopy class in Ω' of the boundary circle ∂D_i traversed once counterclockwise. Then there is an exact sequence

$$Z \star \cdots \star \mathbb{Z} \longrightarrow \pi_1(\Omega') \longrightarrow \pi_1(\Omega) \longrightarrow 1.$$

Don't let the stars frighten you, they stand for something called a free product of groups. If you want to dig into these questions Alan Hatchers book [?] can be recommended. In clear text the sequence means that the fundamental group $\pi_1(\Omega)$ equals the quotient of $\pi_1(\Omega)$ by the normal subgroup generated by the r classes c_i.

The form of this statement, useful for us, is that if γ is a closed path null-homotopic in Ω and avoiding the points a_i, its homotopy class equals an integral combination $n_1c_1 + \cdots + n_rc_r$ of the classes of the little circles round the a_i-s. By applying the homotopy invariance to the different integrals $\int_\gamma (z-a_i)^{-1}dz$, one sees that $n_i = \mathrm{n}(\gamma, a_i)$, so in the fundamental group of Ω', one has the equality

$$[\gamma] = \sum_i \mathrm{n}(\gamma, a_i) c_i$$

whenever γ is a closed and null-homotopic path in Ω (and $[\gamma]$ denotes its homotopy class); indeed, one has

$$\frac{1}{2\pi i}\int_{c_i} (z-a_j)^{-1}dz = \delta_{ij}.$$

(1.4) We have come to the scene of the ultimate formula in the context of counting poles and zeros: The setting is a domain Ω, a function f meromorphic in Ω and a function g holomorphic there. Finally, a closed path null-homotopic in Ω is an important player, and here comes the hero of the play, the ultimate formula:

$$\frac{1}{2\pi i}\int_\gamma g(z) d\log f(z) = \sum_{a\in\Omega} g(a)\, \mathrm{n}(\gamma, a) \mathrm{ord}_a f \tag{1.34}$$

TheUltimateArgPrinc

This formula looks suspiciously like the formula (1.28) on page 47, but the difference is of course the relaxed conditions on the domain and the path. The proof is simple once the preparations are in place.

We know that only for only finitely many points $a_1, \ldots, a_r$ in Ω the following product $\mathrm{n}(\gamma, a)\mathrm{ord}_a f$ is non-zero, hence the sum in the formula is finite. We know that γ is homotopic to an an integral combination $c = \sum_i n_i c_i$, with $c_i = \mathrm{n}(\gamma, a)_i$, and by the homotopy invariance of the integral we can replace $\int_\gamma g d\log f$ by $\sum_i n_i \int_{c_i} g d\log f$. Finally, in each of the terms in the latter sum the integral equals $g(a)\mathrm{ord}_{a_i} f$ by Cauchy's formula for a disk.

1.6 Laurent series

Recall that an annulus is a region in the complex plane bounded by two concentric circle. If the two radii are R_1 and R_2 with R_1 the smaller, and a is their common center, the annulus consists of the points z satisfying $R_1 < |z-a| < R_2$. In case $R_1 = 0$ or $R_2 = \infty$, the annulus is *degenerate* and equals to either the punctured disk $0 < |z-a| < R_2$, the complement of a closed disk $R_1 < |z-a|$ or the whole complex plane (in case $R_1 = 0$ and $R_2 = \infty$).

This section is about functions that are holomorphic in an annulus. They have a development into a double series analogous to the Taylor development of a function holomorphic in a disk.

(1.1) Let a_n be a sequence of complex numbers that is indexed by $\mathbb{Z}$; that is n can take both positive and negative integral values. Consider the *double series*

$$\sum_{n\in\mathbb{Z}} a_n(z-a)^n, \tag{1.35}$$

DobbelSeries

which for the moment is just a formal series. It can be decomposed in the sum of two series , one comprising the terms with non-negative indices, and the other the terms having negative indices. That is we one has

$$\sum_{n\in\mathbb{Z}} a_n(z-a)^n = \sum_{n<0} a_n(z-a)^n + \sum_{n\geq 0} a_n(z-a)^n. \tag{1.36}$$

DekompSeries

One says that the series Σ is convergent for the values of z belonging to set S if and only if each of the two series in the decomposition above converges for z in the given S, and we say that the convergence is uniform on compacts if it is for each of the two decomposing series.

In case the series (1.35) converges for z in the set S, the "positive" and the "negative" series in (1.36) converges to functions f_+ and f_- respectively, and we say that double series converges to the function $f = f_+ + f_-$.

The "positive"series

$$\sum_{n\geq 0} a_n(z-a)^n$$

is an ordinary power series centered at the point a, and has, as every power series has, a radius of convergence. Call it R_2. The series thus converges in the disk D_{R_2} given by $|z-a| < R_2$, and diverges in the region $|z-a| > R_2$. It converges uniformly on compact sets contained in D_{R_2}, and as we know very well, defines a holomorphic function there.

On the other hand, the "negative" series

$$\sum_{n<0} a_n(z-a)^n$$

is a power series in $w = (z-a)^{-1}$; indeed, performing this substitution we obtain the expression

$$\sum_{n>0} a_{-n}w^n$$

for the "negative" series. This power series has a radius of convergence, that we for a reason soon to become clear call R_1^{-1}, so it converges for $|w| < R_1^{-1}$ and diverges if $|w| > R_1^{-1}$. Translating these conditions on w into conditions on z, we see that the "negative" series converges for $|z-a| > R_1$ and diverges for $|z-a| < R_1$. The convergence is uniform on compacts and therefore the sum of the series is a holomorphic function f_- in the region $|z-a| > R_1$.

The interesting constellation of the two radii of convergence is that $R_1 < R_2$, in which case the double series converges in the region sandwiched between the two circles centered at a and having radii R_1 and R_2 respectively, and there it represents the holomorphic function $f = f_+ + f_-$.

(1.2) Now, let $R_1 < R_2$ be two positive real numbers and let a be a complex number. We shall work with a function f that is holomorphic in the annulus $A(R_1, R_2)$, and we are going to establish that f has what is called a *Laurent series* in A, that , it can be represented as double series like the one in (1.35). We shall establish the following result:

Teorem 1.20 *Assume that f is holomorphic in the annulus $A = A(R_1, R_2)$. Then f is represented by a double series*

$$f(z) = \sum_{n\in\mathbb{Z}} a_n (z-a)^n$$

which converges uniformly on compacts in A. The coefficients a_n are given by

$$a_n = \frac{1}{2\pi i}\int_{c_r} f(w)(w-a)^{-n-1}dw$$

where c_r is any circle centered at a and having a radius r with $R_1 < r < R_2$.

Bevis: To begin with, we let r_1 and r_2 be two real numbers with $R_1 < r_1 < r_2 < R_2$. The two circles c_1 and c_2 centered at a and with radii r_1 and r_2 respectively (and both traversed once counterclockwise) are clearly two freely homotopic paths in A, a homotopy being $\phi(s,t) = sc_{1(t)} + (1-s)c_2(t)$ (where c_i as well denotes the standard parametrization of c_i). Hence for any z lying between c_1 and c_2 the general Cauchy formula gives

$$f(z) = \frac{1}{2\pi i}\int_{c_2} f(w)(w-z)^{-1}dw - \frac{1}{2\pi i}\int_{c_1} f(w)(w-z)^{-1}dw \tag{1.37}$$

DiffEqua indeed, the winding number of the composite path $c_2 - c_1$ round z equals one.

Now, the point is that the two integrals appearing in (1.37) above, will be the two functions f_+ and f_-. To see this we shall apply the proposition 1.8 on page 27 twice.

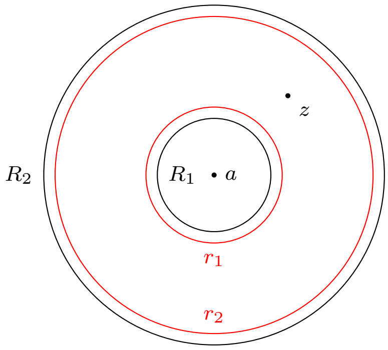
We start be examining the first integral, whose path of integration is c_2, and we take $\phi(w) = f(w)$ in proposition 1.8. Hence

$$f_+(z) = \frac{1}{2\pi i}\int_{c_2} f(w)(w-z)^{-1}dw$$

is holomorphic in the disk $|z-a|<c_2$, and its Taylor series about a has the coefficients

$$a_n = \frac{1}{2\pi i}\int_{c_2} f(w)(w-a)^{-n-1}dw.$$

According to proposition 1.8, the Taylor series converges in the largest disk not hitting the path of integration, that is the disk $|z-a|<c_2$.



Figur 1.8: *The annulus and the two auxiliary circles*

Next we the examine the second integral, and to do this, we perform the substitution $u=(w-z)^{-1}$. Then $dw=-u^{-2}du$, and the new path of integration is $|u|=r_1^{-1}$, a circle centered at the origin which designate by d. Upon the substitution, the integral becomes

$$f_-(z)=\frac{1}{2\pi i}\int_{c_1} f(w)(w-z)^{-1}dw = -\frac{1}{2\pi i}\int_d f(u^{-1}+z)u^{-1}du$$

Applying once more the proposition 1.8, this time with $\phi(u)=-f(u^{-1}+z)$ and the path of integration equal to d (positively oriented), we conclude that the integral is a holomorphic function in the disk $|u|<r_1^{-1}$, or equivalently for $|z-a|>r_1$. Its Taylor series about the origin has, according to proposition 1.8, coefficients b_n given by the integrals below, where we as well, reintroduce the variable w:

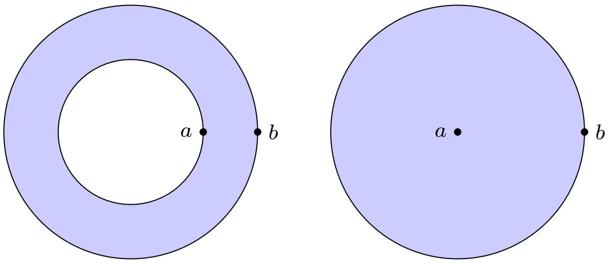
$$b_n=-\frac{1}{2\pi i}\int_d f(u^{-1}+a)u^{-n-1}du = \frac{1}{2\pi i}\int_{c_1} f(w)(w-a)^{n-1}dw,$$

And in fact, that will be all! ❑

OPPGAVE 1.40. Determine the Laurent series of the function $f(z)=(z-a)^{-1}(z-b)^{-1}$ in the annulus $A(|a|,|b|)$ centered at the origin. ✱

OPPGAVE 1.41. Determine the Laurent series of $f(z)=(z-a)^{-1}(z-b)^{-1}$ in the annulus $A(0,|b-a|)$ centered at a. ✱

OPPGAVE 1.42. Let f have an isolated singularity in a and be holomorphic for $0<|z-a|<r$. Show that f has a pole at a if and only if the series for f_- in the Laurent development of f in annulus the $A(0,r)$ centered at a has a finite number of terms. ✱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2. Del

Existence of complex complex functions

Version 0.6— Monday, October 3, 2016 at 10:19:39 AM
Still preliminary version
Have corrected a lot of mistakes. 2016-11-30 13:04:29+01:00

Sometimes it is inestimably useful to be able to approximate functions by polynomials. A local study of differentiable functions is unthinkable without Taylor polynomials. In a somehow more global and advanced setting, the Weierstrass approximation theorem is fundamental. It tells us that any real continuous function on a compact set in euclidean space $\mathbb{R}^n$ can be uniformly approximated by real polynomials to any degree of accuracy.

Our primary primary concern are the holomorphic functions, and the question naturally becomes this: Given a holomorphic function in a domain Ω and a compact set $K \subseteq \Omega$. When can can f be approximated by polynomials uniformly on K? That is, when can one to any $\epsilon > 0$ find a polynomial $P(z)$ with $\sup_{z \in K} |f(z) - P(z)| < \epsilon$?

The first comment is that a positive answer to this, is both a stronger and weaker statement than in Weierstrass' theorem. Complex polynomials are very special compared to the real polynomials, and there are a lot less. The dimension of the real vector space of real polynomials in two variable whose degree is less than n is quadratic in n, but the complex ones form a vector space of real dimension $2n$. Their properties are also very different. The real and complex part of complex polynomials only have saddle points, whereas real ones of course can have local extrema of any kind.

On the other hand, we suppose that f is holomorphic on an open set containing K, whereas in Weierstrass one assumes only that f is continuous on K. There is famous theorem proved by Sergei Nikitovich Mergelyan io 1951, generalizing Weierstrass's result in this respect. It says that if the complement $\mathbb{C} \setminus K$ is connected, any function f continuous in K and holomorphic in the interior K° of K, can be approximated uniformly by complex polynomials. There are also versions where the approximating functions can be rational, *e.g.,* if the complement of K only has finitely many components one can approximate with rational functions.

The second comment is that the answer to our question is not a clear yes or no, it depends on the topology of K. A illustrative example is the closed annulus A centered at the point a with radii 1 and 3 say. Then $(z-a)^{-1}$ is holomorphic in A, but can not be approximated uniformly in A. Indeed, if $P(z)$ is a polynomial, it holds true that $\int_C f(z)dz = 0$, whereas $\int_C (z-a)^{-1}dz = 2\pi i$. So an inequality

$$\left|P(z) - (z-a)^{-1}\right| < \epsilon < 1$$

for points $z \in A$, leads to the contraction

$$2\pi = \left|\int_C P(z) - (z-a)^{-1}\right| < 2\pi\epsilon < 2\pi.$$

And in fact, to some extent, this a constituting example. It describes very well the obstructions to yes being the answer of our question. And it also indicates the next natural question: When can a function holomorphic in K be approximated uniformly by rational functions, holomorphic in Ω? We must alow poles in the "holes" of K, and for the rational function to be holomorphic in Ω, in every hole in K there must be a "hole" in Ω where we can put the pole!, but as long as we comply with that rule we can be very specific about where to locate the poles

PROBLEM 2.1. Let A an annulus and a a point in the "inner circle". Show that any function holomorphic in A can be approximated uniformly by rational functions with only a pole at a. HINT: Treat first the case with a being the center of A; Laurent series is then the key-word. ✷

2.1 A few preparations

We begin with some preparations about the space of holomorphic functions $\mathcal{H}(K)$ and we recall some basic facts about the connected components of complements.

2.1.1 Function spaces

To begin with, let K be a compact subset of $\mathbb{C}$ and recall that a function f is said to be holomorphic in K if it is defined and holomorphic in a domain containing K. In the present context the domain is irrelevant, and does not appear in the notation. The set of functions holomorphic on K is denoted by $\mathcal{H}(K)$. One has to be precise about this, $\mathcal{H}(K)$ is the set of continuous functions which are restrictions of functions holomorphic in an open set containing K.

The space $\mathcal{H}(\mathcal{K})$ is a complex algebra, closed under linear combinations and under product (two functions holomorphic in K are defined and holomorphic on a common domain containing K, *e.g.,* the intersection of the domains where either is holomorphic).

The space K has a topology induced by the norm

$$\|f\|_K = \sup_K |f(z)|;$$

one easily verifies this is a norm, *i.e.,* the triangle inequality is satisfied, and the norm axiom $\|fg\|_K \leq \|f\|_K \|g\|_K$ is as well. The latter inequality will be strict when the maximum of the two functions occur at different points, which would *a priori* bet at. This topology is called *the topology of uniform convergence*, and one sees almost by definition that a sequence $\{f_n\}_n$ converges in $\mathcal{H}(K)$ precisely when it converges uniformly on K. Both addition and multiplication in $\mathcal{H}(K)$ are continuous in this topology and the $\mathcal{H}(K)$ qualifies to be what is called a *normed algebra*.

Problem 2.2. Describe $\mathcal{H}(K)$ if $K = \{a_1, \ldots, a_r\}$. ✷

Problem 2.3. Assume that K is connected and has interior points. Two functions holomorphic in open neighbourhoods of K that restrict to the same continuous function on K must be equal on a neigbourhood of K. ✷

Problem 2.4. One may define the space germs $\mathcal{G}(K)$ of holomorphic functions around a compact set K in the following way. The starting point is the set of pairs (f, U) where U is an open set containing K and f is holomorphic in U. Two pairs (f, U) and (g, V) are equivalent if there is an open, non-empty subset $W \subseteq U \cap V$ such that $f|_W = g|_W$, and the set $\mathcal{G}(K)$ of germs is defined to be the set of equivalence classes. Show that $\mathcal{G}(K)$ is an algebra. If K has a non-void interior, show that $\mathcal{G}(K)$ is a normed algebra. ✷

Problem 2.5. Describe $\mathcal{G}(K)$ when $K = \{a_1, \ldots, a_r\}$. ✷

(2.1) If $\mathcal{A}$ is a subclass of functions in $\mathcal{K}$, we say that every holomorphic function on K can be uniformly approximated by functions from the class A, if $\mathcal{A}$ is dense in with $\mathcal{H}$. Written out, this means that for any function f holomorphic on K and any $\epsilon > 0$ given, there is a function $g \in \mathcal{A}$ with

$$\|f - g\|_K < \epsilon.$$

The following little lemma is now and then useful:

Lemma 2.1 *Suppose $\mathcal{H}$ is normed algebra and that $\mathcal{A} \subseteq \mathcal{H}$ is a subalgebra. Then the closure $\overline{\mathcal{A}}$ is a subalgebra as well.* LilleNyttigLemma

Proof: Elements in the closure $\overline{\mathcal{A}}$ are limits of sequences from $\mathcal{A}$, and we get away with the following reasoning: If $a_i \to a$ and $b_i \to b$, then $a_i + b_i \to a + b$ and $a_i b_i \to ab$. ❑

(2.2) For open sets the topology of uniform convergence on $\mathcal{H}(\Omega)$ is slightly more complicated to define.

2.1.2 Connected components

The connected components of complements $\mathbb{C} \setminus A$ of subsets A of $\mathbb{C}$— A mostly being compact or open—play a prominent role in the Runge-theory, so it is worth while saying a few words about them.

(2.1) Recall that subset of a topological space is *connected* if it is not the disjoint union of two open sets, or equivalently, it is not the disjoint union of two closes sets.

A *connected component* of the topological space X is a maximal connected subset. As the intersection of two connected sets is connected, the connected components of X form a partition of X; that is the space X is the disjoint union of its connected components.

Connected components are always closed subsets of X; for if$C$ is one and x belongs to the boundary of C, any open neigbourhood of x has points in common with C, and whence $X \cup \{x\}$ is connected. It follows that $x \in C$ since C is a maximal connected subset.

However components are not always open. For example, the topological space $\mathbb{Q} \subseteq \mathbb{R}$ consisting of the rational numbers with the topology induced from the reals $\mathbb{R}$, the only connected subsets are the singletons $\{q\}$: Two different rational numbers q of q', can always be separated by an open interval. Just chose a real number r between them; then $(-\infty, r) \cap \mathbb{Q}$ and $(r, \infty) \cap \mathbb{Q}$ are two disjoint open sets either containing q or q'. A space having this property—that the points are the connected components—is said to be *totally disconnected*.

Luckily the connected components of an open subset $\Omega \subseteq \mathbb{C}$ are all open. This follows from disks being connected: If C is one of the components of Ω and $x \in C$ is a point, there is a disk D centered at x contained in Ω. The union $C \cup D$ is connected —both D and C are, and their intersection is non-empty— so $D \subseteq C$ by the maximality of C.

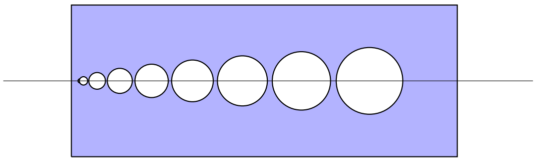
Problem 2.6. Show that the Cantor-set $\mathfrak{c}$ is totally disconnected. ✷

Problem 2.7. Let $\mathbb{Z}_{p^\infty} \subseteq \mathbb{S}^1$ denote the subset whose points are all the p^r-te roots of unity for a natural number r; *i.e.*, those on the form $\exp(2\pi i a/p^r)$ for $a \in \mathbb{Z}$. Show that $\mathbb{Z}_{p^\infty}$ is totally disconnected. ✷

(2.2) The complement $\mathbb{C} \setminus K$ of a compact set is of course an open set, and its connected component are all open subsets of $\mathbb{C}$. There is a unique one that is unbounded; if there where two, K being compact, they would shear a connected neigbourhood of the point at infinity, *e.g.*, the complement $\mathbb{C} \setminus D$ for a disk D of sufficiently large radius.

The other components, have compact closures being compact, but they can very well be infinite in number. Any sequence of disjoints disks whose radii diminish sufficiently quickly, and contained in a compact, would give an example. To be concrete let D_n be the disk centered at $1/n$ and with radius $1/3n$. Then $D = \bigcup_n D_n$ is an open set contained in the square $I \times I$, and $I \times I \setminus D$ is compact with all the disks D_n as components of the complement.

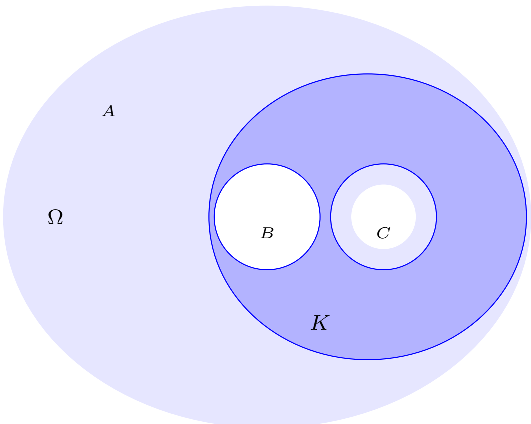
We do not put any further hypothesis on the compact sets, so they need not be connected, and can have uncountable many connected components; like *e.g.,* $\mathfrak{c} \times \mathfrak{c}$ where we as usual $\mathfrak{c}$ stands for the Cantor set. And they do not necessarily have interior points.



(2.3) Recall that a subset Y of a topological space X is called *relatively compact* if the closure $\overline{Y}$ of Y in X is compact. This is equivalent to Y being contained in a compact subset of X.

If Ω is a domain and $K \subseteq \Omega$ is a compact subset, the connected components of $\Omega \setminus K$ are all open in Ω. They come in two flavors. They can be relatively compact in Ω or not, and this distinction is very important in Runge-theory. Let us agree to use the colloquial—but descriptive—term *hole* for a bounded component of the complement $\mathbb{C} \setminus A$ of a set $A \subseteq \mathbb{C}$.

In figure 2.1 below we have depicted a domain Ω containing a compact set K. The complement $\Omega \setminus K$ has three components A, B and C of which only one is relatively compact, namely B. Among the two others, A is the intersection of the unbounded component of K with Ω, and this is generally one way of obtaining components that are not relatively compact. The other way is illustrated by C; heuristically one may describe that phenomenon as a "hole in the hole"; that is Ω has a hole contained in a hole of K.



Figur 2.1: *A domain Ω and a compact subset K*

One also observes that the part of the boundary $\partial\Omega$ that is contained in Ω—in the figure the boundary of the region B—is contained in K. This is general; one has

Lemma 2.2 *Let $K \subseteq \Omega$ be a compact within a domain and let C be a connected compact of the complement $\Omega \setminus K$. Then $\partial C \cap \Omega \subseteq K$. If C is a bounded component, it holds true that C is relatively compact if and only if $\partial C \subseteq K$.* BoudaryComp

Proof: Let z be a point in the intersection $\partial C \cap \Omega$ and assume that z does not belong to K. Then there is a disk D about z contained in $\Omega \setminus K$. Now, $D \cup C$ is connected so D must be contained in C. But C is open in Ω and consequently z does not belong to the boundary ∂C, which is absurd.

A bounded component C is relatively compact if and only its closure in Ω equals its closure in $\mathbb{C}$, which is equivalent to the boundary ∂C being contained in Ω. ❏

2.1.3 An interesting example

The example we are about to describe was found by the Swiss mathematician Alice Roth in 1938 and described in her article [?]. Subsequently it got the nickname "Roth's Swiss cheese ", which you will find appropriate after having seen the construction (take look at figure 2.2 below). Most of the work is left as exercises.

Alice Roth made this example as a counterexample to certain hypothesis about polynomial approximations by rational functions. It is part of the story that her example was forgotten and rediscovered by the armenian mathematician Mergelyan; whose famous approximation theorem we alluded to in the beginning of this chapter.

There is a stronger version of Mergelyan theorem saying that if the diameter of the bounded components of the complement of K are bounded away from zero, uniform approximation by rational functions is possible; and the Roth's Swiss cheese, is an example that the boundedness condition is necessary.

The construct of the example is a compact subset K of the closed annulus $A = \{\, z \mid 1 \leq |z| \leq 3 \,\}$ which is nowhere dens in A and has Lebesgue measure as close to 8π as one wants. The complement is a union of disjoint disks. Additionally the intersection[1] $C_2 \cap K$ is nowhere dens in C_2 and of measure zero.

We begin by choosing a sequence $\{z_n\}$ in the open ring A° which is dens in A, and such that subsequence of $\{z_n\}$ lying on the circle C_2 forms a dens subset of C_2. We then chose a sequence of positive numbers ϵ_n satisfying $\sum_n \epsilon_n = \rho < 1/2$.

The construction is of course recursive. The first step being to chose a disk D_1 centered at z_1 and contained in A° having a radius η_1 less than ϵ_1, and such none of the points $z_2, z_3, \ldots$ lie on the boundary.

The recursive step is as follows. Assume that $D_1, \ldots, D_n$ are constructed. In case z_{n+1} lies in the union $\bigcup_{k\leq n} D_k$, we let $D_{n+1} = \emptyset$. If this is not the case, we let D_{n+1} be a disk contained in A° , disjoint from all the previously chosen disks , centered at z_{n+1} and having a radius η_{n+1} less than ϵ_{n+1}. Furthermore none of the points $z_{n+2}, z_{n+3}, \ldots$ should lie on the boundary. Finally, the "swiss cheese" is as announced defined by $K = A \setminus \bigcup_{k\in\mathbb{N}} D_k$

Problem 2.8. Show that K is nowhere dense in A and that the two-dimensional Lebesque measure μK satisfies $\mu(K) \geq 8\pi - \pi \sum_k \eta_k^2$. Show that we K can have a measure as close as we want to 8π. Show that the total length of the circumferences of ∂D_n is at most π, that is $\sum_n \Lambda(\partial D_n) < \pi$. ✷

[1] We denote by $C_r$the circle centered at the origin and having radius r.

PROBLEM 2.9. Show that $K \cap C_2$ is nowhere dense in C_2. ✷

PROBLEM 2.10. Assume that $f(z)$ is a rational function without poles in the set K. Show, by using Cauchy's residue formula, that

$$\int_{C_1} f(t)t^{-1}dt - \int_{C_3} (f(t)-1)t^{-1}dt + \sum_n \int_{\partial D_n} f(t)t^{-1} = 2\pi i.$$

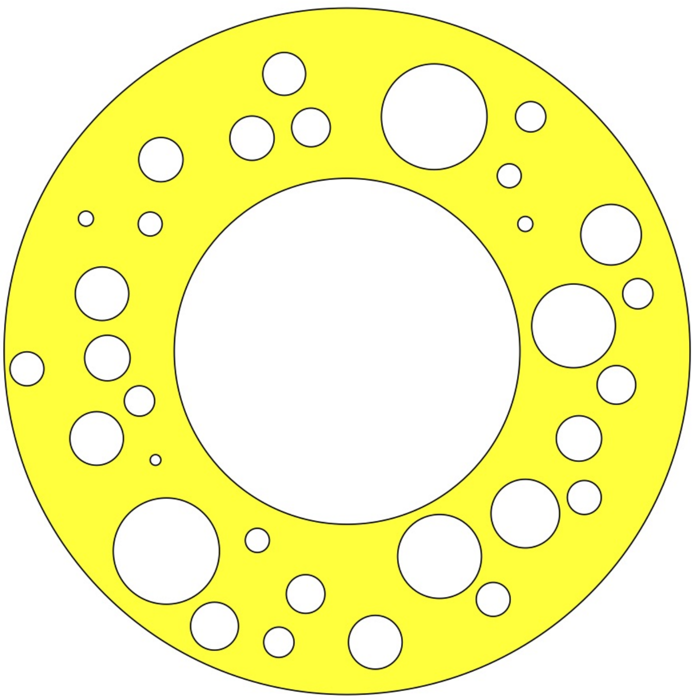
✷

PROBLEM 2.11. Show there is no rational function $f(z)$ such that

- ☐ $|f(z)| < 1$ for $z \in K$,
- ☐ $|f(z)| < 1/4$ for $z \in C_1$,
- ☐ $|f(z) - 1| < 1/4$ for $z \in C_3$.

✷

PROBLEM 2.12. Show there are continuous functions on K that can not be uniformly approximated by rational functions. HINT: Find an appropriates linear combination $\alpha\,|z| + \beta$. ✷



Figur 2.2: *The Swiss cheese of Alice Roth.*

2.2 Runge for compacts

We shall follow the exposition of Reinholdt Remmert as in a book [Rem] closely. He bases the theory on so called *step-polygons* This gives an easy to follow and rather elementary proof of Runge's theorem. Additionally it gives a certain "spin off" in some near by contexts. A draw back is that the method is kind of sensitive to the context, and does not generalize easily, *e.g.,* to Riemann surfaces.

2.2.1 The formulation of Runge's theorem for compacts

The example in the beginning of this chapter can be generalizes easily. If a is any point in the complement of K belonging to one of the bounded components, say C, the function $(z-a)^{-1}$ can not be approximated by polynomials on K (or holomorphic functions for that matter). Indeed, we may chose ϵ according to the prescription $\epsilon < \sup_K |z-a|^{-1}$, which is finite (since K is compact) and positive (as a is not lying in K). Now, if there were a polynomial $p(z)$ such that $|(z-a)^{-1} - p(z)| < \epsilon$ for $z \in K$, we would have $|1-(z-a)p(z)| < \epsilon |z-a| < 1$, for all $z \in C$. And this is a flagrant contradiction as a lies in C.

(2.1) The arguments from the previous paragraph show that in order to have a general approximation theorem, it is necessary to alow poles in every bounded component of $\mathbb{C} \setminus K$, and Runge tells us that this is sufficient as well:

KompaktRunge **Theorem 2.1** *Let K a compact subset and $P \subseteq \mathbb{C}$ be a subset. Every function f holomorphic in K can be approximated uniformly by rational functions whose poles all lie in P if and only if P meets every one of the bounded components of $\mathbb{C} \setminus K$*

The proof will occupy the rest of this section and it has three distinct parts. In one we establish a nice version of Cauchy's formula allowing us to represent the function f as a certain integral. In the second, certain Riemann sums for this integral give us some an approximation by rational function, and finally, a process called "pole-pushing" lets us conclude.

(2.2) But before starting on the proof, we formulate two corollaries:

Corollary 2.1 *Assume that $K \subseteq \mathbb{C}$ is compact. Every function holomorphic on K can be approximated uniformly with the help of polynomials if and only if $\mathbb{C} \setminus K$ is connected*

Proof: As $\mathbb{C} \setminus K$ has no bounded component, the set P is empty. ❑

Corollary 2.2 *Let Ω be a domain and $K \subseteq \Omega$ a compact subset. Every function holomorphic in K can be approximated uniformly by rational functions holomorphic in Ω if and only if every connected component of $\mathbb{C} \setminus K$ meets $\mathbb{C} \setminus \Omega$.*

Proof: To find a set P to use, chose in every component of $\mathbb{C} \setminus K$ chose a point not belonging to Ω. ❑

2.2.2 A useful version of Cauchy's formula

Recall that a polygon is a closed subset of $\mathbb{C}$ whose boundary consists of finite number of line segments $[a_i, a_{i+1}]$, called *edges*, that do not meet anywhere else then in the vertices a_i. One also requires that the edges close up, that is for some k it holds that $a_1 = a_{k+1}$.

Given a *grid* in the plane with *mesh-width* δ. The grid has squares, edges and vertices, the vertices being the points $(n\delta, m\delta)$ and the edges and the squares are what common usage tell you.

A step *step polygon* is a polygon whose edges are either horizontal or vertical. The step polygons we shall most frequently shall meet, will fit a grid; that is, all their edges are edges of the grid as well. The vertices of a step polygon is therefore of the form $a_{n,m} = (n\delta, m\delta)$.

(2.1) To begin with we prove a very special case of Cauchy's theorem. The setting is as follows. We are given a domain Ω and a compact subset K. Our first objective is to construct a closed chain σ in Ω disjoint from the compact K, such that any function holomorphic in Ω has a representation à la Cauchy, that is, is an integral along σ.

Lemma 2.3 *Given a compact K contained in the domain Ω, one may find a cycle in $\Omega \setminus K$ satisfying* FirKantLemma

$$f(z) = \frac{1}{2\pi i}\int_\sigma \frac{f(w)}{w-z}dw$$

for all $z \in K$ and for all functions f holomorphic in Ω.

The chain σ will be a sum $\sigma = \tau_1 + \cdots + \tau_k$ where the terms τ_i are closed step polygons fitting a grid.

Proof: We start by choosing a grid whose mesh-width δ is smaller than the distance from K to the boundary of Ω, that is, it satisfies $0 < \delta < d(\partial\Omega, K)/\sqrt{2}$.The boundary of the squares in the grid has a natural orientation, given by going counterclockwise round the square.

Our chain σ will be constructed as a sum of edges from the grid, and it will be part of the construction to give these edges a good orientation.

To proceed with the construction, we let $\mathcal{Q}$ be the collection of the squares in the grid that meet the compact set K. It is a finite set K being compact. Our interest is primarily in the edges of the grid lying on exactly one of the squares from $\mathcal{Q}$, and we reserve the notation $\mathcal{S}$ for the set of those. They have a natural orientation, the one they inherit from the unique square in $\mathcal{Q}$ they lie on.

The chain σ we are seeking, is the sum of the edges from $\mathcal{S}$ oriented in the natural way; that is we have

$$\sigma = \sum_{s \in \mathcal{S}} s.$$

The orientation an edge of the grid inherits from the two squares it lies on are opposite, and therefore it holds true that

$$\sigma = \sum_{Q \in \mathcal{Q}} \partial Q.$$

indeed, if s lies on two squares from $\mathcal{Q}$ it appears twice in the sum with opposite orientations.

Now, the chain σ is disjoint from K: If a point a lies one on an edges s it would lie on both the squares having s as an edge and would thence not be in $\mathcal{S}$. And, σ is contained in Ω: The distance from s to K is less than the diameter $\sqrt{2}\delta$ of the squares, and in its turn, the diameter is (by choice of the mesh-width δ) less than the distance $d(\partial\Omega, K)$ from K to the boundary $\partial\Omega$.

We proceed by attacking the formula for the integral in the lemma, and start out with a point z lying in the interior of one of the squares Q_0 from $\mathcal{Q}$. Cauchy's formula then gives us

$$f(z) = \frac{1}{2\pi i}\int_{\partial Q_0} \frac{f(w)dw}{w-z} = \sum_{Q \in \mathcal{Q}} \frac{1}{2\pi i}\int_{\partial Q} \frac{f(w)dw}{w-z} = \frac{1}{2\pi i}\int_{\sigma} \frac{f(w)dw}{w-z}. \qquad (2.1)$$

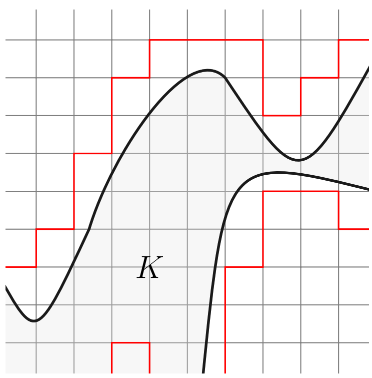
IntFormell

In case z is located on an edge between two squares from $\mathcal{Q}$, we chose a sequence of points $\{z_n\}$ from the interior of one of the squares that converges towards z. Then the equality in (2.1) holds for each of the points z_n, and by continuity, it will still hold in the limit.

Finally, we must show that s is a cycle. One may write $\partial\sigma = \sum_{s \in \mathcal{S}} \partial s = \sum_{c \in \mathbb{C}} n_c c$ where the coefficients n_c are integers only finitely many of which are non-zero. Suppose one of the coefficients is non-zero, say $n_{c_0} \neq 0$. Let $P(z)$ be a polynomial with $P(c_0) = 1$ and vanishing in all the other points c where $n_c \neq 0$. By the integral formula in the lemma applied to the function $(z-a)P'(z)$, where a is any point in K, we find

$$0 = \int_{\sigma} P'(w)dw = \sum_{c} n_c P(c) = n_{c_0} P(c_0) = n_{c_0}.$$

❑



Figur 2.3: *Part of the compact K and the chain σ*

Be aware that the chain s can have several distinct connected cycles. This happens if there are "holes" in the compact set K, at least if the mesh-width is small compared to the diameter of the hole, so that some squares from the grid are entirely contained in the hole. In case K has infinitely many holes, the cycle will only encompass finitely many of them, indeed, there will only be finitely many holes with diameter larger than a given positive constant.

PROBLEM 2.13. Show that if $K \subseteq \Omega$ is compact and $Z \subseteq \Omega$ is closed in $\mathbb{C}$, there is a cycle σ with $\mathrm{n}(\sigma, a) = 0$ for $a \in \mathbb{Z}$ and $\mathrm{n}(\sigma, a) = 1$ for $a \in K$. ✷

Corollary 2.3 *Suppose that the complement of the domain Ω has a compact component. Then Ω is not simply holomorphically connected; i.e., there is a function $f(z)$ holomorphic in Ω and a cycle γ with $\int_\gamma f(z)dz \neq 0$.*

PROOF: Let K be the actual compact component and let σ be a cycle like in lemma 2.3, and pick $a \in K$. Then $(z-a)^{-1}$ is holomorphic in Ω, and we have

$$\int_\sigma \frac{dw}{w-a} = 2\pi i.$$

❏

PROBLEM 2.14. Let K be the compact subset of the closed unit disk $\overline{\mathbb{D}}$ obtained by removing a countable sequence $\{D_n\}$ of pairwise disjoint open disks from $\mathbb{D}$. Show that if Ω is s domain in which K is contained, then D_n will be entirely contained in Ω except for finitely many indices n. ✷

PROBLEM 2.15. Let $K = \mathfrak{c} \subseteq [0,1]$ denote the Cantor set and let $\Omega = \mathbb{C}$. Make a sketch of the chain σ for small values of δ, and convince yourself that Cauchy's formula holds. ✷

PROBLEM 2.16. Show Jordan's curve theorem for closed, connected step polygons; *i.e.*, if τ is a step polygon then the complement $\mathbb{C} \setminus \tau$ has exactly two connected components. HINT: Use induction on the number of edges (or vertices). ✷

2.2.3 Et approximation lemma

The lemma we are about to establish, is by no means deep, it is simply just an explicit description of some of the Riemann sums for the integrals appearing in lemma 2.3, but to be kind to the students, we shall do it in detail.

(2.1) The lemma answers in a way the questions we posed in the introduction, but the answer is unsatisfactory in the sense that we do not control the location of the poles, except they are lying on the chain σ. We shall regain that control pretty well in the next section, but the prise to pay will be that the rational functions appearing in the approximation no more will have simple poles, as they have in the present lemma.

(2.2) Here comes the announced result:

Lemma 2.4 *Let $\Omega \subseteq \mathbb{C}$ be a domain and $K \subseteq \Omega$ a compact subset. If σ is a chain like in lemma 2.3 above, every function holomorphic in Ω, can be approximated uniformly on K by rational functions of the form* ApproxLemma

$$\sum_k c_k(z-a_k)^{-1}$$

where the c_k-s are complex constants and all the poles a_k lie on σ.

Proof: The function $f(w)(w-z)^{-1}$ is uniformly continuous in the variables w and z as long as $w \in \sigma$ and $z \in K$ (the product $\sigma \times K$ is compact). Hence if $\epsilon > 0$ is given, there is a $\delta > 0$ with the following property: If s_i are the segments that constitute σ, and we chop them up in smaller segments s_{ij} all of length at most δ, then

$$\left|f(w)(w-z)^{-1} - f(w_{ij})(w_{ij}-z)^{-1}\right| \leq \epsilon$$

for every choice of constants w_{ij} from s_{ij}, and for every $w \in s_{ij}$ and $z \in K$. Integrating over s_{ij} and summing over i and j, we find

$$\left|f(z) - \sum_{ij} c_{ij}(z-w_{ij})^{-1}\right| < \lambda\epsilon$$

where λ denotes the total length of σ and $c_{ij} = -f(w_{ij})$. Reindexing and rebaptizing the w-s to a-s, gives the lemma. ❑

2.2.4 Pole-pushing

The technic—with the euphonious name 'pole-pushing"—we are about to describe, is the third ingredient in the proof of Runge's theorem. It was invented by Carl Runge and published in his famous paper from 1885 where his approximation theorem appeared for the first time.

(2.1) We are literally going to push the poles around, but only within each connected component of $\mathbb{C} \setminus K$. The salient point is to push all the poles appearing in one component into one specific point in that same component.

PolForskyvning **Lemma 2.5** *Let $K \subseteq \mathbb{C}$ be compact and let C denote one of the components of the complement $\mathbb{C} \setminus K$. If a and b are two points in C the function $(z-a)^{-1}$ can be approximated uniformly on K with polynomials in $(z-b)^{-1}$.*

Proof:

Let U_a denote the set of points b in C such that $(z-a)^{-1}$can be uniformly approximated over K with the help of polynomials in $(z-b)^{-1}$. An elementary observation is that if $b \in U_a$, then $U_b \subseteq U_a$; (use lemma 2.1 if you want).

The tactics are to show that U_a is both open and closed, it then equals C, the set C being connected by hypothesis. It is open since we have the geometric series

$$\frac{1}{z-a} = \sum_{n\geq 0} \frac{(a-b)^n}{(z-b)^{n+1}},$$

to our disposal. The series converges—uniformly on compacts— whenever $|b-a| < |z-b|$, so as long as $|b-a| < d(b,K)$ it converges for all z in K. Hence the disk with center a and radius $d(a,K)/2$ lies within U_a, which combined with the elementary observation shows that U_a is open.

To see that U_a is closed, assume that b lies on the boundary of U_a in C, and let D be a disk with radius $d(b,K)/3$ centered at b. I meets U_a, so let $c \in D \cap U_a$. Observation above implies that $U_c \subseteq U_a$. Then if D' is a disk about c with radius $d(c,K)/2$, then $D' \subseteq U_c$ by what we did above. By the triangle inequality we get

$$d(c,K) \geq d(b,K) - |b-c| \geq 2d(b,K)/3,$$

hence

$$|b-c| \leq d(c,K)$$

and $b \in D' \subseteq U_a$, and we are through. ❏

PROBLEM 2.17. Let a and b be points with $a \in \mathbb{D}$ and $b \in \mathbb{C} \setminus \overline{\mathbb{D}}$. Assume that $g(z)$ is a polynomial in $(z-b)^{-1}$.

a) Show that

$$\int_{\partial\mathbb{D}} \big((z-a)^{-1} - g(z)\big)dz = 2\pi i$$

b) Show that

$$\sup_{z\in\partial\mathbb{D}} \big|(z-a)^{-1} - g(z)\big| \geq 1,$$

and conclude that $(z-a)^{-1}$ can not be approximated uniformly on $\partial\mathbb{D}$ by polynomials in $(z-b)^{-1}$.

✷

2.2.5 Finishing the proof of Runge's theorem for compact sets

Recall the theorem we want to prove:

Theorem 2.2 *Let K be a compact subset and let P be a set that meets every bounded component of $\mathbb{C} \setminus K$. Then every function holomorphic in K can be approximated uniformly with rational functions whose poles all lie in P.*

PROOF: The rational functions having poles only in P form obviously a subalgebra $\mathcal{A}_P$ of $\mathcal{H}(K)$. If a lies in a bounded component of the complement of K, the function

$(z-a)^{-1}$ belongs to the closure $\overline{\mathcal{A}_P}$ by pole-pushing, the assumption that P hits the component of $\mathbb{C}\setminus K$ where a lies, and the little lemma 2.1 on page 69 that tells us that the closure $\overline{\mathcal{A}}_P$ is an algebra. A pole lying in the unbounded component of $\Omega\setminus K$ can be push outside a disk centered at the origin and containing the compact K. The resulting rational function can be uniformly approximated on K by its Taylor polynomials, and hence $(z-a)^{-1}$ belongs to $\overline{\mathcal{A}_P}$.

Let f be a function that is holomorphic in a domain containing K. By choosing a chain σ like in lemma 2.3 and appealing to lemma 2.4, we conclude that f can be approximated uniformly on K by function being linear combinations of terms of the type $(z-a)^{-1}$ where a does not belong to K. But all these lie in the closure $\overline{\mathcal{A}}_P$; hence f lies there as well, and we are through. ❏

2.2.6 Runge's main theorem for compacts

HoloKonvex **Lemma 2.6** *Assume that C is a bounded component of $\Omega\setminus K$ that is relatively compact. Then for any holomorphic function f in Ω, one has $\sup_{z\in C}|f(z)|\leq\sup_{z\in K}|f(z)|$.*

Proof: This is just the maximum principle. Since C is relatively compact by lemma 2.2 on page 71 it holds true that $\partial C\subseteq\Omega$. The function f is therefore holomorphic in $\overline{C}$, hence $\sup_{z\in C}|f(z)|<\sup_{z\in\partial C}|f(z)|$ by the maximums principle. Clearly $\sup_{z\in\partial C}|f(z)|\leq\sup_{z\in K}|f(z)|$, and we are done. ❏

Recall that a subset X in a topological space Y is *relatively compact* if the closure of X in Y is compact. In our situation with $K\subseteq\Omega$ a pair of a compact set contained in a domain; the bounded components of $\Omega\setminus K$ can be relative compact or not. If C is a component, the boundary ∂C

RungeMain **Theorem 2.3** *The following three conditions are equivalent*

1. *None of the components of $\Omega\setminus K$ are relatively compact;* Main1
2. *Every holomorphic function in K can be approximated by rational functions holomorphic in Ω;* Main2
3. *For every point $c\in\Omega\setminus K$ there is an f holomorphic in Ω such that one has $|f(c)|>\sup_{z\in K}|f(z)|$.* Main3

We start by proving the equivalence between 1) and 2 in the theorem. To that end, recall that the second statement is equivalent to the following

- ☐ *Every bounded component of $\mathbb{C}\setminus K$ intersects $\mathbb{C}\setminus\Omega$.*

Proof: We start by proving the implication 1 $\Rightarrow$ 2:
So, assume $*$ to be false; *i.e.*, one of the components C of $\mathbb{C} \setminus K$ is contained in Ω. Then C is a component of $\Omega \setminus K$ as well, and as obviously ∂C lies in K, we conclude by lemma 2.2 on page 71 that C is a relatively compact component in $\Omega \setminus K$, contradicting the first statement.
We then proceed to prove 2 $\Rightarrow$ 1:
Let C be a component of $\Omega \setminus K$ that is relatively compact. Choose a point $a \in C$ and let δ be any number with $\delta > \sup_{a \in K} |z - a|$. The function $(z-a)^{-1}$ is holomorphic on K and can by assumption be approximated by a rational function $f(z)$ holomorphic on Ω to any accuracy, so there is such an f with

$$\left|(z-a)^{-1} - f(z)\right| < \delta^{-1},$$

for all $z \in K$. This gives

$$|1 - (z-a)f(z)| < 1 \tag{2.2}$$

for all $z \in K$. In particular (2.2) holds on ∂C as ∂C lies in K by lemma 2.2, the component C being relatively compact in Ω. By the maximum principle it follows that (2.2) holds for all $z \in C$ as well, but that is absurd, since putting $z = a$ we would get $1 < 1$. SalientInqe
Then comes the implication 3 $\Rightarrow$ 1:
To that end, assume that C is a relatively compact component of $\Omega \setminus K$. By lemma 2.6 it follows that $|f(c)| \leq \sup_K |f(z)|$ for all $c \in C$ and all f holomorphic in Ω, and this contradicts the first statement.
Finally, we prove that 1 $\Rightarrow$ 3
The set $K \cup \{c\}$ is compact and contained in Ω, and its complement in Ω has the same connected components as $\Omega \setminus K$ except that c has been deleted from one of them, but this does not make that component relatively compact. The auxiliary function given by

$$g(z) = \begin{cases} 0 & \text{if } z \in K \\ 1 & \text{if } z = c \end{cases}$$

is holomorphic on $K \cup \{c\}$ and can be approximated by functions holomorphic on Ω; hence there is an $f \in \mathcal{H}(\Omega)$ with

$$|f(z) - g(z)| < 1/2$$

from which it easily follows that $|f(c)| > \sup_K |f(z)|$.

❑

2.3 Runge for domains

Recall that an exhaustion of a domain by compacts is an ascending chain of compacts K_n such that the two following properties are satisfied

- $K_n \subseteq K_{n+1}^{\circ}$ for all n;
- $\bigcup_n K_n = \Omega$.

Every domain has many exhaustion like this, one can for example use the following

$$K_n = \{\, z \in \Omega \mid d(z, \partial\Omega) \geq n^{-1} \text{ and } |z| \leq n \,\}.$$

(2.1) We start with a few words about the space $\mathcal{H}(\Omega)$ of holomorphic functions on a domain Ω. It has topology, the topology of uniform convergence on compact sets, also called the compact-open topology. A sequence f_n in $\mathcal{H}(\Omega)$ converges to g if and only if it converges uniformly on compacts. It is a metric space with the metric given as

$$d(f, g) = \sum_n \frac{d_n(f, g)}{1 + d_n(f, g)} 2^{-n}.$$

A subset A of $\mathcal{H}(\Omega)$ is dense if and only if for every $f \in \mathcal{H}\Omega$, for every compact set $K \subseteq \Omega$ and every $\epsilon > 0$, there is a function $g \in A$ with $\|f - g\|_K < \epsilon$.

2.3.1 The Runge hull

The setting in this paragraph is the usual one with Ω a domain and $K \subseteq \Omega$ a compact set. The components $\Omega \setminus K$ are of two types. Some of them are relatively compact in Ω and some are not. We let $\mathcal{K}$ denote the set of the connected components in $\Omega \setminus K$ that are relatively compact in Ω, and we define the *Runge hull of K in Ω* to be

$$\hat{K}_\Omega = K \cup \bigcup_{C \in \mathcal{K}} C$$

that is, it is the union K and all the relatively compact components in the difference $\Omega \setminus K$.

When we try to approximate functions holomorphic on K by functions holomorphic in Ω the obstructions are precisely the components in $\mathcal{K}$. The idea is to replace K by $\hat{K}_\Omega$ and in that way plug in those holes that create problems, so that the obstructions vanish, but of course we need $\hat{K}_\Omega$ to be compact.

Proposition 2.1 *The Runge hull $\hat{K}_C$ is compact.*

Proof: The first observation is that $\hat{K}_\Omega$ is closed since the complement $\Omega \setminus \hat{K}_\Omega$ is the union of the components of $\Omega \setminus K$ not in $\mathcal{K}$; so our task reduces to establishing that the Runge hull $\hat{K}_\Omega$ is a bounded set.

Let U be an open and bounded set containing K whose closure is contained in Ω and. Such creatures exist; *e.g.,* let U be the set of the points z in Ω with $d(z,K) < d(K,\partial\Omega)/2$.

The boundary ∂U is covered by the connected components of $\Omega\setminus K$. These are all open and ∂U being compact is contained in finitely many of them. Among these say that $C_1,\ldots,C_r$ are the relatively compact ones. We shall see that all the other components in $\mathcal{K}$ must be contained in U: Indeed, let C be one. It has an empty intersection with the boundary ∂U since ∂U is covered by components different from C, and different components are disjoint. The component C being relatively compact it follows from lemma 2.2 on page 71 that $\partial C \subseteq K$, and hence $C\cap U \neq \emptyset$ as C is an open neigbourhood of the points in $\partial C \cap K$. Because $C \cap \partial U$ is empty, one has the representation

$$C = (C\cap U)\cup(C\cap\mathbb{C}\setminus\overline{U})$$

of C as the union of two disjoint open sets. Since C is connected, and $C\cap U$ is not empty, it follows that $C\cap(\mathbb{C}\setminus\overline{U})=\emptyset$ and hence $C\subseteq U$. Thus

$$\hat{K}_\Omega \subseteq U\cup C_1\cup\cdots\cup C_r,$$

and because all the sets in the union to the right are bounded sets, K will be bounded.

❏

Problem 2.18. Show that if $K\subseteq L$ are two compacts contained in Ω then $\widehat{K}_\Omega \subseteq \widehat{L}_\Omega$. Show that if $K\subseteq L^\circ$, then $\widehat{K}_\Omega\subseteq(\widehat{L}_\Omega)^\circ$. ✷

Problem 2.19. Show that the operation of taking the Runge hull is idempotent; that is, if you apply it seconds, you don't anything new; formally, one has

$$\widehat{K}_\Omega = \widehat{(\widehat{K}_\Omega)}_\Omega.$$

✷

Problem 2.20. What is the Runge hull of the unit circle $\partial\mathbb{D}$ in $\mathbb{C}$? ✷

2.3.2 Classical Runge

Theorem 2.4 *Let $\Omega_1\subseteq\Omega_2$ be two domains and let P be a closed subset of $\mathbb{C}$ set intersecting all bounded components of the difference $\Omega_2\setminus\Omega_1$. Then $\mathcal{H}(\Omega_2)$ is dense in $\mathcal{H}(\Omega_1)$.*

We are a little sloppy in formulation as $\mathcal{H}(\Omega_2)$ is not contained in the space $\mathcal{H}(\Omega_2)$, but it is canonically isomorphic (as a topological algebra) to one, the isometry being the restriction map.

The conclusion of the theorem can be formulated as follows. For any compact subset K of Ω_1 and any function holomorphic in Ω_1, and any ϵ there is a $g\in\mathcal{H}(\Omega_2\setminus P)$ with

$$\sup_K |f(z)-g(z)| < \epsilon.$$

Proof: We want to use Runge's theorem xxx for the pair $K \subseteq \Omega_2$, but we merely know that P meets the bounded components of $\Omega_2 \setminus \Omega_1$ and not those of $\Omega_2 \setminus K$. The trick as we shall see, will be to replace K by the Runge hull $\hat{K}_{\Omega_1}$.

Let D be a component of $\Omega_2 \setminus K$ relatively compact i Ω_2. If D intersects $\Omega_2 \setminus \Omega_1$, it contains a component of the latter which obviously will be relative compact in Ω_2. Hence P meets D. In case D does not meet $\Omega_2 \setminus \Omega_1$ it is a component of $\Omega_1 \setminus K$. Hence after having replacing K by $\hat{K}_{\Omega_1}$ there will be no such component. By xxx there therefore is a g with $\sup_{\hat{K}_{\Omega_1}} |f - g| < \epsilon$, and hence also $\sup_K |f - g| < \epsilon$ ❑

Theorem 2.5 *Assume that $\Omega_1 \subseteq \Omega_2$ are two domains. If there is no compact component in the difference $\Omega_2 \setminus \Omega_1$, then $\mathcal{H}(\Omega_2)$ is dense in $\mathcal{H}(\Omega_1)$.*

2.4 Some applications of Runge

The setting will be as follows. We are given a sequence of disks D_n all centered at the origin with strictly increasing radii;so that satisfy the inclusions $\overline{D}_n \subseteq D_{n+1}$ for all $n \in \mathbb{N}$. The disks will eventually form an exhaustion of the unit disk, but for the moment there are no more constraints. We denot by D the union of the disks D_n.

The second ingredient is a sequence of compacts sets Σ_n. They are subjected to two conditions. Firstly, their complements should all be connected, and secondly, they should "lie in-between" the circles ∂D_n and ∂D_{n+1}; in clear text the inclusions $\Sigma_n \subseteq D_{n+1} \setminus \overline{D}_n$ should be valid.

The third ingredient of set up is a sequence of functions σ_n each σ_n being holomorphic on the compact set Σ_n, which by convention means it defined and holomorphic in an open set containing Σ_n.

Using Runge1s approximation theoren, we shall show the following

Proposition 2.2 *Given $\eta > 0$. There exists a function ϕ holomorphic in D such one has for any n the following*

$$\|\phi - \sigma_n\|_{\Sigma_n} < 2^{-n}\eta.$$

Proof: The idea is to recursively construct a sequence $\{\phi_n\}$ of polynomials subjected to the two conditions below where s_n denotes the partial sum $s_n = \sum_{k \leq n} \phi_k$.

- $\|\phi_n\|_{\overline{D}_n} < 2^{-n}\eta$;
- for all $k \leq n$ oner has $\|s_n - \sigma_k\|_{\Sigma_k} < (2^{-(k-1)} - 2^{-n})\eta$.

So assume such a sequence of functions has been constructed for $k = n-1$. Since $\mathbb{C} \setminus \overline{D}_n \setminus \Sigma_n$ is connected and $\overline{D}_n$ and Σ_n are disjoint compact sets, we may define a function on their union by letting it be σ_n on Σ_n and zero on $\overline{D}_n$. By Runge' approximation theorem there is a poly ϕ_n approximating this function uniformly on $\Sigma_n \cup \overline{D}_n$ to any degree of accuracy. Hence we may have

$$\|\phi_n\|_{\overline{D}_n} < 2^{-n}\eta \quad \text{and} \quad \|\phi_n + s_{n-1} - \sigma_n\|_{\Sigma_n} < 2^{-n}\eta.$$

Let $k < n$. Since Σ_k is contained in D_n one obtains by induction the surch for inequality:

$$\|s_n - \sigma_k\|_{\Sigma_k} = \|\phi_n + s_{n-1} - \sigma_k\|_{\Sigma_k} \leq (2^{-n} + 2^{-(k-1)} - 2^{-(n-1)})\eta = (2^{-(k-1)} - 2^{-n})\eta$$

❑

(2.1) If f is a function that is holomorphic in the unit disk, there is no reason why it should be possible to extend it to a continuous function on the closure $\overline{\mathbb{D}}$. The boundary behavior can be very complicated. Let $w \in \partial\mathbb{D}$ be a boundary point, and for different sequences from $\mathbb{D}$ that converges to w the sequences $\{f(a_n)\}$ can behave very differently; some may converge an some may not, and if they converges they can have all kinds of different limits. One introduces the so called *cluster set* $C(f, w)$. The points of the cluster set are the points in the extended complex plane $\hat{\mathbb{C}}$ that occur as limits of sequences $\{f(a_n)\}$ when a_n run through the sequences in $\mathbb{D}$ converging to the point w. A slightly smaller set then the cluster set is the set of *radial limits* at w, which is obtained similarly but the sequences $\{a_n\}$ are confined to lie on the ray emanating from the origin and passing by w.

With the help of Runge's approximation theorem one may construct examples of functions with bad boundary behavior, and we intend do illustrate that with one example. For the function f of the example *all* the cluster sets are equal to $\hat{\mathbb{C}}$, and even the radial limit set at every point $w \in \partial\mathbb{D}$ will equal $\hat{\mathbb{C}}$!

Lemma 2.7 *The cluster set $C(f, w)$ is closed.*

Proof: This follows from the equality

$$C(f, x) = \bigcap_r \text{closure}\,\{\, f(z) \mid |z - w| \leq r \,\}.$$

❑

(2.2) Let $\Sigma_{r,I}$ be the circular arc obtained by deleting the set $\{\, re^{iy} \mid t \in I \,\}$ from the full circle about the origin and of radius r. That is one has

$$\Sigma_{r,I} = \{\, re^{it} \mid t \in [-\pi, \pi] \setminus I \,\}$$

We choose any sequence of positive numbers $\{r_n\}$ the only requirement is that they tend to one as $n \to \infty$. The choice of the intervals is slightly more juicy: Let α_n be a sequence of positive numbers in $[-\pi, \pi]$ monotonically decreasing to zero, and let I_n be a sequence of *disjoint*, open intervals centered at α_n.

With this in place, it easy to find disks D_n that together with the just introduced compacts $\Sigma_n = \Sigma_{r_n, I_n}$ satisfy the hypothesis in the begining of this section. The functions σ_n will all be constant, and we choose them in the following way. Let $Q \subseteq \mathbb{C}$ be any enumerable dense subset (for instance $\mathbb{Q} \times \mathbb{Q}$) listed in any way you want; say

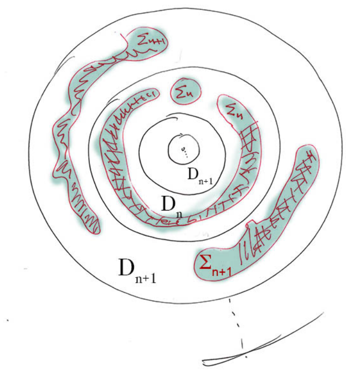


Figure 2.1:

$Q = \{c_n\}$. Then we let σ_n be contsnat equal to c_1 in every other of the n's, equal to c_2 in every other of the remaing n's etc.

The salient point is that any radius of $\mathbb{D}$ with at most one exception intersetcs each Σ_n in exactly one point. Indeed, if the angle the radius makes with the real axis is at most in one of the intevervals, say I_{n_0}, and all the other Σ_n's meet the radius.

Proposition 2.3 *There are functions f holomorphic in the unit disk such that cluster sets $C(f, w)$ all equals $\hat{\mathbb{C}}$.*

Proposition 2.4

2.5 The Mittag-Leffler theorem

2.5.1 Polar parts

We saw in xxx that a function f with an isolated singularity in a point a can be developed in Laurent series. Such a series is split into two parts, a so to say negative part where the summation indices run through negative values, and positive part where they are non-negative. Denote these two parts respectively by f_+ and f_-. The two parts have representation

$$f_- = \sum_{n\geq 1} a_{-n}(z-a)^{-n} \qquad f_+ = \sum_{n\geq 0} a_n(z-a)^n,$$

the sum for f_- converges for all $z \neq a$ hence defines a function holomorphic in $\mathbb{C} \setminus \{a\}$, additionally since the f_- is deprived of the constant term, it tends to zero as z tends to infinity, and of course $f = f_+ + f_-$ in a punctured disk about a. We call f_- *the principal part* of f at a.

The principal part is the only function g holomorphic in $\mathbb{C}\setminus\{a\}$ and tending to zero when z tends to ∞ and with the property that $f-g$ is holomorphic in neigbourhood of a. Indeed, if g_1 and g_2 both are functions like that, their difference would be entire and having limit zero when $z$goes to ∞. Thence the difference is bounded and by Liouville it is constant. Taking the limit when $z \to \infty$, one sees that the constant is zero.

It is convenient to call any function p that is holomorphic in $\mathbb{C}\setminus\{a\}$ with $\lim_{z\to\infty} f(z) = 0$ a *principal part* at a.

2.5.2 Recap on normally convergent series

Given a series $\sum_{k\geq 0} f_k(z)$ where the terms are functions in with only isolated singularities in a domain Ω and suppose that total set of singularities (all singularities for all f_f included) does not have an accumulation point in Ω. Thence a compact subset K of Ω will contain only finitely many of the singularities. In the compact K on may consider the series

$$\sum_{k\geq n} f_k(z) \tag{2.3}$$

where n is so large that the terms in (2.3) all are holomorphic in K, and one can require that it converges absolutely and uniformly in K. If this holds for all compacts K of Ω one says that the series converges *normally* in Ω. By standard technics (*e.g.*, integration term by term combined with Morera's theorem) it follows that the sum is holomorphic in Ω except for possibly isolated singularities where one of f_k's has one. smallsum

2.5.3 The Mittag-Leffler theorem

The question that was pose and answered by Mittag-Leffler was to which extent one prescribe principal parts. More precisely, given a set $A \subseteq \mathbb{C}$, and for each point $a \in A$ a principal part p_a. Can one find a function holomorphic in $\mathbb{C}\setminus A$ whose principal part at a equals the given p_a?

Of course the set A is forced to be locally finite, that is every of its points have a neigbourhood not meeting A in any other point from A. It can still accumulate and there is no hope having f defined and holomorphic in any of the accumulation points of A. The natural domain of definition for f is therefore $\mathbb{C}\setminus\overline{A}$ where $\overline{A}$ as usual is the closure of A.

The result named Mittag-Lefflers theorem is as follows. It was proven in its final form by Weierstrass in 1879, but most of the background was laid by Mittag-Leffler. There are several approaches to the proof. We follow the one by Runge which appeared in his seminal paper from 1885.

Theorem 2.6 *Given a locally finite set $A \subseteq C$ and for each $a \in A$ a principal part p_a. Then there exists a function f holomorphic in $\mathbb{C}\setminus\overline{A}$ whose principal part at a equals p_a for $a \in A$.*

PROOF:

The natural region to work with is $\Omega = \mathbb{C} \setminus A'$, and one may clearly assume that Ω is connected. To begin with, we chose an exhaustion of Ω by compact sets K_n. This is an ascending chain of compacts $\{K_n\}$ in Ω subjected to the three requirements

- $K_n \subseteq K_{n+1}^\circ$;
- $\bigcup_n K_n = \Omega$;
- $\widehat{(K_n)}_\Omega = (K_n)_\Omega$.

For simplicity of the presentation we assume[2] that the difference $K_{n+1} \setminus K_n$ contains exactly one element from A, which we naturally denote by a_n and to simplify the notation the corresponding principal part will be p_n.

The principal part p_n is holomorphic in K_n and since $\widehat{(K_n)}_\Omega = K_n$ we can apply Runge and find a function holomorphic in Ω satisfying

$$\|p_n - f_n\|_{K_n} < 2^{-n}.$$

We claim that the series

$$\sum_n (p_n - f_n)$$

converges normally in Ω and has the prescribed principal parts.

Any compact K is contained in some K_r so it suffices to show normal convergence on K_r. We take a look at the truncated series

$$\sum_{k \geq r} (p_k - f_k) \tag{2.4}$$

trunkSeries

all whose terms are holomorphic in K_r. As $K_r \subseteq K_n$ for $n \geq r$ one has

$$\|p_k - f_k\|_{K_r} \leq \|p_k - f_k\|_{K_k} < 2^{-k}$$

This shows that the series $\sum_{k \geq r} |p_k - f_k|$ is dominated by $\sum_{k \geq r} 2^{-k}$ and the series in (2.4) converges absolutely and uniformly in K_r, that is, it converges normally in Ω to a holomorphic function in Ω.

Since the functions f_k all are holomorphic in Ω, this shows that f is holomorphic outside of A and for $a \in A$ has the principal part p_a at a. ❑

Lemma 2.8 *We may assume that $K_{n+1} \setminus K_n$ has exactly one point from A.*

PROOF: Coming up ❑

[2] By slight modifications of the compacts one can realism this situation.

Problem 2.21. Determine the principal part of $(e^z - z - z^2/2)z^{-6}$ at the origin. ✷

Problem 2.22. Show that fro each natural number k the series

$$A_k = \sum_{m=0}^{\infty} \frac{1}{m!(m+k)!}$$

converges. Show that the principal part of $e^z e^{1/z}$ at the origin is given as

$$\sum_{k\geq 1} A_k z^{-k}.$$

✷

Problem 2.23. Let $\eta = \exp 2\pi i\alpha$ where α is an irrational number. Show that the set $\{\, \eta^n \mid n \in \mathbb{N} \,\}$ of positive powers of η is dense in the unit circle $\mathbb{S}^1$. Hint: Use Dirichlet's theorem on rational approximation that says that for infinitely many natural numbers p and q on has $|q\alpha - p| < q^{-1}$. ✷ PowersDens

Problem 2.24. Let r_n be a sequence of positive real numbers tending to 1 as $n \to \infty$. Let $\eta = e^{2\pi i\alpha}$ where α is not a rational number, and let $A = \{\, r_n\eta^n \mid n \in \mathbb{N} \,\}$.

a) Show that A accumulates at every point on the unit circle, that is $A' = \partial\mathbb{D}$.

b) Show that $\mathbb{C} \setminus \overline{A}$ is not connected. Hint: By exercise 2.23 the powers η^n are dense on the circle.

c) Show that there is function meromorphic in $\mathbb{D}$ having principal part $(z - a_n)^{-1}$ at $a_n = r_n\eta^n$ and is holomorphic everywhere else in the unit disk.

✷

Problem 2.25. Let g be meromorphic function in the unit $\mathbb{D}$ disk all whose poles are of order one and all whose residues are integers.

a) Show that if γ is closed path in $\mathbb{D}$ not passing by any of the poles of g, then the integral $\int_\gamma g(z)dz$ is an integral multiple of $2\pi i$.

For a point z in $\mathbb{D}$ let σ_z be any path joining 0 to z that does not pass by any of the poles of g.

b) Show that

$$G(z) = \exp \int_{\sigma_z} g(z)dz$$

is a well defined (that is, it is independent of the choice of the path σ_z) and meromorphic function in $\mathbb{D}$ whose logarithmic derivative equals g.

c) Show that if all the residues of g are positive, then G is holomorphic.

✷

PROBLEM 2.26. Let r_n be a sequence of positive real numbers tending to 1 as n growths ant $\eta = \exp 2\pi i\alpha$ where α is a rational number. Let $a_n = r_n\eta^n$. Show that there is a function h holomorphic in the unit disk that has a simple zero at a_n for $n \in \mathbb{N}$ and no other zeros. ✷

PROBLEM 2.27. Let $u(z)$ be the series

$$u(z) = \sum_{k\in\mathbb{Z}} (z+k)^{-2}$$

and let et R be the square $R = [-1/2, 1/2] \times [-y, y]$ where y is any positive real number. Further let $v(z)$ denote the function $(z) = \pi^2 \sin^{-2} \pi z$.

a) Show that series $u(z)$ converges normally in $\mathbb{C}$, and that it is periodic with period one.

b) Show that one has $|z+k| \geq |k|/2$ for all k, and conclude that one has

$$\left|u(z) - z^{-2}\right| \leq 2\pi^2/3$$

if $z \in R$ is nonzero.

c) Show that $|\sin \pi z| \leq 16\pi$ on the boundary ∂R and use the maximum principle to conclude that $v(z) - z^2$ is bounded on R.

d) Show the identity

$$\pi^2 \sin^{-2} \pi z = \sum_{k=-\infty}^{\infty} (z+k)^{-2}.$$

✷

2.6 Inhomogeneous Cauchy Riemann

Given a domain Ω in the complex plane and a function ϕ of class C^∞ in Ω. The differential equation

$$\bar{\partial} h = \phi \tag{2.5}$$

dBarEqua has many important applications, and it goes under the name of either the *"$\bar{\partial}$-equation"* or the *"inhomogeneous Cauchy Riemann equation"*. The explanation of the last name being that homogeneous equation associated to (2.5), *i.e.,* the case when $\phi = 0$, is just the Cauchy Riemann equation. Split into its real and imaginary parts the Cauchy Riemann equations are two couples first order differential equations. When the functions h and ϕ are split in their real and imaginary parts; that is $h = u + iv$ and $\phi = \xi + i\eta$, they become

$$\partial_x u - \partial_y v = \xi$$
$$\partial_x v + \partial_y u = \eta.$$

The solutions of the $\overline{\partial}$-equation (2.5) are not unique. The Cauchy Riemann equations tell us that $\overline{\partial} f = 0$ when and only when f is a holomorphic function, hence the solutions of the $\overline{\partial}$-equation are only unique up to the addition of holomorphic functions.

(2.1) The only aim of the present section is to show that the $\overline{\partial}$-equation always has a solution; that is, we shall demonstrate the following.

Theorem 2.7 *Given a domain Ω in the complex plane and a function ϕ of class C^∞ in Ω. Then there exists a function h of class C^∞ in Ω such that* dBar

$$\overline{\partial} h = \phi.$$

There is a slightly sharper version of this theorem. One may relax the regularity condition on ϕ and only assume it be C^k for some $k \geq 2$; but then, of course, one merely gets the weaker conclusion that h is C^k. The proof is basically the same as for C^∞-functions, and we prefer to stick with these.

The proof is a two-step process. The initial step is two solve the ∂-equation with the additional hypothesis that ϕ be a function with compact support, and the final step consists of choosing a compact exhaustion of the domain Ω and Runge's theorem to patch together solutions from each compact of the exhaustion.

2.6.1 The case of compact support

In this paragraph the function ϕ will be of compact support, that is vanishes identically outside of a compact subset K, and it will of class C^∞ that is, twice continuously differentiable. The solution of (2.5) will be given by an explicit integral formula. The proof relies on an argument using Greens' integral formula and the fact that under mild regularity condition on the functions involved, one can differentiate an integral depending on a parameter by differentiating the integrand.

Green's theorem in the complex setting takes the form

$$\int_A \overline{\partial} f(w) d\overline{w} \wedge dw = \int_{\partial A} f(w) dw.$$

Lest A is a domain of a simple kind, one must be careful with how to interpret the boundary ∂A; in our present situation, however, A will just be an annulus. We also remind you that the differential operator $\overline{\partial}$ comply to Leibnitz' rule for the derivative of a product. It follows that $\overline{\partial}(fg) = f\partial g$ whenever f is holomorphic, since in that case $\overline{\partial} f = 0$.

(2.1) Our first observation is of an elementary nature. The function $(w-z)^{-1}$ is integrable over any compact set K, as one sees by switching to polar coordinates; that is, putting $w = r + re^{it}$. The chain rule gives $dw = e^{it}\, dr + rie^{it}\, dt$, and upon conjugating one finds $d\overline{w} = e^{-it}\, dr - rie^{-it}\, dt$ so that $d\overline{w} \wedge dw = 2ir\, dr \wedge dt$. Thence

$$\left| \int_K (w-z)^{-1} d\overline{w} \wedge dw \right| \leq \int_K |w-z|^{-1} |d\overline{w} \wedge dw| = 2 \int_K drdt = 2\mu(K),$$

where as usual $\mu(K)$ denotes the area of K. With this in mind we can formulate the first step in the two-step process, the function h in the statement is well defined by what we just did:

dBarKompSupp

Proposition 2.5 *Let ϕ be a function of class C^∞ with compact support. The function $h(z)$ defined by*

$$h(z) = \frac{i}{2\pi}\int_{\mathbb{C}} \phi(w)(w-z)^{-1} d\overline{w} \wedge dw$$

is of class C^∞ and satisfies the equation $\overline{\partial} f = \phi$.

(2.2) The main ingredient in the proof of the proposition 2.5 just formulated, is a representation of ϕ as an integral, resembling the Cauchy formula for holomorphic functions. We prefer to give as a lemma:

dBarKompFundLemma

Lemma 2.9 *Assume that ϕ is a C^∞-function with compact support. Then*

$$\int_{\mathbb{C}} \overline{\partial}\phi(w)(w-z)^{-1} d\overline{w} \wedge dw = -2\pi i \phi(z)$$

Proof: Fix a complex number z.

Let $r < R$ be two positive real numbers and let $A = A(r,R)$ be the annulus centered at z. The radius R is chosen so big that the compact K where ϕ is supported is contained in the disk $D_R = \{\, w \mid |w-z| < R \,\}$. The smaller radius r will be very small, and eventually we shall let it tend to zero. We put $D_r = \{\, w \mid |w-z| \leq r \,\}$. Under these circumstances the following holds true:

$$\int_{\mathbb{C}} \partial\phi(w)(w-z)^{-1} d\overline{w} \wedge dw =$$
$$= \int_A \overline{\partial}\phi(w)(w-z)^{-1}\phi(w) d\overline{w} \wedge dw + \int_{D_r} \overline{\partial}\phi(w)(w-z)^{-1} d\overline{w} \wedge dw \quad (2.6)$$

the latter integral is bounded above by $4\pi r^2 M$ where $M = \sup_K \left|\overline{\partial}\phi\right|$, and hence tends to zero when $r \to 0$.

The two circles bounding A will be denoted by C_R and C_r. They are parametrized in the standard manner so that the boundary chain of A equals $\partial A = C_R - C_r$. Applying Green's theorem and observing that $\overline{\partial}(\phi(w)(w-z)^{-1}) = \overline{\partial}\phi(w)(w-z)^{-1}$, we obtain the equality

$$\int_A \overline{\partial}\phi(w)(w-z)^{-1} d\overline{w} \wedge dw = -\int_{C_r} \phi(w)(w-z)^{-1} dw = -i\int_0^{2\pi} \phi(z + re^{it}) dt,$$

since the function ϕ vanishes along C_R. The integral to the right tends to $\phi(z)$ as r tends to zero (verifying this is standard and we leave it to the zealous student), and in view of equation (2.6) above, the announced identity in the lemma is established. ❑

This ends the proof of the proposition 2.5 since the integrand is C^∞, we can appeal to xxx in the appendix and switch integration and derivation. That gives indeed

$$\overline{\partial}h(z) = \frac{i}{2\pi}\overline{\partial}\Big(\int_{\mathbb{C}} \phi(w)(w-z)^{-}d\overline{w}\wedge dw\Big) = \frac{i}{2\pi}\int_{\mathbb{C}} \overline{\partial}\phi(w)(w-z)^{-1}d\overline{w}\wedge dw = \phi(z).$$

2.6.2 Proof in the general case

Now let Ω by any domain and let ϕ be a C^∞-function in Ω. The thing to do is to chose an exhaustion of Ω by compacts K_n such that

- ☐ $K_n \subseteq K_{n+1}^\circ$
- ☐ $\bigcup_n K_n^\circ = \Omega$
- ☐ $\widehat{K_{n}}_{\Omega} = K_n$

Secondly, we need a sequence of auxiliary compacts L_n with $K_n \subseteq L_n \subseteq L_{n+1}$, and for each index n we choose a C^∞-function α_n of compact support with $\alpha_n|_{L_n} = 1$—that is, α_n is identically equal to one on L_n—and we let $\phi_n = \alpha_n\phi$. The important things are that ϕ_n has compact support and coincides with ϕ on K_n. By the previous paragraph we can solve the equation $\overline{\partial}h_n = \phi_n$

On K_n one has $\overline{\partial}(h_{n+1} - h_n) = \phi_{n+1} - \phi_n = 0$ since both coincide with ϕ on K_n. Hence the difference $h_{n+1} - h_n$ is holomorphic on K_n and by Runge there is a function g_n holomorphic on Ω satisfying the inequality

$$|h_{n+1}(z) - h_n(z) - g_n(z)| < 2^{-n}$$

for z in the compact K_n. The series $\sum_{m\geq n}(h_{n+1} - h_n - g_n)$ is dominated by the convergent series $\sum 2^{-n}$ on K_n and converges uniformly in K_n to a holomorphic function there.

We looking for a function h on Ω solving the $\overline{\partial}$-equation. The clue is to write down a formula for h on each of compacts K_n, however, for this to be legitimate one must of course verify that the two definitions coincide on $K_n \cap K_m$, and the K_n forming an ascending chain it suffices to do this when $m = n-1$.

On K_n we put

$$h = h_n + \sum_{m\geq n}(h_{m+1} - h_m - g_m) - \sum_{i<n} g_i. \tag{2.7}$$

DefSolut

On K_{n-1} one has

$$h = h_{n-1} + \sum_{m\geq n-1}(h_{m+1} - h_m - g_m) - \sum_{i<n-2} g_i =$$
$$= h_{n-1} + (h_n - h_{n-1} - g_{n-1}) + \sum_{m\geq n}(h_{m+1} - h_m - g_m) - \sum_{i<n} g_i =$$
$$= h_n + \sum_{m\geq n}(h_{m+1} - h_m - g_m) - \sum_{i<n} g_i.$$

Since h_n is C^∞ clearly h is C^∞ in K_n° for all n, hence in Ω, and over K_n° one has $\overline{\partial}h = \overline{\partial}h_n = \phi_n = \phi$ as both series in (2.7) above are holomorphic in K_n°. This finishes the proof of theorem 2.7.

2.6.3 Convolution

We end this section by a paragraph that more has the flavour of an appendix than main stream part of the course. We treat in our restricted context, the problem of smoothing

$$h(z) = \int_{\mathbb{C}} \alpha(w+z)\beta(x)d\overline{w}\wedge dw$$

where we suppose that β is integrable over the entire complex. For example if β has compact support this is certainly the case. For the sake of α we suppose from the outset that it is twice differentiable. T

(2.1) The letter D stands for a first order differential operator with constant coefficients. Any complex linear combination of the partial derivative operators ∂_x and ∂_y will do, of particular interest is the $\overline{\partial}$-operator.

Lemma 2.10 *If α is C^2, then $h(z)$ is C^1 and*

$$Dh(x) = \int_{\mathbb{C}} D\alpha(w+z)\beta(w)d\overline{w}\wedge dw$$

that is, we can compute the derivative of the integral by differentiating the integrand.

PROOF: Assume that D is the derivative in the direction of the vector ξ in the plane (*e.g.*, either ∂_x or ∂_y). That α is D-differentiable at $w+z$ means that is an expression

$$f(w+z+t\xi) = f(x+z) + D_\xi(w+z)t\xi + \epsilon(w+z,t)t$$

were the salient point is that $|\epsilon(w+z,t)|$ tends to zero when $t\to 0$. This limit is in fact uniform in both w and t as long as w is restricted to a compact set.

Integrating gives

$$\int_{\mathbb{C}} f(w+z+t\xi)\beta(w)d\overline{w}\wedge dw =$$
$$= \int_{\mathbb{C}} f(x+z)\beta(w)d\overline{w}\wedge dw + t\int_{\mathbb{C}} D_\xi(w+z)\beta(w)d\overline{w}\wedge dw + t\int_{\mathbb{C}} \epsilon(w+z,t)\beta(w)d\overline{w}\wedge dw,$$

and we are trough once we know that the absolute value of the last integral to right tends to zero with t, but as $|\epsilon(w+z,t)|$ tends uniformly to zero, this is certainly the case. ❑

A successive application of this lemma shows that in case α is of class C^∞, the integral will be as well, and we may compute any derivative of any order by differentiating the integrand.

(2.2) The second point we want to make is the existence of a smooth function α with compact support being constant equal to one on a given compact K. It is easy to find a continuous function with these properties. Just use

$$\psi_K(z) = \max\{1 - d(z, K), 0\}.$$

The tactics is then to smoothen this function by convolution, *i.e.*, integrating it against a smooth bell shaped function κ of unit mass supported on a (small) disk D_r about the origin (the convolution is smooth by the previous lemma). That is, we define α by the integral

$$\alpha(z) = \int_{\mathbb{C}} \kappa(w - z)\psi_K(w) d\overline{w} \wedge dw,$$

well, almost! It is convenient to replace the compact K by the slightly lager compact set L of the points whose distance to K is less than or equal to a small chosen threshold r, and let

$$\alpha(z) = \int_{\mathbb{C}} \kappa(w - z)\psi_L(w) d\overline{w} \wedge dw.$$

If the disk D_r is centered at K, it is entirely contained in L. Thence ψ_L is constant and equal to unity in D_r, and we get

$$\int_{\mathbb{C}} \kappa(w - z)\psi_L(w) d\overline{w} \wedge dw = \int_{D_r} \kappa(w) d\overline{w} \wedge dw = 1.$$

On the other hand, the disk D_r is entirely contained in the complement of L when z belongs to the compact $\{ w \mid d(L, w) \leq 1 + r \}$, and as then ψ_L vanishes identically in D_r, the convolution α vanishes in z.

Problem 2.28. Let κ_a be a positive smooth function of one real variable such that $\kappa(a) = 0$ for $t \geq a$. Let $K(t) = \kappa(t)\kappa(-t))$.

a) Show that K is smooth and vanishes outside the interval $< -a, a >$. Show that $K(t)e^{i\theta}$ is a smooth function supported in the disk $\{ w \mid |w| < a \}$.

b) Show that function

$$\kappa(t) = \begin{cases} \exp(-(t-a)^{-2}) & \text{for } t \leq a, \\ 0 & \text{for } t > a, \end{cases}$$

is smooth.

✷

2.7 Weierstrass products

The first infinite product to appear in the history of mathematics seems to be the Viéte's formula for π:

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\frac{\sqrt{2+\sqrt{2}}}{2}\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots.$$

Francois Viéte published this in 1593. He was a french lawyer (as an other famous french mathematician) and we can thank him for having introduces the use of letters like x and y in the algebraic notation. The formula is a special case of the identity

$$\frac{\sin x}{x} = \cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}\ldots \tag{2.8}$$

EulerId

due to Euler. Nowadays infinite products have large number of applications, and there are many peculiar identities. In the the end of this section we'll give a few of the most known and may be most frequently applied products. Their influence in the realm of holomorphic functions, comes from they making it possible to construct holomorphic functions with prescribed zeros. They were first systematically exploited by Weierstrass, in an article from 1876, where he introduced the so called "convergence producing" factors to force convergence of products. This works has had an enormous influence and, to site Reinholdt Remmert, revolutionized the thinking of function theorist. ([Rem], page 79)

2.7.1 Infinite products

There seems to be no standard of introducing the infinite products. The different text deviate substantially. We follow to a large extent Rudin in his book [?]—it is a short way to the Weierstrass products.

(2.1) Consider a sequence $a_1, a_2, \ldots$ of complex numbers. When can one give a reasonable meaning to the infinite product $\prod_i a_i = a_1 a_2 \ldots$? The obvious try is, in analogy to what one does with infinite series, to consider the *partial products* $p_n = \prod_{i\leq n} a_i = a_1 \ldots a_n$ and the require that the sequence

$\{p_n\}$ they form converges. However this approach has some flaws; the most serious one being that the limit can vanish without any of the factors vanishing. So if we insist on keeping the good old rule that a a product whose factors are non-zero do not vanish, we must proceed slightly differently.

The clue is to disregard any finite number of factors when taking the limit; the formal way to do this is by the *modified partial products* belonging to the sequence $\{a_i\}$. They are defined as $p_{m,n} = \prod_{m\leq i\leq n} a_i$. With those in place, we are ready to define what convergence of a product should mean and what the limit should be:

Defenition 2.1 *The product $\prod_i a_i$ of a sequence $\{a_i\}$ of complex numbers is said to converges if for some m the modified partial product $\{p_{m,n}\}$ converges to a value different from zero. If P is this value, we let product $\prod_i a_i$ be equal to $a_1, \ldots, a_{m-1}P$.*

The first obvious comment is that only finitely many of the a_i's can be zero when the product converges. The second is that the number m occurring only plays an auxiliary role; as long as it is big enough any natural number will do and will give the same value of the product—and big enough means a_m being beyond the last vanishing a_i; *i.e.*, m is such that $a_i \neq 0$ for $i \geq m$. With this in mind, one is convinced that the good old principle survive: The product $\prod_i a_i$ vanishes if and only one of the factors a_i equals zero.

Example 2.1. The product $\prod_{2\leq i}(1 - 1/i)$ diverges to zero. To use a term from the theory of series as taught in calculus course, it is "telescoping":

$$p_n = \prod_{2\leq i\leq n} (i-1)/i = 1/n.$$

On the other hans $\prod_{2\leq i}(1-1/i^2)$ converges to $1/2$. By an easy induction one finds the partial products to be given by

$$p_n = \frac{n+1}{2n}.$$

❄

Problem 2.29. Prove the identity 2.8 by establishing that

$$\sin x = 2^n \sin 2^{-n} x \prod_{1\leq k\leq n} \cos 2^{-k} x$$

and use that $\sin x/x \to 1$ when $x \to 1$. ✷

(2.2) A necessary condition for convergence of the product is that a_n tend to one as $n \to \infty$. Indeed, or $m >> 0$ one clearly has $a_n = p_{m,n}/p_{m,n-1}$, and $p_{m,n}$ and $p_{m,n-1}$ tend to the same non-zero limit. Writing $a_i = 1 + u_i$, the necessary condition translates into the condition $\lim_{i\to\infty} u_i = 0$. It turns out that convergence of the product $\prod_i(1+u_i)$ is narrowly related to the convergence of the series $\sum_i u_i$. To exploit that relation, the following technical lemma will be crucial.

Lemma 2.11 *Let $p_n = \prod_{i\leq n}(1+u_i)$ and let $\widetilde{p}_m = \prod_{i\leq n}(1+|u_i|)$. The one has* dette TregngsOgs

1. $|p_n| \leq \exp(\sum_{i\leq n} |u_n|)$; DenneTrengs
2. $|p_n - 1| \leq \widetilde{p}_n - 1$.

Proof: Recall the inequality $1 + x \leq e^x$ which you learned in your first calculus course. Immediately it gives

$$\prod_{i\leq n}(1+u_i) \leq \prod_{i\leq n}(1+|u_i|) \leq \prod_{i\leq n} \exp|u_i| = \exp\sum_{i\leq n} |u_i|\,.$$

The second estimate is slightly more subtle, and we resort to induction to prove it. For $n = 1$ the inequality reduces to $|u_1| \leq |u_1|$, so assume $|p_k - 1| \leq \widetilde{p}_k - 1$. One finds

$$\begin{aligned} p_{k+1} - 1 = (p_k - 1)(1 + u_k) + u_k &\leq |p_k - 1|\,(1 + |u_k|) + |u_k| \\ &\leq (\widetilde{p}_k - 1)(1 + |u_k|) + |u_k| = \widetilde{p_{k+1}} - 1. \end{aligned}$$

❑

SumProd

Proposition 2.6 *When the series $\sum_i |u_i|$ converges, then product $\prod_i (1+u_i)$ converges.*

PROOF: We study the modified partial product with $M > N > m$

$$|p_{m,N} - p_{m,M}| = |p_{m,N}|\,|p_{N+1,M} - 1| \leq |p_{m,N}|\,(\widetilde{p}_{N+1,M} - 1) \leq |p_{m,N}|\,(e^{\sum_{N+1 \leq i \leq M} u_i} - 1). \tag{2.9}$$

NyttigUlik

By the first inequality in lemma 2.11 it follows that $p_{m,N}$ is bounded; that is $|p_{m,N}| < e^A$ where $A = \sum_i |u_i|$. Given ϵ there is—since the exponential function is continuous—an η such that one has $(e^t - 1) < e^{-A}\epsilon$ once $0 \leq t \leq \eta$, and we may of course require that $\eta < 1/2$. By assumption the series $\sum_i |u_i|$ converges. This means that for N sufficiently big one has $\sum_{N \leq i \leq M} |u_i| < \eta$, and it follows that the modified partial products form a Cauchy sequence. It remains to bee seen that its limit is non-zero. Since $|u_i|$ tends to zero when $i \to \infty$, only finitely many of the u_i's can equal -1; hence for m large none of the partial products $p_{m,N}$ vanish. From (2.9) above we find

$$|p_{m,M}| \geq |p_{m,N}|\,(2 - e^{\eta}) \geq (1 - 2\eta)\,|p_{m,N}| > 0$$

as $\eta < 1/2$ and $e^x \leq 1 + 2x$ for $x < 1$. ❑

(2.3) In this course we mostly concerned with holomorphic functions and their infinite products of functions are of interest. The proof of proposition 2.6 in the previous paragraph works *mutatis mutandis* in a setting where $\{u_i\}$ is a sequence of functions in a domain Ω, and it gives the following result:

KonvergensProp

Proposition 2.7 *Given a sequence of functions holomorphic in a domain Ω.Assume that the series $\sum_i |u_i(z)|$ converges uniformly on compacts in Ω. Then the product $\prod_i (1 + u_i(z))$ converges uniformly on Ω to a holomorphic function $p(z)$ in Ω. The product p vanishes at a point if and only if one of the factors vanish there.*

EXAMPLE 2.2. Let $(1 + x^{2^p}) = (1 + x)(1 + x^2)(1 + x^4)\ldots$ converges to $(1 - x)^{-1}$. Since $\sum_i x^{2^i}$ converges absolutely, the product converges. For the partial products one has

$$(1 + x)(1 + x^2)\ldots 1 + x^{2^n} = 1 + x + \cdots + x^{2^{n+1}-1}$$

as is easily seen by induction:

$$(1 + x^{2^{n+1}}) \sum_{1 \leq i \leq 2^{n+1}-1} x^i = \sum_{1 \leq i \leq 2^{n+1}-1} x^i + \sum_{1 \leq i \leq 2^{n+1}-1} x^{i+2^{n+1}} = \sum_{1 \leq i \leq 2^{n+2}-1} x^i.$$

❄

2.8 Weierstrass products

We come to important question of finding holomorphic functions with prescribed zeros. The must be interpreted in the wide sense including multiplicities of the zeros. Zero sets are isolated in the domain of definition, so we start with gibing a set A that is discrete in a domain Ω and to each point $a \in A$ we give a natural number $m(a)$. The big question is: Can one find a function holomorphic in Ω having a zero at a of multiplicity $m(a)$ for each a in A and no other zeros?

The question can be phrased in a slightly different way by use of the order function. Given a function m in Ω whose values are non-negative integers, and whose support is locally finite (*i.e.*, every point in Ω has a neighbourhood where m vanishes except in a finite set). Can one find a function f holomorphic in Ω with $\mathrm{ord}_a f = m(a)$ for all $a \in \Omega$?

As Weierstrass showed and we shall see in this section, the answer is yes.

2.8.1 The Weierstrass factors

The convergence producing factors of Weierstrass are built on functions of type

$$E_n(z) = (1-z)\exp(z + z^2/2 + \cdots + z^n/n). \tag{2.10}$$

WeierstrassEn

One recognizes the sum in the exponential as the initial part of the Taylor series for $\log(1-z)^{-1}$; indeed, it holds true that

$$\log(1-z)^{-1} = \sum_{1 \geq i} \frac{z^i}{i} = z + z^2/2 + \cdots + z^n/n + \sum_{n<i} z^i/i.$$

The exponential in 2.10 gets closer and closer to $(1-z)$ as n grows, and hence $E_n(z)$ is close to one when n is large.

Lemma 2.12 $E'(z) = -z^n \exp(z + z^2/2 + \cdots + z^n/n)$.

Proof: This could be an exercise in any first year calculus course; and the hint would be to introduce $t_n(z) = z + z^2/2 + \cdots + t^n/n$ and use that $t_n'(z)(1-t) = 1 - z^n$. ❑

Lemma!WSums

Lemma 2.13 *For $|z| \leq 1$ one has the estimate*

$$|E_n(z) - 1| \leq |z|^{n+1}. \tag{2.11}$$

ViktigUlikhet

PROOF: The trick is to use the formula

$$E_n(z) - 1 = z \int_0^1 E_n'(zt)dt.$$

Using that $|e^w| \leq e^{|w|}$ and that $|t_n(w)| \leq t_n(|w|)$ one finds

$$\left| \int_0^1 E_n'(zt)dt \right| \leq \int_o^1 |E_n'(zt)|\, dt \leq |z^n| \int_0^1 t^n \,|e(t_n(zt))|\, dt \leq |z^n| \int_0^1 t^n e(t_n(t))dt = |z^n|\,,$$

as the integrand in the last integral equals $-E_n'(t)$ by lemma 2.13 above and thence the integral equals one. ❑

(2.1) Now, let $\{a_k\}$ is a sequence of complex numbers tending to infinity. We search for a holomorphic function vanishing precisely at the a_k-s. The naive try would be the product $\prod(z - a_k)$ which obviously is a very bad try, as the general factor does not approach one. A better try would be $\prod(1 - z/a_k)$. However, neither this works in general. Just take $a_k = -k$, thence the series $\sum_k z/a_k$ diverges, and by 2.7 the product diverges as well.

The ingenuous idea of Weierstrass was to remedy this by introducing convergence promoting factors, replacing the simple minded factors $1 - z/a_k$ by smart factors $E_{n_k}(z/a_k)$. The liberty to chose n_k depending on the behavior of the sequence $\{a_k\}$ is a clue to this ; choosing n_k large enough will make $E_{n_k}(z/a_k)$ tend to 1 sufficiently fast to make the product converge.

WProdMain1

Theorem 2.8 *Assume that $\{a_k\}$ is a sequence of non-zero complex numbers tending to ∞ as k tends to ∞. Assume that n_k is a sequence of natural numbers such that the series*

$$\sum_k (r/\,|a_k|)^{n_k}$$

converges for all r. The the Weierstrass product

$$\prod_k E_{n_k}(z/a_k)$$

converges normally in $\mathbb{C}$. The product defines an entire function whose zeros are precisely the elements in the sequence $\{a_k\}$, and the multiplicity of a zero a of the product equals the number of times a appears in the sequence $\{a_k\}$; i.e., one has $\mathrm{ord}_a f = \#\{\, k \mid a_k = a \,\}$.

The series in the theorem resembles a power series in r, and when it is, the condition is that the radius of convergence be infinity. If n_k all are equal, say to m, the condition simply imposes that the series $\sum_k |a_k|^{-m}$ converge.

PROOF: We are to show that the product converges uniformly on compacts, and as usual it suffices to consider disks about the origin. So let $r > 0$ be given, and let z be a point with $|z| < r$. By hypothesis $a_k \to \infty$ as $k \to \infty$, so one has $|a_k| > r$ for $k >> 0$, and as $\sum_k (r/|a_k|)^{n_k}$ converges, it follows that $\sum_k (|z|/|a_k|)^{n_k}$ converges. By lemma 2.11 the series $\sum_k |1 - E_{n_k}(z/a_k)|$ converges and consequently, appealing to proposition 2.7 on page 98, we are through. ❏

(2.2) If the sequence $\{a_n\}$ does not tend to ∞, it must be finite; indeed, it is discrete so no infinite subsequence can be bounded. In this case an appropriate polynomial will have the prescribed zeros with the correct multiplicities.

For any sequence $\{a_k\}$ tending to infinity, there are plenty of sequences n_k that fulfill the condition of the theorem. For instance, one may take $n_k = k$. To see this, observe that $a_k > 2|z|$ for $k >> 0$. Hence $|z/a_k|^k < 2^{-k}$, and the the series $\sum_k |z/a_k|^k$ converges, being dominated by the convergent series $\sum_k 2^{-k}$.

Combined with theorem 2.8 these considerations give the following result.

PresCZeroEntire

Theorem 2.9 (Prescribed zeros) *Let $m(a)$ be a function in $\mathbb{C}$ taking non negative integral values. Assume that m has locally finite support. There exists an entire functions whose orders satisfy $\mathrm{ord}_a f = m(a)$ for all a. Such a function is unique up to a factor of the form $e^{g(z)}$ where $g(z)$ is an entire function.*

PROOF: Most of this is done; it remains to allow for a zero at the origin. So if $m = 0$, the search for function will be

$$f(z) = z^m \prod_k E_k(z/a_k).$$

For the unicity statement, suppose that f_1 and f_2 have the same zeros with the same multiplicities. Then the fraction f_1/f_2 vanishes nowhere in $\mathbb{C}$, and consequently has a logarithm in $\mathbb{C}$, so we just put $g(z) = \log f_1/f_2$. ❏

(2.3)

Proposition 2.8 *Any function f in the plane meromorphic is the fraction $f = g/h$ of two entire functions.*

PROOF: Let h be the Weierstrass product formed by the poles of f. Then $g = fh$ is entire, since the zeros of h kill the poles of f. ❏

Proposition 2.9 *Given an entire function f. A necessary and sufficient condition for f to have an n-th root is that $n|\mathrm{ord}_a f$ for all a.*

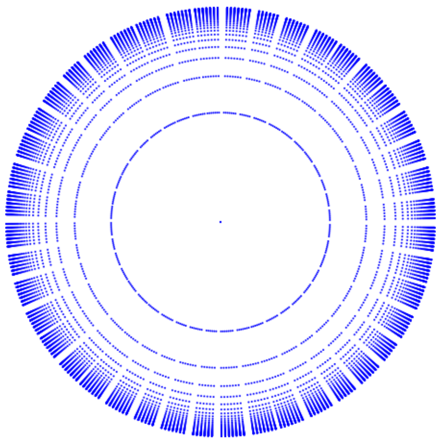
PROOF: Apply the theorem about prescribed zeros (theorem 2.9 on page 101) to the integral valued function $m(a) = \mathrm{ord}_a f/n$ to find a Weierstrass product w with $n\mathrm{ord}_a w = \mathrm{ord}_a f$. The function f/w^n is then entire and with out zeros and consequently has as an n-root, say g. Thence $w^n g^n = f$. The other implication is obvious: One has $\mathrm{ord}_a f^n = n\,\mathrm{ord}_a f$. ❏

2.8.2 Weierstrass products in domains

The next result we go form is the generalization to any domain Ω. So let $m(a)$ be any function whose values are non-negative integers with locally finite support. The support A has no limit points in Ω but certainly can very well have such in $\mathbb{C}$.

EtGodEksempel

Example 2.3. A good example is the following. Let A be the set of points $q^{-1}e^{2p\pi i/q}$ where p and q are two relatively prime natural numbers. Then A is contained in the open unit disk $\mathbb{D}$ and is locally finite: Given $0 < r < 10$ there are only finitely many natural numbers q with $q^{-1} < r$, and for each such q there are only finitely many residue classes mod q. Hence in the disk $D_r = \{ z \mid |z| < r \}$ there are only finitely many points from A. However, every point on unit circle $\partial\mathbb{D}$ is an accumulation point for A; indeed if $\eta = e^{2\alpha\pi i}$ with α an irrational number, there will be infinitely may natural numbers p and q such that $|\alpha - pq^{-1}| < q^{-1}$. ❄



Figur 2.4: *10 000 of the points in A.*

Theorem 2.10 *Given an open subset Ω of $\mathbb{C}$, and a function m in Ω taking non-negative integral values. Assume that m has a locally finite support in Ω. Then there is a Weierstrass product f holomorphic in Ω with $\operatorname{ord}_a f = m(a)$ for all $a \in \Omega$.*

Proof: To begin with we assume that the complement of Ω is compact. We may also assume that the sequence $\{a_n\}$ is bounded: Let R be so large that $\mathbb{C} \setminus \Omega$ is contained in $|z| < R$, and let $\{a_{n_k}\}$ be the subsequence of $\{a_n\}$ with $|a_{n_k}| > R$. Then $\{a_{n_k}\}$ tends to infinity, (if not, there would be an accumulation point Ω), and we can take care of the zeros located at that subsequence by theorem 2.9.

Let $\{a_k\}$ be a listing of the points with $m(a_k) > 0$ each one repeated $m(a_k)$ times. Then $d(a_k, \mathbb{C} \setminus \Omega) \to 0$ as $k \to \infty$ (if this were not the case, the set $\{a_k\}$ would have a limit point in Ω) and we may find a sequence $\{b_k\}$ of points not in Ω with $|a_k - b_k| \to 0$ as $k \to \infty$.

Let K be a compact in Ω, and let $d = d(K, \mathbb{C}\setminus\Omega)$. For k sufficiently large $|a_k - b_k| < 2^{-1}d$, hence for such k's we have $|a_k - b_k| < 2^{-1}|z - a_k|$ for all $z \in K$. This gives

$$\left|(a_k - b_k)(z - b_k)^{-1}\right|^k < 2^{-k}$$

and the series $\sum_k |(a_k - b_k)(z - b_k)|^k$ converges. By 2.8 and **??** the Weierstrass product

$$\prod_k E_k((a_k - b_k)(z - b_k)^{-1})$$

converges normally in K. The factors in the product have the form

$$E_k((a_k - b_k)(z - b_k)^{-1}) = \frac{z - a_k}{z - b_k}\exp(t_k((a_k - b_k)(z - b_k)^{-1}))$$

so they vanish simply in a_k, and we are through since each a_k is repeated the correct number of times.

Finally, if the complement of Ω is not bounded, pick a point $a \in \Omega$ and let $\Omega' = \{\,(z-a)^{-1} \mid z \in \Omega, z \neq a\,\}$. Then $\mathbb{C}\setminus\Omega'$ is bounded, indeed let $|z - a| < \epsilon$ be disk in Ω, then the points w with $|w| > \epsilon^{-1}$ all lie in Ω'. Do the construction for Ω' with discrete function $m(w^{-1} + a)$ to find a function $f(w)$ solving the problem. Then $f((z - a)^{-1})$ will solve the problem for m and Ω. ❑

2.8.3 Domains of holomorphy

As an application of the Weierstrass products we discuss the concept of "domain of holomorphy". When f is a holomorphic function in a domain Ω it is a frequently surfacing question whether f can be extended to a holomorphic function in a lager domain. For example a Weierstrass product W associated to the set A in example 2.3, can not be extended across the boundary $\partial\mathbb{D}$ to bigger open set. The point is that the zeros of W accumulate at every point of the boundary: So if there were a domain Ω containing D and a holomorphic function W_1 extending W, at least one boundary point would be contained in Ω, and the zeros of W_1 would accumulate in that point. This is impossible because zeros of holomorphic functions are isolated.

One says that a domain Ω is a *domain of holomorphy* if there is a holomorphic function in Ω that does not extend to any larger domain. One may show that all domains in Ω are domains of holomorphy. One technic to show this is similar to what we did for the unit disk with the set A, and we shall illustrate this by showing the relatively compact case

Proposition 2.10 *Every domain Ω in $\mathbb{C}$ is a domain of holomorphy*

Proof(for the compact case): Assume that $\partial\Omega$ is compact. For each natural number k we cover $\partial\Omega$ by finitely many disks $D_{i,k}$. They shall be centered at a point in $\partial\Omega$, and their diameter shall be $1/k$. Each one of these disks meets Ω, and we may pick a point a_{ik} in $D_{ik} \cap \Omega$.

The first salient point is that the sequence $\{a_{ij}\}$ is locally finite in Ω. Indeed; for any natural number n the set $U_n = \{\, z \in \Omega \mid d(z, \partial\Omega) > 1/n \,\}$ is open and can contain only those a_{ik} with $k < n$, and those are finite in number. The second salient point is that the a_{ik}-s accumulate at every point in the boundary $\partial\Omega$: For every k at least one of the disks D_{ik} contains a given point $a \in \partial\Omega$ and D_{ik} contains the point a_{ik} as well.

The same argument as we gave for the case of the set A in the unit disk, works generally. A Weierstrass product constructed with basis in the set $\{a_{ij}\}$ does not extend to any open set larger than Ω. ❑

2.8.4 Blaschke products

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3. Del

Normal families

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3.1 Convergence uoc

(3.1) Frequently one encounters sequences of functions that are not defined in the same domain but nevertheless one wants to have the limit-functions at one's disposal. At first sight this might seem paradoxical, but it is meaningful under certain conditions. To be more precise, let the functions be $\{f_\nu\}$ with f_n defined in the domain Ω_ν, and let $\Omega = \bigcup_\nu \Omega_\nu$. Of course, for our search for a limit to have a meaning, there must be tight relations between the domains Ω_ν—in the extreme case they being disjoint, for instance, there is not much hope. The crucial assumption is that any point $z \in \Omega$ has a neigbourhood U_z which from a certain index on lies in all the Ω_n-s; that is, there is an N_z with $U_z \subseteq \Omega_\nu$ for $\nu \geq N_z$.

When this condition is fulfilled, any result about convergence which is local in nature, is applicable to the sequence $\{f_\nu\}_{\nu \geq N_z}$ in the neighbourhood U_z. For instance, saying that the sequence $\{f_\nu\}$ converges UOC is meaningful: Any compact $K \subseteq \Omega$ is covered by finitely may of the open sets U_z above. Hence $K \subseteq \Omega_\nu$ for $\nu >> 0$, and the good old definition of uniform convergence applies. When the all the functions f_ν are holomorphic from the start, for these ν-s all functions f_ν are holomorphic on K, and the limit function will be holomorphic by Weierstrass convergence theorem.

One common situation when the crucial condition is fulfilled is when the domains Ω_n form an ascending chain, that is, $\Omega_\nu \subseteq \Omega_{\nu+1}$ for all ν.

Another situation that that one continually meet is that the functions are not holomorphic everywhere in the domain Ω, but have isolated singularities. The condition then becomes that every point z has neighbourhood where all but finitely, many of the functions are holomorphic.

(3.2) The impact of Weierstrass' work during the last quarter of the 19-th century on the theory function was immense, and among the many result of his we recall one that is fundamental in function theory (and in fact, we have already used it several times). We call it the *Weierstrass convergence theorem*, and its statement is the very natural that UOC-limits of holomorphic functions are holomorphic, and more over, the sequence of derivatives converges UOC to the derivative of the limit function. The grounds behind such a theorem are rather clear: As order of the integration and the limit process is immaterial when the convergence is UOC, any property expressible in terms integrals will be inherited by the limit.

Theorem 3.1 *If the sequence $\{f_\nu\}$ of functions holomorphic in Ω converges* UOC *towards f, then f is holomorphic throughout Ω. The sequence f'_ν of derivatives converges to f'* UOC *in Ω.*

Proof: The first part follows from Morera's theorem. Let γ be a closed path in Ω. The convergence being uniform on γ limits and integrals can be swapped, thence it holds true that

$$0 = \lim_\nu \int_\gamma f_\nu(\zeta)d\zeta = \int_\gamma (\lim_\nu f_\nu(\zeta))d\zeta = \int_\gamma f(\zeta)d\zeta.$$

The statement about the derivatives is derived by the help of Cauchy's formula for the derivative. When γ is the boundary, of a small disk in Ω encompassing z, traversed once counterclockwise, his formula reads

$$f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^2}d\zeta = \frac{1}{2\pi i}\int_\gamma \lim_\nu \frac{f_\nu(\zeta)}{(\zeta - z)^2}d\zeta =$$
$$= \frac{1}{2\pi i}\lim_\nu \int_\gamma \frac{f_\nu(\zeta)}{(\zeta - z)^2}d\zeta = \lim_\nu f'_\nu(z).$$

❑

(3.3) By repeatedly applying this theorem one sees that the sequence $\{f_\nu^{(k)}\}$ of derivatives of any order k converges UOC to $f^{(k)}$. With anti-derivatives the situation is slightly more complicated as there are constants of integration involved, which seen from the point of view of the derived series are quit arbitrary. For instance, the series $f_n(z) = n$ does not converge, but of course, the derived series does. However if the constants are taken care of, there are nice results, like:

Problem 3.1. Given a sequence $\{f_\nu\}$ of holomorphic functions that converges UOC to f. For each ν let F_ν be a primitive for f_ν. Show by an example that the sequence $\{F_\nu\}$ dos not necessarily converge. Assume further that for one point $a \in \Omega$ the sequence $\{F'_\nu(a)\}$ converges to $f(a)$. Show that the sequence $\{F_\nu(z)\}$ then converges UOC to a primitive of f. ✱

3.1.1 Hurwitz about limit values

Pursuing the philosophy that properties expressible by integrals pass to uniform limits, one can prove a series of strong and useful results. We are about to give a few, all due to Adolf Hurwitz, and circling about relations between values of the limit and values of the functions in the sequence.

(3.1) The first Hurwitz' theorem that we shall cite is about zeros of the limit, and it is easily extended to a statement about values: Any value taken by the limit is eventually taken by the functions in the sequence, but to begin with, we treat only the zeros:

Theorem 3.2 *Let the sequence $\{f_\nu\}$ of holomorphic functions converge* UOC *to f in the domain Ω. If all the functions f_ν are without zeros in Ω, then f is without zeros in Ω as well unless it vanishes identically.*

Proof: Assume that f is not identically zero. The logarithmic derivative $d\log f = f'/f$ is then meromorphic in Ω, and we may count the number $n(f,D)$ zeros of f in any (small) disk $D \subseteq \Omega$ by the formula

$$n(f,D) = \frac{1}{2\pi i}\int_{\partial D} d\log f.$$

The sequence of derivatives $\{f'_\nu\}$ converges toward f', and hence $d\log f_\nu$ converges toward $d\log f$. Integrating along ∂D we find, swapping the limit and the integral, the equalities

$$0 = \lim_\nu \int_{\partial D} d\log f_\nu = \int_{\partial D} d\lim_\nu d\log f_\nu = \int_{\partial D} d\log f.$$

❏

Applying theorem 3.2 above to the sequence $f_\nu - f(a)$ one obtains on the fly the version of Hurwitz's theorem about values alluded to the top of the paragraph:

Theorem 3.3 *For any $a \in \Omega$ and any neigbourhood U of a there are points a_ν in U such that $f_\nu(a_\nu) = f(a)$ for $\nu >> 0$.*

Proof: As already indicated, use the previous theorem with $\Omega = U$ and with $\{f_\nu - f(a)\}$ as sequence of functions. ❏

(3.2) Hurwitz has also a result about injectivity of the limit; the UOC-limit of injective functions is injective or constant:

Theorem 3.4 *Let $\{f_n\}$ be a sequence of holomorphic functions in the domain Ω converging* UOC *toward f. Assume that the functions f_ν are injective for $\nu >> 0$, then f is either constant or injective.*

Proof: The clue is to apply 3.2 to the domain $\Omega \setminus \{a\}$. Since f_ν are supposed to be injective in Ω, the functions $f_\nu - f_\nu(a)$ are without zeros in $\Omega \setminus \{a\}$, and they clearly converge uoc toward $f - f(a)$. Thence, by 3.2, $f - f(a)$ is either without zeros in $\Omega \setminus \{a\}$, in which case f is injective, or vanishes identically, in which case f is constant. ❑

Problem 3.2. Assume that $\{f_\nu\}$ converges uoc toward f in Ω. Show that for any disk $D \subseteq \Omega$ there is a natural number N_D such that

$$\sum_{z \in D} \operatorname{ord}_z f = \sum_{z \in D} \operatorname{ord}_z f_n$$

for $n > N_D$. ✷

Problem 3.3. Let $\{f_\nu\}$ be a sequence of holomorphic functions in the domain Ω converging uoc to f and let $a \in \Omega$ be a point.

a) By studying the function $f_\nu - f(c)$ show that in any disk D contained in Ω and containing a, there is a natural number N_d and points a_ν such that $f_\nu(a_\nu) = f(a)$ for $\nu > N_D$

b) Show that there is sequence of points a_ν in Ω converging to a with $f(a_\nu) = f(a)$.

✷

Problem 3.4. Show by exhibiting examples of sequences of real analytic functions, that neither of the three theorems of Hurwitz' above is valid in a real setting, that is, for real functions of a real variable. ✷

3.2 Arsela-Ascoli

The classical Bolzano-Weierstrass-theorem tells us that every bounded sequence of numbers—real or complex— possesses a convergent subsequence. It is natural to wonder wether a similar result applies to sequences of functions as well, and this question was answered by the two Italian mathematicians Cesare Arzelà and Giulio Ascoli. Among the two, Ascoli was the first contribute. He established the sufficient condition, and about ten years later Arzelà tidied up the formulation and proved the necessity.

Their famous result gives a necessary and sufficient condition for a sequence of functions to have convergent subsequences, and of course, by convergence we then understand uniform convergence on compacts. We intend to describe this result— which strictly speaking belongs to real analysis— without giving the proof, and we shall do that in a rather general setting. The Arzelà and Ascoli is precursor of Montel's criterion for convergence which is a corner-stone in the theory of holomorphic functions.

(3.1) Let X and Y de two metric whose metrics are d_X and d_Y respectively. We assume that X locally compact and that Y be complete, innoxious assumptions for us as in the applications we have in mind X will be a domain in the complex plane equipped with the usual euclidean metric and Y will be an open subsets of the Riemann-sphere $\hat{\mathbb{C}}$ with spherical metric.

The concepts of uniform convergence on compacts is meaningful in this general setting, the definitions are word for word the same as in the case of complex functions, but with the metrics d_X and d_Y replacing the good old distance function $|z - w|$. The set of $\mathcal{C}(X, Y)$ of continuous functions from X to Y has a topology called the *topology of uniform convergence* with the property that a sequence of functions converges in that topology if and only if it converges UOC. A subset, or as we shall say, a family $\mathcal{F} \subseteq \mathcal{C}(X, Y)$ is called *normal* if every sequence of elements from $\mathcal{F}$ has a convergent subsequence. This means that the closure of $\mathcal{F}$ is compact, and In more topological terms one says that $\mathcal{F}$ is *precompact* or *relatively compact*. One do not request that limit function lie in $\mathcal{F}$, but of course it will be continuous.

EXAMPLE 3.1. As a first example we let the domain Ω be the unit disk $\mathbb{D}$ and let the family $\mathcal{F}$ consist of the fractional linear transformations

$$\phi_\nu(z) = \frac{z - c_\nu}{\bar{c}_\nu z - 1}$$

where c_ν is a sequence in the unit disk converging to a point $c \in \overline{\mathbb{D}}$. If c belongs to the boundary $\partial\mathbb{D}$, observing that $\bar{c} = c^{-1}$ one realizes swiftly that $\{\phi_\nu\}$ converges UOC to the constant function with value c.

However, if the point c does not lie on the boundary but in the (open) unit disk itself, the limit is the function $\phi(z) = (z - c)(\bar{c}z - 1)^{-1}$. ❄

(3.2) The Arzelà-Ascoli theorem involves the concept of *equicontinuous families*. If $\mathcal{F}$ is a family of functions from X to Y — in other words a subset of $\mathcal{C}(X, Y)$— it is said to be equicontinuous in a set $A \subset X$ if there for any $\epsilon > 0$ is possible to find a $\delta > 0$ such that the implication

$$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon$$

holds true for all pair of points x, y in A and all functions f from the family $\mathcal{F}$. Loosely expressed, in a slogan-like manner: “The same δ works everywhere in X and for all members f of the family”. As we shall see, the sufficient condition in the Arzelà-Ascoli-theorem will be that $\mathcal{F}$ be equicontinuous on every compact subset of X.

(3.3) An observation that occasionally is useful is at being equicontinuous on compact sets is a local property of a family: Once one can establish that $\mathcal{F}$ is equicontinuous in all members of an open covering of X, one knows it to be equicontinuous on compact sets.

Lemma 3.1 *Let $K \subseteq X$ be a compact set and $\mathcal{F}$ a family of continuous functions from X to Y. If there exists an open covering $\mathcal{U}$ of K such that $\mathcal{F}$ is equicontinuous on every member $U \in \mathcal{U}$, then $\mathcal{F}$ is equicontinuous in K.*

Proof: The set K being compact, we may assume that $\mathcal{U}$ is finite. The covering has a so called Lebesgue-number, that is a $\rho > 0$ with the property that any open ball in X of diameter less than ρ intersects K in set contained in one of opens from the covering.

If now $\epsilon > 0$ is given. For each $U \in \mathcal{U}$ there is a δ_U with $d_Y(f(x), f(y)) < \epsilon$ once $d_X(x, y) < \delta$ for any f from the family. It is then clear that any positive number δ less than ρ and less than all the δ_U-s (which are finite in number) works! ❏

(3.4) Finally we have come to the Arzelà-Ascoli-theorem, and as we said, we content ourselves with the formulation of the theorem and do not give the proof:

Theorem 3.5 *Let X and Y be two metric spaces and assume that Y is complete. Assume that $\mathcal{F}$ is a family of continuous maps from X to Y. Then $\mathcal{F}$ is a normal family if and only if the following to conditions are satisfied.*

- *$\mathcal{F}$ is equicontinuous on compacts;*
- *For every point $x \in X$ the set of values $\{\, f(x) \mid f \in \mathcal{F} \,\}$ is contained in a compact set.*

3.3 Montel's criterion for normal families

The french mathematician Paul Montel was the one who introduced the name "normal families" in a paper in 1912. His thesis from 1907 was about families of holomorphic functions, and subsequently he devoted a large part of his scientific life to the study of such families. One of his famous result, if not the most famous, is that family of holomorphic functions whose values avoid two numbers form a normal family, and we shall come back to that in due course. This section is about a necessary and sufficient condition for a family to be normal, proven by Montel in his thesis.

(3.1) Before starting on the Montel criterion, we recall a few nuances about bounded families. A family $\mathcal{F}$ of functions in a domain Ω is *uniformly bounded*, or for short just *bounded*, in a set A if its members have a common upper bound in A; that is, for a suitable constant M_A it holds true that $|f(z)| \leq M_A$ for all $z \in A$ and all $f \in \mathcal{F}$. Expressed in sup-norm-lingo this reads $\|f\|_A < M_A$ for all members f of the family.

The family is said to be *locally bounded* in Ω if one around every point may find a neighbourhood over which $\mathcal{F}$ is uniformly bounded. This implies that $\mathcal{F}$ is bounded in all compacts; indeed, a compact K is covered by finitely many such neighbourhoods, and the largest of the corresponding upper bounds will be a common upper bound for all functions in $\mathcal{F}$.

(3.2) With these nuances in place, we are ready for Montel's criterion for normality, one of the more important results in function theory, and which will be used over and over again.

Theorem 3.6 *If a family $\mathcal{F}$ of holomorphic functions in a domain Ω is locally bounded, it is normal.*

Proof: As announced this relies on the Arzelà-Ascoli-theorem. There are two conditions to be checked. The first one comes for free, the family is obviously pointwise bounded being locally bounded. Our concern is therefore to show that a locally bounded family is equicontinuous on compacts. Equicontinuity on compacts being a local property, we can restrict ourself to disks. So let D be disk in Ω whose radius we denote by r, and let M be a common upper bound over D for the functions in $\mathcal{F}$.

Cauchy's integral formula gives

$$f(z) - f(z') = \frac{1}{2\pi i}\int_{\partial D} f(w)((w-z)^{-1} - (w-z')^{-1})dw = \tag{3.1}$$

$$= \frac{z-z'}{2\pi i}\int_{\partial D} f(w)(w-z)^{-1}(w-z')^{-1}dw \tag{3.2}$$

for two points $z, z' \in D$. When $w \in \partial D$ and z is confined to the disk D' that is concentric with D and with radius $r/2$, the inequality $|w-z| > r/2$ is valid. Combining this with 3.1 we obtain the estimate

$$|f(a) - f(b)| < |a-b|\, 4M/r^2.$$

The bound $4M/r^2$ does not depend on f and $\mathcal{F}$ is equicontinuous in D', and it is clear that the disks D' corresponding to the disks D from an appropriate covering of Ω, cover Ω. ❑

Example 3.2. The family consisting of all holomorphic functions in Ω having a common upper bound is normal, while the family whose members are the bounded holomorphic functions is not normal. ❄

(3.3) A sequence of numbers converges if and only its many subsequences all converge to the same value. It is even fairly easy to see that it suffices that all convergent subsequences converges to the same point, for in that case the set of accumulation points is reduced to a singleton, which means that sequence converges. The corresponding statement is true for sequences of holomorphic functions as well, a result that goes under the name of *Montel's criterion for convergence*:

Theorem 3.7 *A locally bounded sequence of holomorphic functions all of whose* uoc-*convergent subsequences converge to the same function, is* uoc-*convergent.*

Proof: Let the sequence be $\{f_n\}$ and assume it does not converges uoc to f. Then there is a compact K, an $\epsilon > 0$ and a subsequence $\{f_{n_k}\}$ with $\|f - f_{n_k}\|_K > \epsilon$.

After Montel's normality criterion (theorem 3.6 on page 113) the sequence $\{f_{n_k}\}$ has a subsequence that converges uoc, and we can as well assume that the sequence $\{f_{n_k}\}$ it self converges uoc. Thence $f_{n_k} \to f$ uniformly on compacts *per hypothesis*. Of course this implies that $\lim_k \|f - f_{n_k}\|_K = 0$, in flagrant contradiction with the inequality $\|f - f_{n_k}\|_K > \epsilon$, valid for all k. ❏

(3.4) Montel's criterion has a consequence that many might find astounding. Sequences of holomorphic functions has a clear tendency to converge, at least convergence frequently can be *contagious* as Reinholdt Remmert writes in [Rem], it can spread from a subset to the entire domain of definition, and he cites George Pólya and Gàbor Zsegö who gave a pertinent characterization of the phenomenon: "The propagation of convergence can be compared to the spread of an infection". One striking example is *Vitali's convergence theorem*, convergence on set with an accumulation point implies convergence everywhere:

Theorem 3.8 *Let $\{f_n\}$ be a locally bounded sequence of holomorphic functions in the domain Ω. Suppose that the set A of those z i Ω where the limit $\lim_n f_n(z)$ exists,has an accumulation point. Then the sequence $\{f_n\}$ converges* uoc *i Ω.*

Proof: The limits of two convergent subsequences must coincide on the set A, since the whole sequence converges there. Both are holomorphic in Ω by Weierstrass' convergence theorem (theorem 3.1 on page 108), and coinciding on the set A, which possesses an accumulation point, they are equal by the identity theorem. Hence the sequence converges uoc after Montel's criterion for convergence. ❏

Problem 3.5. Let $\mathcal{F}$ be a pointwise bounded family of complex valued continuous functions in the domain Ω. The aim of this exercise is to demonstrate that there exists a subdomain $U \subseteq \Omega$ such that $\mathcal{F}$ is locally bounded over U.

a) Under the assumption that $\mathcal{F}$ is not locally bounded, show that one might find a sequence of functions $\{g_\nu\}$ from $\mathcal{F}$ and a descending chain of compact disks K_ν with $\|g_\nu\|_{K_\nu} > \nu$.

b) If $\mathcal{F}$ is not locally bounded, exhibit a point $a \in \Omega$ such that $|g_\nu(a)| > \nu$.

c) Finally, show that there exists a subdomain $U \subseteq \Omega$ such that $\mathcal{F}$ is locally bounded over U.

✷

Problem 3.6. The aim of this exercise is to show the following result due to American mathematician William Fogg Osgood. If $\{f_\nu\}$ is a sequence of holomorphic functions converging to a *continuous* function in the domain Ω, there is an open, dense subset $U \subseteq \Omega$ over which the sequence converges uoc.

a) Let $D \subseteq \Omega$ be any disk. Use problem 3.5 above to find an open subset U_D of D where the sequence $\{f_\nu\}$ is locally bounded.

b) Show that $U = \bigcup_D U_D$ is an open, dense subset of Ω where $\{f_\nu\}$ converges UOC.

✷

PROBLEM 3.7. In this exercise the task is to exhibit an example of a sequence of entire functions that do converge to a continuous function in $\mathbb{C}$ but do not converge UOC everywhere in $\mathbb{C}$.

Recall the Mittag-Leffler function $F(z)$ constructed in problem xxx. It is an entire function whose limit at ∞ along any ray emanating from the origin equals zero, that is $\lim_{r\to\infty} F(re^{it}) = 0$ for every t. Let $f_n(z) = F(nz)/n$

a) Show that $\lim_{n\to\infty} f_n(z) = 0$ for all z.

In the rest of the exercise, we assume that sequence $\{f_n\}$ is bounded near 0 (with the intention to arrive at an absurdity)

b) Show (with the assumption above) that there are constants $r > 0$ and $M > 0$ with $|F(z)| \leq nM$ for $|z| \leq nr$ and all n.

c) Denote by a_k the k-th Taylor coefficient of F about the origin. Use Cauchy's estimates to prove that the inequality

$$|a_k| \leq nM/(nr)^k,$$

holds for all $k \geq 0$ and $n \geq 1$.

d) Show that $a_k = 0$ for $k \geq 2$ and arrive at a contradiction.

✷

3.4 Spherical convergence

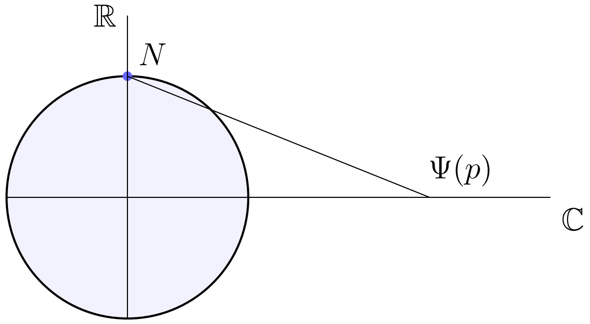
So far we have mostly spoken about families of *holomorphic functions*,but it is quite natural and of great interest to extend the theory to comprise families of *meromorphic functions* as well. Habitually we think about meromorphic functions as functions mapping Ω to the Riemann sphere $\hat{\mathbb{C}}$, and the Riemann sphere comes equipped with the *spherical metric* —the distance function inherited from the standard metric on unit sphere in $\mathbb{R}^3$ through the stereographic projection—and this metric turns out to be a convenient tool for studying families of meromorphic function.

3.4.1 The spherical metric

The metric on $\mathbb{S}^2$ is the one used by navigators for centuries; the distance between two points being the angular measure of the smaller great circle arc connecting the two points. Transporting this metric to the Riemann sphere requires some computations, not very intricate though, but sufficiently draggy (and in our context uninteresting) to make us skip them. Instead we just give an expression for the metric and argue directly that it has the properties we want, *i.e.*, that it gives the correct topology on the extended plane and is equivalent to the euclidean metric in the finite part.

(3.1) When describing the stereographic projection we prefer to identify $\mathbb{R}^3$ with $\mathbb{C} \times t$ and in the latter the points are (z, t). The unit sphere is given by the equation $|z|^2 + t^2 = 1$. The north pole N is the point $(0, 1)$.

The stereographic projection Ψ sends a point on the unit sphere, other than the north pole, to the point where the complex plane (that is the plane $t = 0$) meets the line joining the point to the north pole. The north pole is sent to the point at infinity in $\hat{\mathbb{C}}$. One easily verifies that this defines a homeomorphism between $\hat{\mathbb{C}}$ and $\mathbb{S}^2$.



Figur 3.1: *A sectional view of the stereographic projection*

Problem 3.8. If Φ denotes the inverse of the stereographic projection, show that

$$\Phi(z) = (2z(1 + z\overline{z})^{-1}, (z\overline{z} - 1)(z\overline{z} + 1)^{-1}). \tag{3.3}$$

Show that the great circles on the sphere projects to the (generalized) circles in $\hat{\mathbb{C}}$ given by

$$z\overline{z} + a\overline{z} + \overline{a}z = 1,$$

and show that for such a circle the centre is $-a$ and the radius equals $(1 + |a|^2)^{1/2}$. Show that the pairs of points z and $-\overline{z}^{-1}$ in $\hat{\mathbb{C}}$ (with a liberal interpretation if $z = \infty$) correspond to the pairs of antipodal[1] points on the sphere. ✷

(3.2) The *spherical length* of a path γ in $\hat{\mathbb{C}}$ is given by the integral

$$\Lambda(\gamma) = \int_\gamma \frac{2\,|dz|}{1 + |z|^2}, \tag{3.4}$$

which is easily seen to converge in case one (or both) of the end points lies at infinity.

[1]This involve all of the three natural involutions on $\mathbb{C}$, and would make any semi-serious cosmologist with conspiratorial tendencies smile widely

Indeed, the change of variable $z = w^{-1}$ gives $dz = -w^{-2}dw$, and this substitution does not alter the form of the integral, *i.e.,* it becomes

$$\Lambda(\gamma) = \int_{\gamma'} \frac{2\,|dw|}{1 + |w|^2}. \tag{3.5}$$

The *spherical distance* between two points z and z' is determined as the infimum of the spherical lengths of paths joining the two points:

$$\rho(z, z') = \inf_{\gamma} \Lambda(\gamma),$$

where as we said, the infimum is taken over connecting paths.

(3.3) The spherical metric is equivalent to the euclidean metric in the finite part of $\hat{\mathbb{C}}$ (but of course, they are not equal). The argument goes like this: Let z and z' be two points both lying in a disk D of radius R about the origin, and let γ be a path in D connecting the two, then the inequalities beneath holds

$$\frac{l(\gamma)}{1 + R^2} < \int_{\gamma} \frac{2\,|dz|}{1 + z\bar{z}} < l(\gamma),$$

where $l(\gamma)$ denotes the good old euclidean length of the path γ. Taking infimum over paths and keeping in mind that the shortest path between the two points in the euclidean sense is the line segment joining them, we obtain the inequalities

$$\frac{|z - z'|}{1 + R^2} < \rho(z, z') < |z - z'|\,.$$

By this we have established that the two metrics are equivalent in the finite part of the extended plane; but as the form of the integral giving the spherical metric is insensitive to the change of variables $w = z^{-1}$, the argument is even valid in neighbourhoods about the point at infinity. Hence the two metrics are topologically equivalent.

Problem 3.9. Verify that $\rho(z, z')$ satisfies the axioms for a metric. That is, it is symmetric, the identity of indecernibles holds, and the triangle inequality is fulfilled. ✷

(3.4) Great circles through the north pole—also called meridians our hour circles—project onto lines through the origin, and their spherical length is particular easy to compute. As an illustration we offer the following computation

$$\int_a^{\infty} \frac{2dz}{1 + z\bar{z}} = \int_{|a|}^{\infty} \frac{2du}{1 + u^2} = \pi - 2\arctan|a| = 2\arctan 1/\,|a|$$

giving the spherical length from a point a to the point at infinity. We recommend the students to verify that this coincides with the *polar distance* of the corresponding point on the sphere, *i.e.,* the angular measure along the meridian from the north pole to the point. This also explains the factor 2 in the formula for the spherical distance.

Problem 3.10. Consider the fractional linear transformation ψ given by

$$\psi(z) = \frac{az + b}{-\bar{b}z + \bar{a}},$$

where a and b are two complex constants satisfying $|a|^2 + |b|^2 = 1$. Show that the spherical metric is invariant under ψ; that is, show that for any path in $\hat{\mathbb{C}}$ one has $\Lambda(\psi \circ \gamma) = \Lambda(\gamma)$. ✱

3.4.2 Weierstrass' spherical convergence theorem

The general concepts of convergence, and of the more specific convergence UOC, of sequences of maps between metric spaces applies to the current situation. A sequence of meromorphic functions in a domain Ω is viewed as sequence of maps $\Omega \to \hat{\mathbb{C}}$ with $\hat{\mathbb{C}}$ equipped with the spherical metric, and we can speak about convergence UOC in this situation. To distinguish this concept from the habitual concept of convergence UOC, we shall refer to it as *spherical convergence* UOC. Any holomorphic function may be considered being meromorphic, so in the case all the functions in the sequence are holomorphic we have two concepts of convergence, and the two differ slightly — a stupid example being the sequence of the constant functions $f_n(z) = n$, which clearly diverges in the finite plane, but converges UOC to the the point ∞ in the extended plane. The sequence $\{z^\nu\}$ gives a somewhat more substantial example when considered on the domain $\{ z \mid |z| > 1 \}$. There it converges UOC in the spherical sense to the constant function with value ∞, but since high-school we learned that it diverges when viewed as a sequence of maps into $\mathbb{C}$.

One word of warning. One must be very careful about the derivative of members of a spherically normal family, they do necessarily form a spherically normal sequence. You will find an example in exercise 3.17 on page 122 below,

(3.1) Before attacking the spherical Weierstrass version, we make an obvious general observation. Let $\{f_\nu\}$ be a sequence of continuous functions from one metric space X into another Y, and assume it converges UOC toward the function f. Let $a \in X$ be a point. Denote by U an open neighbourhood of the image point $f(a)$ and choose a compact subset K of the inverse image $f^{-1}(U)$ containing a in its interior. It then holds that $f_\nu(K) \subseteq U$ for $\nu >> 0$. To verify this just perform a classical ϵ-δ exercise (or may be one should say an ϵ-δ-ν exercise) with ϵ equal to the distance from $f(a)$ to the boundary ∂U.

(3.2) A fundamental tool on which we up to now have based the theory, is the Weierstrass convergence theorem, and to pursue the development we need a version for spherical convergence. The slight discrepancy between the two types of convergence for holomorphic functions—depending upon they being viewed as maps into $\mathbb{C}$ or into $\hat{\mathbb{C}}$—deserves an explanation. The two examples above are illustrative; in the spherical world functions are allowed to converges to the constant infinity.

Proposition 3.1 *Let $\{f_\nu\}$ be a family of meromorphic functions in the domain Ω in the finite plane, and assume that it converges spherically* uoc *to the function f. Then it hods true that*

- *the function f is meromorphic in Ω;*
- *if all the functions f_ν are holomorphic in Ω then either f is holomorphic or constant equal to ∞.*

Proof: Let $a \in \Omega$ be a point. There are two cases to treat.

Firstly, assume that $f(a) \neq \infty$. Choose a disc U about $f(a)$ not containing ∞, and let D be a disk about a whose closure is contained in the inverse image $f^{-1}(U)$. By the obvious observation in paragraph (3.1) above it holds that $f_\nu(D) \subseteq U$ for $\nu >> 0$, and consequently the functions f_ν are all holomorphic in D. The spherical and euclidean metrics being equivalent in D the sequence converges uoc in the euclidean sense in D, and by the habitual Weierstrass' convergence theorem (theorem 3.1 on page 108), the limit is holomorphic.

Secondly, assume that $f(a) = \infty$. In a similar manner as in the first case, fix a disk U about the point at infinity that does not contain the origin, and choose a disk D about a with $f_\nu(D) \subseteq U$ for $\nu >> 0$. For such ν's the functions f_ν will be uniformly bounded away from zero in D. Hence their inverses $1/f_\nu$ are all holomorphic there and form a bounded sequence that converges towards $1/f$, uoc in the euclidean sense. By the classical Weierstrass' convergence theorem $1/f$ is holomorphic in D; that is, f is meromorphic near a.

Finally, assume additionally that all the functions f_ν are holomorphic. This means that the inverse functions $1/f_n(z)$ have no zeros in D, and by Hurwitz' theorem of zeros (theorem 3.2 on page 109) we conclude that either $1/f$ is constant equal to zero or without zeros in D. Correspondingly, f is either constant equal to ∞ or holomorphic. ❑

3.4.3 The spherical derivative

The setting is a domain Ω in the finite plane where our function f is meromorphic. Like the derivative, the spherical derivative is not insensitive to the change of variables $w = z^{-1}$, there will be a factor $|w|^2$ appearing. Hence, to keep life reasonably simple, we shall stick to the setting described and only work with domains in the finite plane.

As a start we define the spherical derivative in points where f is holomorphic, and the definition goes as follows.

$$f^\sharp(z) = \frac{|f'(z)|}{1 + |f(z)|^2}.$$

We observe that $(1/f)^\sharp = f^\sharp$ in points where f does not vanish; indeed the little calculation beneath shows this:

$$(1/f)^\sharp = \frac{|-f'/f^2|}{1 + |1/f|^2} = f^\sharp.$$

We exploit that equality, to define the spherical derivative at poles of f, by defining $f^\sharp$ to be $(1/f)^\sharp$ near a pole. The spherical derivative $f^\sharp$ is a function with real positive values defined and continuous throughout the domain Ω where f is meromorphic.

(3.1) To acquaint ourself with the spherical derivative, let us examine when it vanishes. Clearly at points where f is holomorphic this happens where and only where the derivative $f'(z)$ vanishes. We shall see that it vanishes at a pole if and only if the pole has order at least two, and in case f has a simple pole at a, one has $f^\sharp(a) = |\mathrm{res}_a f|^{-1}$.

We may clearly assume that a is the origin. So assume that f has a pole of order n at the origin. Near the origin it holds that $f(z) = z^{-n}g(z)$ with $n \geq 1$ and we find by simple computations that

$$f^\sharp(z) = \frac{|-nz^{-n-1}g(z) + z^{-n}g(z)|}{1 + |z^{-n}g(z)|^2} = \frac{|-nz^{n-1}g(z) + z^n g'(z)|}{|z|^{2n} + |g(z)^2|}.$$

If $n > 1$ the limit of this expression when $z \to 0$ is equal to zero, but when $n = 1$ the limit becomes equal to $1/|g(0)|$, and in fact this value equals $|\mathrm{res}_0 f|^{-1}$.

Problem 3.11. Show that $(\exp z)^\sharp = 2/\cosh \mathrm{Re}\, z$ and that $(\exp iz)^\sharp = 2/\cosh \mathrm{Im}\, z$. ✱

Problem 3.12. Show that if α is a Möbius-transformation, then $(\alpha \circ f)^\sharp = f^\sharp$. Find an example such that $(f \circ \alpha)^\sharp$ does not equal $f^\sharp$. ✱

Problem 3.13. Show that if f is meromorphic in a neighbourhood of ∞, then $\lim_{z\to\infty} f^\sharp(z) = 0$. ✱

Problem 3.14. Show that if we let $w = z^{-1}$ and $g(w) = f(1/w)$, then one has $g^\sharp(w) = f^\sharp(1/w)\,|w|^2$ where the involved quantities are defined. ✱

(3.2) In the euclidean case integrating $|f'(z)|$ along a path γ gives us the euclidean length $l(f \circ \gamma)$ of the image of γ under f. Correspondingly, in the spherical case the integral of the spherical derivative $f^\sharp(z)$ along γ gives half of the spherical length of the image $f \circ \gamma$, that is, one has the formula

$$2^{-1}\Lambda(f \circ \gamma) = \int_\gamma f^\sharp(z)\,|dz|\,.$$

Indeed, by the simple substitution $w = f(z)$ it holds that

$$2^{-1}\Lambda(f \circ \gamma) = \int_{f\circ\gamma} \frac{|dw|}{1 + |w|^2} = \int_\gamma \frac{|f'(z)|}{1 + |f(z)|^2}\,|dz| = \int_\gamma f^\sharp(z)\,|dz|\,.$$

Problem 3.15. Assume that f is meromorphic and injective in the $A \subseteq \Omega$. Show that the spherical area of $f(A)$ equals $4^{-1}\int_A (f^\sharp(z))^2 dz d\bar{z}$. ✱

3.4.4 Marty's theorem

Marty's theorem is the spherical version of Montel's theorem of convergence:

Theorem 3.9 *A family $\mathcal{F}$ of meromorphic functions in the domain Ω in the finite plane is normal if and only if the family formed by the spherical derivatives $f^\sharp$ for $f \in \mathcal{F}$ is bounded on compacts.*

Proof:

We appeal once more to the theorem of Arzelà-Ascoli. As the Riemann sphere $\hat{\mathbb{C}}$ is compact the second condition of Arzelà and Ascoli is automatically fulfilled, and our task reduces to checking that a family is equicontinuous on compacts in Ω when being spherically bounded on such compacts.

Equicontinuity on compacts being a local property by lemma 3.1 on page 112 we can concentrate on disks D whose closure lies in Ω. So let z and z' be two points from D and let denote by γ the standard parametrization of the line segment between them, *i.e.*, $\gamma(t) = tz + (1-t)z'$. Furthermore, let M denote a common upper bound in D for the functions in $\mathcal{F}$. We find for $f \in \mathcal{F}$ the inequality

$$\rho(f(z), f(z')) \leq \int_{f\circ\gamma} \frac{|dz|}{1+z\bar{z}} = \int_\gamma f^\sharp(z)\,|dz| \leq M \int_\gamma |dz| = M\,|z-z'|\,,$$

from which the equicontinuity is evident (again we use that the two metrics are equivalent in the finite plane).

The other implication is standard. The family $\{\, f^\sharp \mid f \in \mathcal{F} \,\}$ not being bounded on compacts means that there is a compact $K \subseteq \Omega$ and a sequence $\{f_\nu\}$ from $\mathcal{F}$ with $\|f_\nu^\sharp\|_K > \nu$. Replacing $\{f_\nu\}$ by a subsequence if necessary, we may assume that $\{f_\nu\}$ converges UOC in Ω towards a function f that by the spherical version of Weierstrass' convergence theorem is meromorphic. It is easy to see that the sequence $\{f_\nu^\sharp\}$ then converges UOC to $f^\sharp$ and we can conclude that $f^\sharp$ is continuous on the compact K. Therefore it has a maximum there, which contradicts that $\|f_\nu^\sharp\|_K > \nu$. ❑

Example 3.3. The spherical derivatives of functions in the family $\{z^\nu\}$ are uniformly bounded on compacts contained in $\Omega = \{\, z \mid |z| > 1 \,\}$. Indeed, if $|z| > R > 1$ we find

$$(z^\nu)^\sharp = \frac{\nu\,|z|^{\nu-1}}{1+|z^{2\nu}|} = \frac{\nu\,|z|^{-\nu-1}}{|z|^{-2\nu}+1} < \nu R^{-\nu-1}$$

which stays bounded as ν growths since $xe^{-x\log R} \to 0$ as $x \to \infty$ when $R > 1$. Hence $\{z^\nu\}$ is a normal family in $\mathbb{C} \setminus \overline{\mathbb{D}}$ in the spherical sense. The sequence converges spherically UOC to the constant ∞. However, the family is not normal in the traditional euclidean sense since no subsequence converges UOC. ❋

Problem 3.16. Is $\{z^\nu\}$ a normal family in the spherical sense in the disk $|z+i| < 1$? What about the euclidean sense? ✷

Problem 3.17. This exercise is an example (borrowed from Carathéodory's book [Car54], page 188) of spherically normal sequence such that the derived sequence is not spherically normal. This should be contrasted with the traditional Weierstrass' convergence theorem (theorem 3.1 on page 108).

a) Show, by using Marty's theorem, that sequence $f_\nu(z) = \nu^2/(1-\nu^2 z^2)$ of meromorphic functions converges spherical UOC to $f(z) = -z^{-2}$.

b) Compute the spherical derivatives of $f'_\nu(z)$ and show that the sequence they form is not bounded at the origin. Conclude, again by Marty's theorem that $\{f'_\nu\}$ is not a spherically normal family. Hint: In pure mercy with the students, the derivative of f_ν is: $f'_\nu(z) = -2\nu^4(1+3\nu^2 z^2)(1-\nu^2 z^2)^{-3}$ (Don't trust me, check it!!).

✷

3.5 Zalcman's lemma of Bloch's principle

Most of what we have done so far was developed in the last quarter of the nineteenth century or the beginning of the twentieth. The scientific activity around those questions had a golden age in the first haft of the twentieth century.

In his article [Zal98] from 1998,Lawrence Zalcman modestly says that he proved his "little lemma" to give Bloch's principle precise form. This he already in 1975 in the paper [Zal75]. Twenty years later the interest in normal families bloomed again, and Zalcman's lemma "proved amazingly versatile", as he himself expresses it in [Zal98].

3.5.1 Zalcman's lemma

Zalcman's "little lemma"—now upgraded to the status of theorem—is a criterion for a family not to be normal (and we speak about families normal in the spherical sense). Marty's theorem (theorem 3.9 on page 121) is one of the main ingredients of the proof. The lemma deals with families of functions in the unit disk, but normality being a local property, this is sufficient for the applications.

(3.1) The statement in the lemma involves functions of type $g(z) = f(a+\rho z)$—where a is a complex constant and ρ a real number—deduces from a function f holomorphic in a disk D about the origin by *rescaling* and *translation*. If D has radius r, the function g will be holomorphic in the disk D' given by $|z| < (r-|a|)\rho^{-1}$, and provisionally we denote the radius of D' by R'; that is, $R' = (r-|a|)\rho^{-1}$. (In fact it will be holomorphic in greater disk, whose centre is $-\rho^{-1}a$ and radius $\rho^{-1}r$, but we shall be content with using D'.)

The following easy estimate will be useful. Fix positive real number R with $R < R'$. For points z satisfying $|z| < R$ it holds true that

$$r - |a+\rho z| \geq r - |a| - |\rho z| \geq r - |a| - \rho R \geq (r-|a|)(1-\frac{R}{R'}). \tag{3.6}$$

(3.2) Here comes the lemma:

Theorem 3.10 *Let $\mathcal{F}$ be a family of holomorphic (resp. meromorphic) functions in the unit disk $\mathbb{D}$ that is not spherically normal. Then there exist a sequence $\{f_\nu\}$ of function belonging to $\mathcal{F}$, a positive real number r and a sequence of points $\{a_n\}$ with $|a_\nu| < r$, and a sequence $\{\rho_\nu\}$ of positive real numbers tending to zero such that following holds true: The sequence formed by the functions*

$$g_\nu(z) = f(a_\nu + \rho_\nu z)$$

converges to a non-constant entire (resp. meromorphic in $\mathbb{C}$) function g with $g^\sharp(z) \leq g^\sharp(0) = 1$.

Before starting the proof, we remark that the functions g_ν are not all defined in the same domain. However, g_ν is defined in the disk D_ν about the origin of radius $R_\nu = (r - |a_\nu|)\rho_\nu^{-1}$, and it will appear during the proof that the radii R_ν tend to infinity with ν. This implies that any point in the complex plane $\mathbb{C}$ eventually will be contained in all the D_ν's, that is will be lying in D_ν for $\nu >> 0$.

Proof: The family $\mathcal{F}$ not being normal, Marty's theorem (theorem 3.9 on side 121) tells us that the family formed by the spherical derivatives $f^\sharp$ of functions from $\mathcal{F}$ is not bounded on compacts in $\mathbb{D}$. In clear text this means that there is a compact K contained in $\mathbb{D}$ and a sequence $\{f_\nu\}$ from $\mathcal{F}$ with the sequence of sup-norms $\|f_\nu^\sharp\|_K$ tending to ∞ with ν. The compact K has a positive distance ρ to the boundary $\partial\mathbb{D}$ strictly less than one, hence we may choose a radius r with $r < 1$ so that the disk D_r given by $|z| < r$ contains K.

The crux of the proof is to consider the spherical derivatives $f_\nu^\sharp$ modified by the "cut-off-factor" $(1 - |z|^2/r^2)$, and their maximum values in D_r. We put

$$M_\nu = \max_{z \in D_r} (1 - |z|^2/r^2) f_\nu^\sharp(z),$$

and we let a_ν be a point in D_r where the maximum value is achieved. Since $\|f^\sharp\|_K \to \infty$ one easily sees that $M_\nu \to \infty$ when $\nu \to \infty$; indeed, one has $M_\nu \geq (1 - \rho^2/r^2)\|f^\sharp\|_K$.

We put $\rho_\nu = 1/f^\sharp(a_\nu)$, so that $\rho_\nu = (1 - |a_\nu|^2/r^2)/M_\nu$. Hence $\rho_\nu \to 0$ when $\nu \to \infty$. Furthermore it holds true that

$$R_\nu = (r - |a_\nu|)/\rho_\nu = M_\nu r^2/(r + |a_\nu|) \to \infty$$

as M_ν growths beyond limits when $\nu \to \infty$, whereas the denominator in the fraction to the right stays greater then r. The functions

$$g_\nu(z) = f_\nu(a_\nu + \rho_\nu z),$$

are holomorphic in the disks D_ν with radii R_ν about the origin, and plan is to use Marty's convergence theorem to show that these functions form a normal family. To that end, observe using the chain rule that $g_\nu^\sharp(z) = \rho_\nu f_\nu^\sharp(a_n+\rho_\nu z)$, and hence by simple manipulations using the definitions we obtain the estimate:

$$g_\nu^\sharp(z) \leq \frac{r^2 M_\nu}{f^\sharp(a_\nu)(r^2 - |a_\nu + \rho_\nu z|^2)} \leq \frac{r - |a_\nu|}{r - |a_\nu + \rho_\nu z|} \cdot \frac{r + |a_\nu|}{r + |a_\nu + \rho_\nu z|} \qquad (3.7)$$

The last factor clearly stays bounded as $\nu \to \infty$ and tends to one as $\nu \to \infty$. Attacking the first factor we confine z to a given compact disk $|z| \leq R$, thence $R_\nu > R$ if $\nu >> 0$. For such ν the denominator stays uniformly bounded away from zero by the estimate (3.6), and hence the first factor tends to one as well. Consequently $g_\nu^\sharp$ is bounded in the disk $|z| < R$, independently of ν.

The limit function g is thus entire (meromorphic in $\mathbb{C}$) by Marty's theorem. The inequality $g^\sharp \leq 1$ follows from (3.7) when we let ν go to infinity—both factors tend to one. ❑

Example 3.4. An illustrative example is the following. The domain will be the unit disk and the family will be

$$\mathcal{F} = \{\, f_\nu(z) = 2^\nu z^\nu \mid \nu \in \mathbb{N} \,\}.$$

This family is not normal in the region $|z| > 1/2$, it is not even pointwise bounded there. We short cut the recipe in the proof, and take $a_\nu = 1/2$ for all ν and $\rho_\nu = \alpha/2\nu$, where α is any real number. With this data we find $g_\nu(z) = (1 + \alpha z/\nu)^\nu$ which approaches $g(z) = e^{\alpha z}$ when $\nu \to \infty$. ❄

3.5.2 Bloch's principle

According to Robert Osserman as he tells in the very readable article [Oss99], that André Bloch is probably best known on three counts, one is his tragic story. He killed his brother, his aunt and his uncle and passe most of his life in a psychiatric hospital. A second one called "Bloch's principle", which is a very vague statement. As Osserman says, it is more a heuristic device than a result. it states in essence that whenever one has a global result, there should be a stronger, finite version from which the global result follows. The origin of Zalcman's lemma seems to be his wish to make this foggy principle into a theorem, and indeed he did. Below we give a very short description of Zalcman's version of Bloch's principle.

(3.1) Let $\mathcal{P}$ be a property of holomorphic functions in domains U. Properly speaking, it is a property of pairs (f, U) where f is holomorphic in the open set U. Such pairs are frequently called *function elements*. Example of such a property could be "bounded in U" or "injective in U" or "locally injective i U"— there are plenty of possibilities. Formally one may say that a property $\mathcal{P}$ is just a set function elements; that is, of pairs of the type (f, U) above.

(3.2) There are three conditions on function elements that enter into Zalcman's version of Bloch's principle.

- ☐ The first condition is that the property be *compatible with restrictions*. This ought to be self-explanatory, but means that the restriction $f|_{U'}$ has the property in U' whenever f has it in U and U' is any open subset of U. Or phrased differently, if $U' \subseteq U$ and the function element (f, U) belongs to $\mathcal{F}$, then $(f|_{U'}, U')$ belongs to $\mathcal{F}$ as well.
- ☐ One may call the second *compatibility with affine coordinate changes*. This means that if $\phi(z) = a + \rho z$ is any affine change of coordinates with $a \in \mathbb{C}$ and $\rho \in \mathbb{R}$, then the function $f(\phi(z)) = f(a + \rho z)$ has the property in the open set $\rho^{-1}(U - a) = \phi^{-1}(U)$ whenever $f(z)$ has the property in U; or phrased with functions elements: If (f, U) belongs to $\mathcal{F}$, then $(f \circ \phi, \phi^{-1}(U))$ belongs there also.
- ☐ The third and final property is somehow more subtle and one may call it *compatible with* UOC-*convergence*. For any ascending chain $\{U_\nu\}$ of open sets, and any sequence $\{f_\nu\}$ of a function each holomorphic U_ν and converging UOC toward f, the limit function f_ν is required to have the property on $\bigcup_\nu U_\nu$ whenever each f_ν has it in U_ν.

These three conditions are of course made to perfectly match the setting coming out of Zalcman's lemma, and hence the following proposition—which is Zalcman's version of Bloch's principle—is an immediate consequence of it.

Proposition 3.2 *Assume that a property P fulfils the three requirements above. Assume further that the only entire functions having the property are the constants. Then for any domain Ω the family of functions having the property $\mathcal{P}$ in Ω is spherically normal.*

PROBLEM 3.18. Show that the family of derivatives of univalent functions in Ω is normal. HINT: Show that any entire univalent function is of the form $az + b$, hence has a constant derivative. ✷

3.6 Picard's big and Montel's second

We now turn to the second theorem of Montel's which may be is the more famous one, being equivalent to Picard's big theorem. Recall, when we studied isolated singularities we proved the theorem of Casorati and Weierstrass saying that a function comes arbitrarily near every complex value in every neighbourhood of the singularity. Picard's big theorem is a significant strengthening of this result. It states that the function in fact achieves every value, with at most one exception, infinitely often. The function $e^{1/z}$ has no zeros, so one must accept an exception.

(3.1) Frequently Picard's big theorem is cited as a statement about entire functions, and the conversion is not particularly deep. If $f(z)$ is entire, the function $f(1/z)$ has an isolated singularity at the origin. This singularity is essential if and only if $f(z)$ is not a rational function—and non-rational functions go under name of transcendental functions. So Picard's big theorem about entire functions reads says that an entire, transcendental function takes on all complex values infinitely often with at most one exception.

(3.2) As there is a big Picard theorem, there must also be a little one. And indeed, there is one: Given two different complex numbers a and b. If an entire function avoids the two complex numbers, then it is constant. There is a version for meromorphic functions as well, but in that case three values are need. If a, b and c are three complex numbers and f a meromorphic functions avoiding all three, then f is constant. As any two sets of three different complex numbers can be mapped to each other by a Möbius transformation, both these statements are equivalent to the statement that the only entire functions not assuming the values 0 and 1 are the constants.

3.6.1 Montel's second theorem

With Picard's theorem in mind, it quit natural to study the family of function in a domain Ω avoiding the values 0 and 1, and one of Paul Montel's main results is that this is a normal family. Fairly easy and standard arguments, that we shall give below, show that this implies Picard's big theorem. But, for the moment it is about Montel's result:

Theorem 3.11 *Let Ω be a domain. And let $\mathcal{M}$ be the family of holomorphic functions in Ω avoiding 0 and 1; that is holomorphic functions $f\colon \Omega \to \mathbb{C} \setminus \{0, 1\}$. Then $\mathcal{M}$ is a spherically normal family.*

Proof: Since being normal is a local property of families we may very well assume that Ω is a disk, and after a translation and a rescaling, we can without loss of generality assume that the disk is the unit disk $\mathbb{D}$.

For any natural number n one has the n-th roots of unity. They constitute a set μ_n contained in the unit circle $\partial\mathbb{D}$; one has $\mu_n = \{ z \mid z^n = 1 \}$. Their union is dense in the circle, and in fact for any prime p (for instant 2) the sets μ_{p^n} form an ascending chain whose union (frequently denoted by μ_{p^∞}) is dense.

For a natural number n we let $\mathcal{M}_n$ be the family of holomorphic functions in $\mathbb{D}$ that in addition to avoiding 0 avoid all the n-roots of unity; *i.e.*, functions $\phi\colon \mathbb{D} \to \mathbb{C} \setminus \mu_n \cup \{0\}$. It is clear that $f \in \mathcal{M}_n$ if and only if $f^n \in \mathcal{M}_n$.

Now, when $|f(z)|$ is bounded by M in a set A clearly $|f(z)^n|$ is bounded by M^n. It follows that the family $\mathcal{M}_n$ being locally bounded in $\mathbb{D}$ entails that $\mathcal{M}$ is locally bounded as well. The converse holds equally true since members of $\mathcal{M}$ all avoid zero and consequently have an n-th root. The end of the story is that $\mathcal{M}$ is normal if and only if $\mathcal{M}_n$ is.

So assume that $\mathcal{M}$ is not normal. By Zalcman's lemma there is for each n an entire function g_n which is the limit of scaled and translated versions of functions from $\mathcal{M}_n$:

$$g_n(z) = \lim_k f_k(a_k + \rho_k z).$$

They satisfy $g_n^\sharp(0)$=1, and $g_n^\sharp(z) \leq 1$. So by Marty's theorem they form a normal family (in the spherical sense)!

We concentrate on the indices n being powers of 2. The functions g_{2^k} form a normal family, and hence there is an entire function G being the spherical UOC-limit of functions of this type; it is not constant since $G^\sharp(0) = g_{2^n}^\sharp(0) = 1$ (hence it can not be constantly equal ∞ either and is entire). Now, each g_n avoids μ_n by Hurwitz' theorem on values (theorem 3.3 on page 3.3), and as the μ_{2^k}-s form an ascending chain, the function G must avoid all 2^k-roots of unity, that is μ_{2^∞}.

The image $G(\mathbb{C})$ of G is an open set avoiding the set μ_{2^∞} that is dense in $\partial\mathbb{D}$, and must therefore be disjoint from $\partial\mathbb{D}$. It follows that $G(\mathbb{C})$ is either contained in the unit disk $\mathbb{D}$ or in its complement. In both cases Liouville's theorem implies that G is constant, which is a contradiction. ❏

3.6.2 Picard's big theorem

There are several version of this theorem, the backbone being the following:

Theorem 3.12 *Let f have an isolated and essential singularity et the point $a \in \mathbb{C}$. Then f assumes all complex values in every neighbourhood of a with at most one exception.*

Proof: We can without loosing generality assume that $a = 0$. If f is a function for which the conclusion does not hold, there are two numbers b and c and a disk D about the origin where f does not assume the value b and c. Clearly, replacing f by $(f-b)(c-b)^{-1}$, we can assume that the two values avoided by f are 0 and 1, and by scaling the variable, we may also assume that D is the unit disk.

It therefore be suffices to see that if f avoids 0 and 1 in the unit disk, either f or f^{-1} is bounded near the origin; indeed, this entails that f is regular or has a pole there, and the singularity is not essential.

The family $\{f(z/n)\}$, being contained in the family of functions avoiding 0 and 1, is spherically normal in the domain $\Omega \setminus \{0\}$ by Montel's second theorem, . Hence there is subsequence, say $\{f(z/n_k)\}$, that converges spherically UOC—either to a holomorphic function or to the constant ∞.

The circle $|z| = 1/2$ is compact, and either $f(z/n_k)$ or $1/f(z/n_k)$ is uniformly bounded there; that is, there is a constant M such that either $|f(z/n_k)| < M$ or $1/f(z/n_k) < M$ holds for $|z| = 1/2$. Indeed, if $\{f(z/n_k)\}$ converges UOC to a holomorphic function f this is clear as f has a maximum on $|z| = 1/2$. If $f(z/n_k)$ tends to ∞ uniformly on compacts, it holds that $f(z/n_k) > 1$ for $|z| = 1/2$ and $k > N$ for some N.

With is in place assume that $|f(z/n_k)| < M$ for $|z| = 1/2$. Hence $|f(z)| < M$ when $z\,|z| = 1/2n_k$, and the by the maximum principle it holds that $|f(z)| < M$ in the annulus $1/2n_{k+1} \leq |z| \leq 1/2n_k$. As n_k tends to infinity with k, it follows that $|f(z)|$ is bounded in $\partial\mathbb{D} \setminus \{0\}$. The case that $|f(z/n_k)| < M$ on the circle $|z| = 1/2$ is treated *mutatis mutandis* in the same way. ❑

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4. Del

Riemann's mapping theorem

version 0.21 — Tuesday, October 25, 2016 6:47:46 PM
very preliminary version— more under way.

One dares say that the Riemann mapping theorem is one of more famous theorem in the whole science of mathematics. Together with its generalization to Riemann surfaces, the so called Uniformisation Theorem, it is with out doubt the corner-stone of function theory. The theorem classifies all simply connected Riemann-surfaces uo to biholomopisms; and list is astonishingly short. There are just three: The unit disk $\mathbb{D}$, the complex plane $\mathbb{C}$ and the Riemann sphere $\hat{\mathbb{C}}$!

Riemann announced the mapping theorem in his inaugural dissertation[1] which he defended in Göttingen in 1851. His version a was weaker than full version of today, in that he seems only to treat domains bounded by piecewise differentiable Jordan curves. His proof was not waterproof either, lacking arguments for why the Dirichlet problem has solutions. The fault was later repaired by several people, so his method is sound (of course!).

In the modern version there is no further restrictions on the domain than being simply connected. William Fogg Osgood was the first to give a complete proof of the theorem in that form (in 1900), but improvements of the proof continued to come during the first quarter of the 20th century. We present Carathéodory's version of the proof by Lipót Fejér and Frigyes Riesz, like Reinholdt Remmert does in his book [Rem], and we shall closely follow the presentation there.

This chapter starts with the legendary lemma of Schwarz' and a study of the biholomorphic automorphisms of the unit disk. In this course the lemma ended up in this

[1]The title of his thesis is "Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse". It starts with the Cauchy-Riemann equations and ends with the Mapping Theorem.

chapter. It certainly deserves a much broader treatment, but time is short these days! Finally, it is may be not the worse place to include it.

The automorphisms of the disk play important role in the proof of the Mapping theorem, as in many other branches of function theory.

4.1 Swartz' lemma and automorphisms of the disk

Both these themes could be the subject of a book, if not several. So this modest sections gives a short and bleak glimses of two utterly richt and manfold worlds.

We start by some examples of automorphism and then passe to prove Schwarz' lemma. With that lemma establish, we determine the group $\mathrm{Aut}(\mathbb{D})$. It consists of all Möbius transformations mapping the unit disk into itself, so the examples we gave generate the group.

4.1.1 Some examples

We shall describe two classes of automorphisms of the unit disk, and in the end it will turn that these two classes generate all the automorphism of $\mathbb{D}$. To be precise, any automorphism is a product of one from each class.

(4.1) The first class of examples are the obvious automorphisms, namely the *rotations* about the origin. They are realized as multiplication by complex numbers of modulus one, *i.e.,* they are given as $z \mapsto \eta$, and if $\eta = e^{i\theta}$ the angle of rotation is θ. Such a rotations is denoted by ρ_η so that $\rho_\eta(z) = \eta z$. The rotations obviously form a subgroup of $\mathrm{Aut}(\mathbb{D})$ canonically isomorphic to the circle group $\mathbb{S}^1$. They of course all have 0 as a fixed point, and are, as we shall see, characterized by this.

(4.2) The other class of automorphism is less transparent, they will however all be Möbius transformations of a special kind. For any $a \in \mathbb{D}$ we define the function

$$\psi_a(z) = \frac{z-a}{\overline{a}z-1}.$$

It is a rational functions with a sole pol at $\overline{a}^{-1}$, and hence it is holomorphic in the unit disk. There are several ways to check that ψ_a maps the unit disk into itself, one can for instance resort to the maximum principle. A more elementary way is to establish the inequality

$$1 - |\psi_a(z)|^2 = \frac{(1-|z|^2)(1-|a|^2)}{|\overline{a}z-1|^2},$$

a matter of simple algebraic manipulations.

The map ψ_a has two fixed points, one lying in the unit disk and the other one outside. The fixed points are determined by solving the equation $\psi_a(z) = z$, which is equivalent to the quadratic equation

$$\overline{a}z^2 - 2z + a = 0.$$

So the fixed points are the two points $\overline{a}^{-1}(1 \pm \sqrt{1-|a|^2})$.

Clearly it holds true that $\psi_a(a) = 0$ and $\psi_a(0) = a$; so 0 and a are swapped by ψ_a. Since a Möbius transformation that fixes three points equals the identity, it follows[2] that $\psi_a \circ \psi_a = \mathrm{id}$. One says that ψ_a is an *involution*. The derivative $\psi_a'(z)$ is easily computed and is given as

$$\psi_a'(z) = \frac{1-|a|^2}{(\overline{a}z-1)^2}. \tag{4.1}$$

In particular we notice that the derivative at zero, $\psi_a'(0) = 1-|a|^2$, is real and positive.

It is worth noticing that the four most important points for ψ_a, that is the zero, the pole and the two fixed points, all lie on same line through the origin. And in some sense they are pairwise "conjugated", the product of the pole and the zero, and the product of the two fixed points are both unimodular and equal to $a\overline{a}^{-1}$.

Problem 4.1. Show that if $|\eta| = 1$ one has $\rho_\eta \circ \psi_{\overline{\eta}a} = \psi_a \circ \rho_\eta$. ✷

Problem 4.2. Show that any Möbius transformation $\phi(z) = (az+b)(cz+d)^{-1}$ not reduced to the identity, has at least one but at most two fixed points. Show that it has one fixed point if and only if $(a+d)^2 \neq 4(ad-bc)$. Prove that two Möbius transformations coinciding in three points are equal. ✷

Problem 4.3. Let $\psi = \rho_\eta \circ \psi_a$ with η unimodular. Show that the product of the two fixed points equals $\eta a\overline{a}^{-1}$, and conclude that ψ has at most one fixed point in $\mathbb{D}$ unless it reduces to the identity. ✷

Problem 4.4. Show that if $\psi = \rho_\eta \circ \psi_a$ where η is unimodular, one has

$$1 - |\psi(z)|^2 = (1-|z|^2)\,|\psi'(z)|$$

for all $z \in \mathbb{D}$. Show that if b is a fixed point for ψ, then the value $\psi_a'(b)$ of the derivative at b is unimodular. ✷

Problem 4.5. Show that the fixed points of $-\psi_a$ are the two square roots of $a\overline{a}^{-1}$. Hence these maps do not have fixed points in $\mathbb{D}$. ✷

Problem 4.6. Let $f\colon \mathbb{D} \to \mathbb{D}$ be holomorphic and assume that $f(\mathbb{D})$ is realtively compact in $\mathbb{D}$ (the closure in $\mathbb{D}$ is compact). Show that f has a fixed point. Hint: Use Rouché's theorem with the functiopns $f(z)$ and $f(z) - z$. ✷

4.2 Schwarz' lemma

Karl Hermann Amandus Schwarz have given many significant contributions to complex function theory among them the result called "Schwarz' lemma". It appeared for the

[2] This could of course as well be viewed by a direct substitution.

first time in 1869 during a course Schwarz gave at ETH in Zürich about the Riemann mapping theorem, which at that time, although being in its infancy, was the cutting edge of mathematical science. It seems therefore appropriate to treat Schwarz' lemma in a chapter about Riemann's mapping theorem; that said, Schwarz' lemma has so many applications and extension that it certainly would have deserved its own proper chapter. Both the formulation and the proof of the lemma has developed, and it found its modern form in 1905, published by Carathéodory, though the proof of to day is due to Erhardt Schmidt.

Theorem 4.1 (Schwarz's lemma) *Let $f\colon \mathbb{D} \to \mathbb{D}$ be a holomorphic map having 0 as a fixed point. Then it holds that $|f(z)| < |z|$ and $|f'(0)| <$ unless f is a rotation, i.e., on the form $f(z) = \eta z$ with $|\eta| = 1$.*

Proof: The function $z^{-1}f(z)$ has a removable singularity at the origin since f vanishes there. Hence if we let

$$g(z) = \begin{cases} z^{-1}f(z) & \text{when } z \neq 0 \\ f'(0) & \text{when } z = 0, \end{cases}$$

g will be holomorphic in $\mathbb{D}$. The idea is to apply the maximum principle to g in disks D_r given by $|z| < r$ with $r < 1$. On the boundary ∂D_r one has

$$|g(z)| = |f(z)|\, r^{-1} \leq r^{-1},$$

and consequently it holds that $|g(z)| \leq r^{-1}$ for $z \in D_r$. In the limit when r tends to 1 one finds $|g(z)| \leq 1$.

To finish the proof, assume that $|g|$ takes the value 1 at a $a \in \mathbb{D}$. Then the modulus $|g|$ has a maximum at a; indeed, if $|g(b)| > 1$ for some other point in the unit disk, the above argument with $r > \max\{1/|g(b)|, |b|\}$ would give $|g(b)| < |g(b)|$. So by the maximum principle g is a constant η, and clearly $|\eta| = 1$. ❏

Problem 4.7. Assume that f is holomorphic that maps $\mathbb{D}$ to $\mathbb{D}$ and vanishes to the n-th order at the origin (that is f and the derivatives $f^{(i)}$ vanish at 0 for $i < n$). Show that $|f(z)| < |z|^n$ unless $f(z) = \eta z^n$ with $|\eta| = 1$. ✷

Problem 4.8. Let f be a holomorphic map from $\mathbb{D}$ to $\mathbb{D}$, and let $a \in \mathbb{D}$ be any point. Study the function $\psi_{f(a)} \circ f \circ \psi_a$, which maps zero to zero, and prove that

$$\left|\frac{f(z) - f(a)}{1 - \overline{f(a)}f(z)}\right| \leq \left|\frac{z-a}{1-\bar{a}z}\right|.$$

Let z tend to a to obtain

$$\frac{|f'(a)|}{1 - |f(a)|^2} \leq \frac{1}{1-|a|^2}.$$

What happens in case of equality in a point? ✷

4.2.1 The automorphisms of $\mathbb{D}$

Our first applications of Schwarz' lemma is to determine all automorphisms of the unit disk. They are all compositiuons of maps from the two calsses of examles we begun with. ne has

Proposition 4.1 *A biholomorphic map $\psi\colon \mathbb{D} \to \mathbb{D}$ can be factored as $\psi(z) = \eta\psi_a(z)$ in a unique way. The numbers a and η are invariants of ψ determined by a being the only zero of ψ and $\psi'(0) = \eta(1 - |a|^2)$.*

Proof: Since ψ is bijective it has exactly one zero, call it a. Then the composition $\psi \circ \psi_a$ maps $\mathbb{D}$ to $\mathbb{D}$ and takes zero to zero. Therefore it suffices to show that an automorphism ψ fixing zero is a rotation. To do that we apply Schwarz' lemma to both ψ and to ψ^{-1} and obtain the two inequalities

$$|\psi(z)| \le |z| \text{ and } \left|\psi^{-1}(z)\right| \le |z|.$$

Replacing z by $\psi(z)$ in the first, we obtain

$$|z| = \left|\psi^{-1}(\psi(z))\right| \le |\psi(z)| \le |z|,$$

and so $|\psi(z)| = |z|$. From Schwarz' lemma we deduce then that ψ is a rotation. The uniqueness follows since the function ψ_a is the only one in its class that vanishes at a. The statemant about the derivative follows trivially from the formula (4.1) above. ❑

All elements in $\mathrm{Aut}(\mathbb{D})$ are therefore Möbius transformations, and $\mathrm{Aut}(\mathbb{D})$ can be described as the group of Möbius transformations that leave the unit disk invariant.

(4.1) The rotations are precisely those automorphisms that have zero as fixed point. A group theorist would say the the rotations constitute the *isotropy group* of the origin. It is of course isomorphic to the circle group $\mathbb{S}^1$.

Any other point a in $\mathbb{D}$ has an isotropy group as well. It is denoted by $\mathrm{Aut}_a(\mathbb{D})$ and consists of the automorphisms leaving a fixed. As generally true in groups acting transitively, different points have conjugate isotropy groups. Hence $\mathrm{Aut}_a(\mathbb{D})$ is conjugated to the group of rotations; indeed, $\psi_a \circ \psi \circ \psi_a$ fixes 0 if and only if ψ fixes a.

Problem 4.9. Show that the map $\mathrm{Aut}_a(\mathbb{D}) \to \mathbb{C}$ defined by $\psi \mapsto \psi'(a)$ is a group homomorphism which induces an isomorphism between $\mathrm{Aut}_a(\mathbb{D})$ and the circle group $\mathbb{S}^1$. ✷

(4.2) The particular maps ψ_a can be characterized among all the automorphisms in several ways. They are the only ones whose derivative at zero is real and positive. Indeed, if $\psi(z) = \eta\psi_a(z)$, one has $\psi'(a) = \eta\psi_a'(a) = \eta(1 - |a|^2)$ which is real and positive if and only if $\eta = 1$.

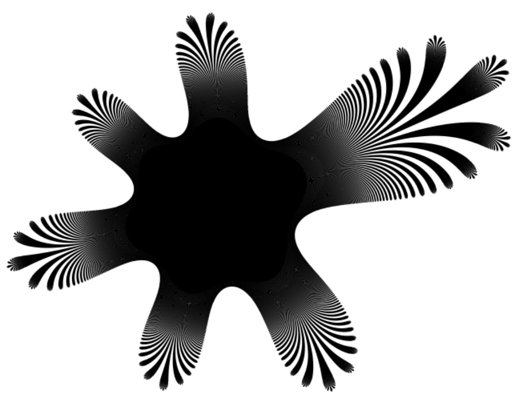
They are also the only involutions in $\mathrm{Aut}(\mathbb{D})$. To see this assume that ψ is an involution, so that $\psi \circ \psi = \mathrm{id}$, and factor ψ as $\psi = \rho_\eta \circ \psi_a$. Then $\psi(a) = 0$, and since ψ is an involution, it follows that $\psi(0) = \psi(\psi(a)) = a$. This gives that $a = \psi(0) = \eta\psi_a(0) = \eta a$, and hence $\eta = 1$.

Problem 4.10. Show that if ψ and ϕ are two commuting automorphisms of $\mathbb{D}$ then they have a common fixed point. Show that if $\psi \in \mathrm{Aut_a}(\mathbb{D})$ is of finite order (as a group element) larger than 2, then ψ is conjugate to the rotation ρ_η with η a n-th root of unity. ✷

4.3 The Riemann mapping theorem

The theorem states that any simply connected domain in the complex plane, that is not the entire plane, is biholomorphic to the unit disk. Thinking about what enormously number of different simply connected domains there are and that they can be almost of infinite complexity, the theorem is at the least extremely deep and impressive. There is a generalization, of even greater depth, called the "Uniformization Theorem" stating that among the Riemann surfaces only $\mathbb{D}$, $\mathbb{C}$ and the Riemann sphere $\hat{\mathbb{C}}$ are simply connected. So the universal cover of any domain in $\mathbb{C}$ is either $\mathbb{C}$ or $\mathbb{D}$, and with some exceptions (that is, those with have $\mathbb{C}$ as the universal cover, but they are not so intricate) open sets in $\mathbb{C}$ are tightly connected to locally injective functions on $\mathbb{D}$! A clear indication that holomorphic function in $\mathbb{D}$ are worth a close study.

Camille Jordan was a great french mathematician, and proved the theorem that any closed, simple curve divides the plane in two connected components, and the bounded one is simply connected. Such curves are called *Jordan curves*, the theorem is called "Jordan's curve theorem". The domains bounded by Jordan curves form a very important class of simply connected domains, but they can also be extremely complicated. There are Jordan curves having positive area! Well, one should say positive two dimensional Lebesgue measure to be precise. In iteration theory beautiful and intricate simply connected domains appear. They are called Siegel domains and picture of them you can see everywhere (also here!)



4.3.1 Examples

To give a modest indication of the depth of Riemann's mapping theorem, we offer a few concrete example of complicated simply connected domains — one could be tempted to call it a "horror show" of domains except that the author finds these examples as beautiful as the pictures! But, still, we can not resist showing John Lennon as a Jordan curve! The big question is what lies outside and what lies inside?

Figur 4.1: John Lennon as a Jordan curve

EXAMPLE 4.1. Let p/q be a positive rational number on reduced form so that p and q are positive relatively prime integers. We also assume that p/q lies between zero and one.

Denote by $L_{p/q}$ the part of the ray making an angle $2\pi p/q$ with the real axis whose points have a distance from the origin larger than q. That is one has $L_{p/q} = \{\, re^{2\pi i p/q} \mid |z| \geq q \,\}$.

Clearly $L_{p/q}$ is closed, but the union $L = \bigcup_{p/q} L_{p/q}$ is also closed. The union being an infinite union, this is slightly subtle; the point is that given any complex number z not in L, there is only finitely many rationals p/q between zero and one such that $q \leq |z| + 1$. Hence if D denotes the disk about z of radius one, the intersection $L \cap D$ is a finite union of closed subset, and consequently is closed in D.

The complement $\mathbb{C} \setminus L$ is open and it is star-shaped with apex at the origin. Hence it is simply connected.

If one wants a finite version of this example here is one. We remove a "hedgehog-like" set from the open square $Q = < -1, 1 > \times < 0, 1 >$. Let $M_{p/q}$ be the ray $\{\, re^{2\pi i p/q} \mid 0 \leq r \leq 1/q \,\}$ and let U be the sqaure Q with all the rays $M_{p/q}$ that lie in Q removed. ❄

Example 4.2. A similar construction is as follows. Take a copy of the cantor set $\mathfrak{c}$ lying on the unit circle (for instance the image of $\mathfrak{c}$ by the standard parametisation) and let L_c be the partial ray $L_c = \{\, rc \mid r \geq 1 \,\}$. The Cantor set being closed, it is not difficult to see that the union $L = \bigcup_{c \in \mathfrak{c}} L_c$ is a closed subset of $\mathbb{C}$, even if it is a uncountable union of closeds. And again, the complement $\mathbb{C} \setminus L$ is star-shaped with the origin as apex.

This domain also has the virtue that its complement in the Riemann sphere $\hat{\mathbb{C}}$ is connected, removing the point at infinity, the complement disintegrates into an uncountable union of components. ❋

Example 4.3. The third example is the so called "comb-set". We again start with an open square, say $Q = <0,1> \times <0,1>$. and the set to be removed is the union of the sets $T_n = \{\, 1/n + yi \mid 0 \leq y \leq 1/2 \,\}$ for $n \in \mathbb{N}$. The result is an open domain that is simply connected being a deformatable to say the segment $<0,1> \times \{2/3\}$. ❋

4.3.2 A motivation

Sometimes it is good strategy to explore a hypothetical solution to problem, to get a clue how to solve the problem. It turns out to be smart to somehow normalize the situation: Fix a point $a \in \Omega$ and confine the maps we are interested inn to those sending a to zero.

So assume that Ω is a simply connected domain and assume f is the solution we are striving for; a biholomorphic map $f\colon \Omega \to \mathbb{D}$ sending a to 0. We want to compare it to any other holomorphic map $g\colon \Omega \to \mathbb{D}$ with $f(a) = 0$, and to that end, consider the composition $g \circ f^{-1}$. It sends the disk $\mathbb{D}$ into it self and fixes the origin. Hence it is prone to a treatment by Schwarz' lemma, that gives the inequality $g(f^{-1}(z)) \leq |z|$, or if one replaces z by $f(z)$, it becomes $|g(z)| \leq |f(z)|$. The solution we seek is therefore a solution to a optimalisation problem: Find the function being maximal in modulus among the those mapping Ω to $\mathbb{D}$ and sending a to 0.

4.3.3 The formulaton and the proof

After these preliminary skirmishes it is high time formulation the theorem in a precise manner. The formulation includes a uniqueness statement that basically says that the Riemann mapping is unique up to automorphisms of the unite disk; so imposing normalization requirement on the function it will be unique.

Theorem 4.2 (The Riemann mapping theorem) *Let Ω be a simply connected plane domain is not the entire plane and let a be a point in Ω. Then there is a unique biholomorphic map $\phi\colon \Omega \to \mathbb{D}$ such that $f(a) = 0$ and $f'(a) > 0$*

Proof: The crux of this proof is to search for functions $f\colon \Omega \to \mathbb{D}$ having maximal modulus in one point different from a. So choose a point $b \in \Omega$ other than a, and consider the set

$$\mathcal{P} = \{\, f\colon W \to \mathbb{D} \mid f(a) = 0, \text{ anf } f \text{ is injective } \,\}$$

where we audaciously also insist on the functions being injective.

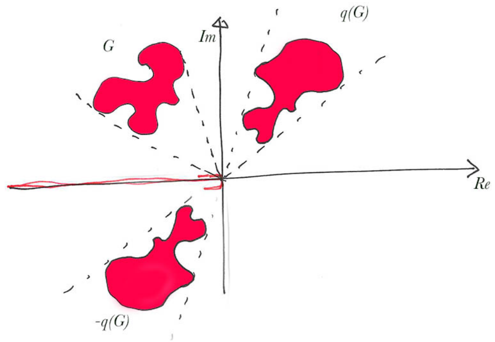
There are three steps on the road to the Riemann mapping theorem. i) Prove that the set $\mathcal{P}$ above is non-empty, ii) show that there is a function f in $\mathcal{P}$ with $|f(b)|$ maximal, and iii) show that f is biholomorphic.

(4.1) We start by showing that $\mathcal{P}$ is non-empty, *i.e.,* we are looking for an injective holomorphic map f from Ω to $\mathbb{D}$; The requirement that a maps to zero is easy to fulfil, we just follow up by an appropriate Möbius transformation sending $f(a)$ to 0.

Pick a point c outside Ω. Then $z-c$ never vanishes in Ω and since Ω is supposed to be a Q-domain, there is a well defined square root of $z-c$ in Ω, that is a holomorphic function q with $q(z)^2 = z-c$.

We claim that $q(\Omega)$ and $-q(\Omega)$ are disjoint. Assume the contrary that is $q(z) = -q(w)$ for two points z and w from Ω. Squaring gives $z = w$, and hence $z = c$, which is impossible since z lies in Ω but c does not.

The set $-q(\Omega)$ is open by the open mapping theorem and therefore contains a disk$D$, say the disk $|z-d| < R$. Then the function $h(z) = R(z-d)^{-1}$ is an injective function, holomorphic for $z \neq d$ and mapping the complement of D, where $q(\Omega)$ lies, into $\mathbb{D}$. The composition $h \circ q$ is a function like we want.



(4.2) The next step is to prove there is a function in $\mathcal{P}$ with $|f(b)|$ maximal. By definition of the supremum there is a sequence of functions f_ν in $\mathcal{P}$ with $f_\nu(p)$ converging to $\alpha = \sup_{f\in\mathcal{P}} |f(p)|$. Montel's first theorem implies that the family $\mathcal{P}$, which is bounded, is a normal family. Hence there is subsequence of f_ν converging UOC in Ω, and we may as well assume that the sequence f_ν itself converges. The limit function f is holomorphic by Weierstrass' convergence theorem, it is injective by Hurwitz' injectivity theorem since each f_ν is, and it takes values in $\mathbb{D}$; *a priori* just in the closure $\overline{\mathbb{D}}$, but $f(\Omega)$ is open. Of course $f(a) = 0$, since $f_\nu(a) = 0$. So the limit function f is our guy!

(4.3) Finally, we have come to the point where to show that f is biholomorphic. By definition f is injective so only the surjectivity is lacking. The salient point is that with the help of a (potential) point d in $\mathbb{D}$, but not in $f(\Omega)$, one can construct an *expanding map* $h\colon f(\Omega) \to \mathbb{D}$. This is a holomorphic map sending $f\Omega$ to $\mathbb{D}$ whose main property is that $|h(z)| > |z|$, with the subsidiary properties of being injective and sending 0 to 0. Such a map would contradict the maximality of $|f(b)|$, since $h \circ f$ is a member of $\mathcal{P}$ and $|h(f(b))| > |f(b)|$.

An obvious expanding map in the unit disk is the square root. However there is the problem with the square root that it can not be a defined in the whole unit disk. To remedy this, we introduce the Möbius transformation ψ_d. It has a sole zero at d and hence does not vanish in $f(\Omega)$.

As $\psi_d(f(\Omega))$ is a Q-domain, a branch q of the square root is well defined there; that is, the composition $q \circ \psi_d$ is well defined in $f(\Omega)$. However, it does not send 0 to 0, but the function $h = \psi_{\sqrt{d}} \circ q \circ \psi_d$ does. This last function h has as inverse the function $\psi_d \circ \kappa \circ \psi_{\sqrt{d}}$ (at least over $f(\Omega)$) where κ is the quadratic function $\kappa(z) = z^2$. One easily checks this using that the ψ_a-s are involutions.

The inverse of a contracting map is expanding, and the function $\psi_d \circ \kappa \circ \psi_{\sqrt{d}}$ is indeed contracting! By Schwarz' lemma this is clear since it maps $\mathbb{D}$ into $\mathbb{D}$, sends 0 to 0 and is not a rotation! Hence h is expanding, and it does the job. That finishes the proof of the existence part of Riemann's mapping theorem.

Finally the statement about the positivity of the derivative is easy to establish .One just follows f by an appropriate rotation; one replaces f by the function $\rho_\omega \circ f$ with $\theta = -\arg f$ which will have a positive derivative at a.

(4.4) To prove the uniqueness statement of the theorem assume that f and g are two biholomorphic maps from Ω to $\mathbb{D}$, both sending a to 0 and both having maximal modulus at b. Then of course $|f(b)| = |g(b)|$.

The composition $f \circ g^{-1}$ maps $\mathbb{D}$ to $\mathbb{D}$ and have 0 as a fixed point, and moreover $|f(g^{-1}(g(b)))| = |f(b)| = |g(b)|$. Due to the last equality we deduce from Schwarz' lemma that the composition $f \circ g^{-1}$ is a rotation and one can write $f(z) = f(g^{-1}(g(z))) = \eta g(z)$ with $\eta \in \partial\mathbb{D}$. Taking derivatives we obtain $f'(b) = \eta g'(b)$ and as both $f'(b)$ and $g'(b)$ are real and positive, it follows that $\eta = 1$. ❑

Problem 4.11. Show that $\psi(x) = \frac{z^2-i}{z^2+1}$ maps the first quadrant biholomorphically onto the unit disk. Determine the inverse map. ✷

Problem 4.12. Find a map that maps a half disk biholomorphically to a full disk. Do the same for a quarter of a disk. ✷

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5. Del

Riemann surfaces

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The idea of a Riemann surface surfaced already in Riemann's inaugural dissertation from 1851. Functions defined by equations tend to be multivalued, as the old-timers expressed it. This occurs even for the simples case $w = z^2$ where the well known ambiguity in sign appears. For other equations equation, for instance $e^w = z$, the situation can be more severe. As we know, there are infinitely many branches of the logarithm. The Riemann surfaces were and are means to resolve this problem. They furnish places where multivalued functions become single valued! In their infancy the definitions of a Riemann surface, and there were a variety, reflected this point of view. The modern definition was strongly promoted by Felix Klein, and it is now ubiquitous in the literature; not only for defining Riemann surfaces, but is almost a universal device for defining geometric structures.

The idea is to use local coordinate charts and impose conditions on how they patch together. Doing calculations on such a space is a little like commanding a submarine. There is no help in looking out of the window on the real world, you are forced to navigate by the maps!

Of course, this idea goes far back in history at least to the Greeks. They understood that it is impossible to have one flat map covering the entire globe. One needs an atlas, that is a collection of maps.

To revert to a more serious tale, the Riemann sphere is an illustrative example. We habitually use two sets of coordinates to describe functions on it. Near the origin—in the southern part in the stereographic picture—we use the familiar coordinate z, but close to north pole—in the vicinity of the point at infinity— we use a coordinate w related to z by the equation $w = z^{-1}$.

5.1 The definition of a Riemann surface

With the example of the globe in mind, a Riemann surface has an underlying topological space X. By a *chart* in X, or we understand an open set U and a homeomorphism z_U from U onto an open subset $z_U(U)$ of $\mathbb{C}$. So the chart is the pair (U, z_U). The open set U will frequently be called *a coordinate neighbourhood*, or a *coordinate patch*. If the open set $z_U(U)$ happens to be a disk, we shall sometimes refer to the chart as a *coordinate disk*.

We call z_U a *coordinate* of the chart, so z_U is a map $z_U\colon U \to z_U(U) \subseteq \mathbb{C}$. In analogy with the commonplace real world, one may think of U as part of the terrain and the open subset $z_U(U)$ as the map[1]. The function z_U gives us the coordinates of the points in U, and the inverse function z_U^{-1} gives the points on X when the coordinates are known—the inverse coordinate function is sometimes called *a parametrization*.

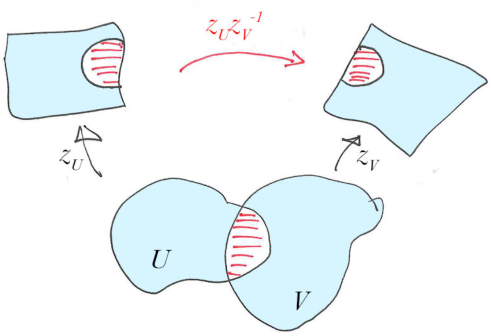
As an example consider the *Riemann sphere* $\hat{\mathbb{C}}$. It has the two open sets U_0 and U_∞, respectively the complement of $\{\infty\}$, that is the finite plane, and the complement of $\{0\}$. On the former one has the canonical coordinate z and on the latter one has the coordinate w given as $w = z^{-1}$ in the finite part of U_∞ and equal 0 at infinity.

(5.1) Given two charts $z_U\colon U \to z_U(U)$ and $z_V\colon V \to z_V(V)$ on X. They both survey the intersection $U \cap V$, and is of course of paramount interest to know which points of the two maps correspond to the same point in the terrain! The answer to that question is encoded in the so called *transition function*, that is the composition

$$z_U \circ z_V^{-1}|_{z_V(U\cap V)}\colon z_V(U \cap V) \to z_U(U \cap V).$$

Not to overload our notation we shall just write $z_U \circ Z_V$ for this function, with the tacit understanding it is defined on $z_V(U \cap V)$.

We say that two charts are *analytically compatible* if the corresponding transition function $z_U \circ z_V$ is holomorphic. This is perfectly meaningful, the transition function being a map between two open subsets of $\mathbb{C}$. As an example, on the intersection $U_0 \cap U_\infty$ in the Riemann sphere, the transition function $z_{U_\infty} \circ z_{U_0}^{-1}$ is the map $z \to z^{-1}$.



[1] In everyday language the map is frequently the piece of paper on which the map is printed, *i.e.*, the set $z_U(U)$. For us, as in the real real life, the map, or the chart, is the pair (U, z_U).

(5.2) By an *atlas* $\mathcal{U}$ on X we understand a collection of charts that together survey the whole topological space X, that is $\mathcal{U}$ is an open covering of X. The atlas is said to be an *analytic atlas* if additionally every two charts from the atlas are analytically compatible. Phrased differently, all the transition function arising in the atlas are holomorphic.

The set of analytical atlases on X are in the a natural way ordered by inclusion; one atlas is smaller than another if every chart in the former also is a chart in the latter. An analytic atlas is *maximal* if, well, it is maximal in this order. The existence of maximal atlases is an easy consequence of Zorn's lemma. If $\mathcal{U}_i$ is an increasing chain of analytical atlases, the union will be one, and by Zorn there is then a maximal one.

Defenition 5.1 *Let X be a connected, Hausdorff topological space. By an* analytic structure *on X, we understand a maximal analytical atlas on X. The pair of the space X and the maximal analytic atlas is called a* Riemann surface*.*

There are several comments to be made. First of all, it is common usage to let Riemann surfaces be connected by definition, mostly to avoid repeating the hypotheses that X be connected all the time. Some authors incorporate the hypothesis that X be second countable (that is, it has a countable basis[2] for the topology) but most do not, for the simple reason that universal covers of open plane sets are not *a priori* second countable—an illustrative example can be the complement of the Cantor set. It is however a relatively deep theorem of the Hungarian mathematician Tibor Radó (1895–1965) in 1925 that any Riemann surface is second countable. The third comment is that our definition works in any dimensions, one only has to replace charts in $\mathbb{C}$ by charts in $\mathbb{C}^n$.

(5.3) Let $\mathcal{U}$ be an analytic atlas on X and let V and W be two charts with coordinate functions z_V and z_W not necessarily belonging to the atlas $\mathcal{U}$. Assume that each one of them is analytically compatible with all charts from the atlas $\mathcal{U}$. Below we shall see that this implies that V and W are compatible as well, and and hence we can append them to $\mathcal{U}$ and get a bigger analytical atlas. And not stopping there, we can adjoin to $\mathcal{U}$ any chart being compatible with all charts in $\mathcal{U}$. In that way we get a gigantesque maximal atlas, and it is the unique maximal atlas containing $\mathcal{U}$.

Proposition 5.1 *Let X be a connected Hausdorff space. Every analytical atlas $\mathcal{U}$ on X. is contained in a unique maximal atlas, and consequently gives X a unique structure as a Riemann surface.*

Proof: After what we said just before the proposition, the poof is reduced to checking that if V and W are two charts both analytical compatible with all charts in $\mathcal{U}$ they

[2] There are many topological manifolds that are not second countable, even of dimension one! Hausdorff's so called "long line" is an example. In dimension two there are a great many examples, but none of them can be given the structure of a Riemann surface. However, in dimension two or more there are analytical spaces that do not have a countable basis for the topology. If you are interested in these outskirts of geometry, [**?**] is a nice reference.

are analytically compatible among themselves; that is, we must verify that $z_V \circ z_W^{-1}$ is holomorphic on $z_W(V \cap W)$. But for any chart U from $\mathcal{U}$ we obviously have the identity $z_V \circ z_W^{-1} = (z_V \circ z_U^{-1}) \circ (z_U \circ z_W^{-1})$ over $z_W(U \cap V \cap W)$, and as the coordinate neighbourhoods from $\mathcal{U}$ cover $V \cap W$, and being holomorphic is a local property, we are through. ❑

Two of the advantages with working with maximal atlases are that we are free to shrink coordinate neighbourhoods at will and that we can perform arbitrary biholomorphic coordinate changes. However, these maximal atlases are awfully large. In the complex plane for instance, the maximal analytical atlas consists of the pairs (U, ϕ) where U is *any* open subset and ϕ is *any* function biholomorphic in U! Luckily, results like proposition 5.1 above allows us to work with very small atlas when we work explicitly; for example on $\mathbb{C}$ we have the *canonical*[3] *atlas* with merely one chart, namely $(\mathbb{C}, \mathrm{id})$!

The Riemann sphere $\hat{\mathbb{C}}$ has as we saw a small atlas consisting of the two open sets U_0 and U_∞ with the coordinates z and w. On the intersection $U_0 \cap U_\infty$ the transition function is given as $w = z^{-1}$.

(5.4) When we are working in $\mathbb{C}$, disks are in use all the time. Similarly on a Riemann surface we shall frequently work with charts such that $z_U(U)$ is a disk, and for convenience we shall call such coordinate neighbourhoods for disks as well. If $z_U(U)$ is a disk about the origin and x is point in the disk with $z_U(x) = 0$ we say that U is *disk about x* or a disk *centered* at x. And of course we shall drop the index U pretty soon and only write z (or any other convenient letter) for the coordinate function.

(5.5) To analytic atlases are said to be *equivalent* if every chart in one is analytically compatible with every chart in the other. Two equivalent atlases are contained in the same maximal atlas, and hence they define the same structure as Riemann surface on X.

5.1.1 Other geometric structures

In the definiton one may impose other conditions on the transistion functions. For instance, the weaker condition that they C, gives us a structure of a smooth surface (or manifold how higher doimenson if the charts take values in $\mathbb{R}^n$) on X, and if addidinolly the Jaboian determinants of $z_U \circ z_V^{-1}$ all are positive, the smooth surface is orientable, and it becomes oriented once we make up our minds and choose one of the orientations of the plane.

Riemann surfaces are orientable because the jacobian of a biholomorphic map is positive. This follows by the Cauchy-Riemann equations, since

$$\det \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} = u_x^2 + v_y^2 > 0$$

[3] Once you have chosen your favorite model for the complex numbers, this is rally canonical. Be aware that the mapping $\mathrm{id}\mathbb{C}$ is the function normally denoted by z in complex function theory.

where u and v are the real and the imaginary part fo the map.

One also strengthen the conditions on the transposition functions, and thus impose further constraints on the surfaces. For example, one can request the transition functions to be affine, that is of the form $z \mapsto az + b$ and one then speaks about an *affine structure* subordinate to the given analytic structure. Or one may ask that they are Möbius transformations. In that case the structure is called a *projective structure*.

As a final example, by a *real analytic structure* on a Riemann surface X, we understand an analytic atlas such that the coordinate domains $z_U(U)$ are symmetric about the real axis, and such if $f(z) = z_U \circ z_V^{-1}$ is a transition function, then $\overline{f(\overline{z})} = f(z)$. This last condition means that the Taylor development of f about real points have real coefficients.

Problem 5.1. Show that X has a real structure if and only if it has an anti-holomorphic involution (Part of the exercise is to find out what this means!). ✷

Problem 5.2. Let X be a Riemann surface with maximal atlas $\mathcal{U}$ with partchs (U, z_U). One defines the *conjugate* Riemann surface in the following way. The maximal atlas $\overline{\mathcal{U}}$ consists of the patches $(U, \overline{z_U})$ and the transitions functions are $\overline{z}_U \circ \overline{z}_V^{-1}$. Check that this is a Riemann surface. ✷

5.2 Holomorphic maps

The study of Riemann surfaces is to a great extend the study of maps between them, and if the maps are going tell us anything about the relation between the analytic structures on X and Y, these maps must be compatible with those structures. That is, they must be holomorphic in some sense. Being holomorphic is a local concept, so to tell what it means that a continuous map is holomorphic, is a local business, and charts are made for that.

(5.1) Assume that X and Y are two Riemann surfaces and that $f : X \to Y$ is a continuous map. Let V be a coordinate patch in Y and U one in X such that $f(U) \subseteq V$. Thence one may consider the map $z_V \circ f \circ z_U^{-1}$ which is a map from $z_U(U)$ to $z_V(V)$. Both these are open subsets of $\mathbb{C}$ so it is meaningful to require that $z_V \circ f \circ z_U^{-1}$ be holomorphic; and if there is a patch (V, z_V) in Y so that this is case, we say that f is *holomorphic in the patch* (U, z_U). This set up of coordinates patches round x and $y = f(x)$ adapted to f may be illustrated with a diagram like this

$$\begin{array}{ccccc} z_U(U) & \xleftarrow[z_U]{\simeq} & U & \hookrightarrow & X \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f|_U} & & \downarrow{\scriptstyle f} \\ z_V(V) & \xleftarrow[z_V]{\simeq} & V & \hookrightarrow & Y, \end{array} \tag{5.1}$$

where $\tilde{f} = z_V \circ f \circ z_U^{-1}$.

The above definition is just an auxiliary definition, here comes the serious one:

Defenition 5.2 *Let X and Y be two Riemann surfaces and $f: X \to Y$ a continuous map between them. The map f is said to be* holomorphic *if it is holomorphic in every coordinate patch of the maximal analytic atlas on X.*

One says that f is *biholomorphic* or an *isomorphism* if f is bijective and the inverse is holomorphic. The composition of two holomorphic maps is holomorphic. Once you have grasped the definition this is quit clear, so it might be a good exercise to check in detail.

PROBLEM 5.3. Show that a Riemann surface X has a real structure if and only it is isomorphic to its conjugate surface $\overline{X}$. ✷

(5.2) Just like for defining analytic structures small atlases can be used to check that a map is holomorphic:

Proposition 5.2 *Let X and Y be two Riemann surfaces and $f: X \to Y$ a continuous map between them. If there is one analytic atlas $\mathcal{U}$ on X such that f is holomorphic in every patch of U, then f is holomorphic.*

PROOF: If $U' \subseteq U$, we have $z_V \circ f \circ z_{U'}^{-1} = z_V \circ f \circ z_U^{-1} \circ z_U \circ z_{U'}^{-1}$ ❑

(5.3) Local properties of traditional holomorphic functions we know from the beginning of the course, frequently have a counterpart for maps between Riemann surfaces. When being accustomed to the abstract definitions one transfers most local properties to Riemann surfaces with ease, once you have the standard set up on the retina it goes almost by itself, but we give detailed proofs at this stage of the course.

Transferring the "Open mapping theorem", gives us the following:

Proposition 5.3 *A non-constant holomorphic map between two Riemann surfaces is an open map.*

PROOF: This is just an exercise with the standard local set up, and of course, the substance comes from the open mapping theorem. Let A be open in X and let $y = f(x) \in f(A)$ be any point. As f is holomorphic near x, there is a patch (U, z_U) around x where f is holomorphic and we can, by shrinking U if necessary, assume that U is contained in A, thus we have the usual local set up like in 5.2:

$$\begin{array}{ccccc} z_U(U) & \xleftarrow[z_U]{\simeq} & U & \hookrightarrow & X \\ \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f|_U} & & \downarrow{\scriptstyle f} \\ z_V(V) & \xleftarrow[z_V]{\simeq} & V & \hookrightarrow & Y \end{array} \qquad (5.2)$$

where $\tilde{f} = z_V \circ f \circ z_U^{-1}$ and where $U \subseteq A$. By the Open mapping theorem we know that $\tilde{f}$ is an open map. Then $f|_U$(U) is open, which is what we need since $f(U) \subseteq A$. ❑

An important corollary is when X is compact;

Corollary 5.1 *Assume that f is a holomorphic map from a compact Riemann surface X to a Riemann surface Y. Then f is surjective and Y is compact.*

Proof: On one hand the image $f(X)$ is closed X being compact, and on the other hand, after the proposition $f(X)$ is open. Hence $f(X)$ is a connected component of Y, and as Y by definition is connected, it follows that $f(X) = Y$. ❑

Proposition 5.4 *The fibres of a non-constant holomorphic map between Riemann surfaces are discrete.*

Proof: Let $x \in X$ and let $y = f(x)$. It suffices to prove that x is isolated in $f^{-1}(y)$; that we have to find an open $U \subseteq X$ such that $U \cap f^{-1}(y) = \{x\}$. Again we resort to the standard set up with U a coordinate patch containing x. From before we know that the fibers of $\tilde{f}$ are discrete, so there is an open U' in $z_U(U)$ intersecting the fibre of $\tilde{f}$ in $z_U(x)$; and moving U' into X, we get our search for open set; *i.e.*, $z_U^{-1}(U') \cap f^{-1}(y) = \{x\}$. ❑

5.2.1 Tangent spaces and derivatives

The derivative of a map between two Riemann surfaces at point is not a number like we are used to when studying functions of one variable, but like most derivatives of functions of several variables it is a linear map, and since we are doing analysis over $\mathbb{C}$ it turns out to be complex linear map—the subtle point is naturally between which vector space. So to begin with, we must define the tangent space $T_{X,x}$ of a Riemann surface X at a point $x \in X$. The definition follows the now standard lines for defining tangent spaces in intrinsic geometry.

(5.1) Recall the ring $\mathcal{O}_{X,x}$ of *germs* of holomorphic functions near x. The elements are equivalence classes $[(\phi, U)]$ where U is an open neighbourhood of x and f a holomorphic function in U, two such pairs (ϕ, U) and (ψ, V) being equivalent if $W \subseteq U \cap V$ on which f and g coincides; that is $\phi|_U = \psi|_V$. One easily checks that this a ring with pointwise addition and multiplication as operations.

Choosing a coordinate patch U with coordinate z centered at x (recall that this means that $z(x) = 0$) one finds an isomorphism between $\mathcal{O}_{X,x}$ and the ring $\mathbb{C}\{z\}$ of powerseries in z with a positive radius of convergence. This i nothing more that the fact that any holomorphic function near the origin can be developed in a Taylor series and this series is unique.

The local ring is functorial. Given a holomorphic map $f \colon X \to Y$ and let $y = f(x)$. If $[\phi, U]$ is a germ of holomorphic function near y, the composition $\phi \circ f$ is holomorphic on $f^{-1}(U)$ and induces a germ $[\phi \circ f, f^{-1}(U)]$ near x. It is left to the zealous students to convince themselves that is a well defined and is a ring homomorphism.

The maximal ideal in $\mathcal{O}_{X,x}$ consisting of functions that vanish at x will be denoted by $\mathfrak{m}_x$.

(5.2) The *tangent space* $T_{X,x}$ is by definition the set of *point derivations* of $\mathcal{O}_{X,x}$, and point derivation $\tau\colon \mathcal{O}_{X,x} \to \mathbb{C}$ is a $\mathbb{C}$-linear map satisfying a product rule à la Leibnitz:

$$\tau(\alpha\beta) = \alpha(x)\tau(\beta) + \beta(x)\tau(\alpha).$$

It follows that $\tau(1) = 0$ (indeed, $1 \circ 1 = 1$!), and by linearity τ vanishes on constants. A point derivation vanishes as well on the square $\mathfrak{m}_x^2$ of the maximal ideal $\mathfrak{m}_x$; by Leibnitz's rule is obvious that if both $\alpha(x) = 0$ and $\beta(x) = 0$, it holds that $\tau(\alpha\beta) = 0$. Consequently every point derivation induces a map $\mathfrak{m}_x/\mathfrak{m}_x^2 \to \mathbb{C}$ and there is a map as in the following lemma. It is a good exercise to prove that it is an isomorphism.

Lemma 5.1 *There is a canonical isomorphism of complex vector spaces.* $T_{X,x} = \operatorname{Hom}_{\mathbb{C}}(\mathfrak{m}_x/\mathfrak{m}_x^2, \mathbb{C},)$*. In particular it holds that* $\dim_{\mathbb{C}} T_{X,x} = 1$*.*

Proof: We have already define a map one way, so let us define a map the other way; that is, a map from $\operatorname{Hom}_{\mathbb{C}}(\mathfrak{m}_x/\mathfrak{m}_x^2, \mathbb{C},)$ to the tangent space $T_{X,x}$. Assume that $\phi\colon \mathfrak{m}_x/\mathfrak{m}_x^2 \to \mathbb{C}$ is a $\mathbb{C}$-linear map and let $\alpha \in \mathcal{O}_{X,x}$ be a germ. We are supposed to associate a point derivation, say τ_ϕ, to ϕ. The germ $\alpha - \alpha(x)$ obviously vanishes at x and belongs to $\mathfrak{m}_x$, so it is legitimate to put $\tau_\phi(\alpha) = \phi(\alpha - \alpha(x))$. One has the equality

$$(\alpha - \alpha(x))(\beta - \beta(x)) = (\alpha\beta - \alpha(x)\beta(x)) - \alpha(x)(\beta - \beta(x)) - \beta(x)(\beta - \beta(x)). \quad (5.3)$$

Since ϕ vanishes on $\mathfrak{m}_x^2$ and the left side of equation (5.3) above lies in $\mathfrak{m}_x^2$, we obtain

$$\tau_\phi(\alpha\beta) = \alpha(x)\tau_\phi(\beta) + \beta(x)\tau_\phi(\alpha),$$

that is Leibnitz's rule, and hence τ_ϕ is a point derivation. It is left as an exercise to show that one in this way obtains the inverse to the already defined map. ❑

(5.3) The map f^* induced a map, and that is the derivative of f at x, from $T_{X,x} \to T_{Y,y}$ simply by composition. That is we define the derivative $D_x\colon T_{X,x} \to T_{Y,y}$ by the assignment $D_x f(\tau) = \tau \circ f^*$. There is as always some checking to be done, but as always we leave that to the zealous students.

(5.4) A n important point is that the derivative is functorial. Id $f\colon X \to Y$ and $g\colon Y \to Z$ are two holomorphic maps with $f(x) = y$ and $\gamma(y) = z$ it holds true that

$$D_x f \circ g = D_x f \circ D_y g.$$

The formula boils down to the traditional chain rule after the mappings having been expressed in local coordinates. To become accustomed to the formalism of tangent space and derivatives in the intrinsic setting it is a good exercise to check this in detail

(5.5) The choice of a local coordinate z_U centered at the point x, *i.e.*, coordinates such that x corresponds to the origin, induces an isomorphism $\mathcal{O}_{X,x} \simeq \mathbb{C}\{z_U\}$, a germ corresponding to the Taylor series of a function representing the germ. In this correspondence the maximal ideal $\mathfrak{m}_x$ of functions vanishing at x corresponds to the ideal $(z_U)\mathbb{C}\{z_U\}$. Therefore $\mathfrak{m}_x/\mathfrak{m}_x^2$ is one dimensional with as basis the class of z_U, that we baptize dz_U. The basis of $T_{X,x}$ induced by the isomorphism in 5.3 and dual to dz_U is denoted by $\hat{dz_U}$.

(5.6) The usual set up of coordinates round x and $y = f(x)$ is as follows

$$\begin{array}{ccccc} z_U(U) & \xleftarrow[z_U]{\simeq} & U & \hookrightarrow & X \\ {\scriptstyle \tilde f}\downarrow & & \downarrow{\scriptstyle f|_U} & & \downarrow{\scriptstyle f} \\ z_V(V) & \xleftarrow[z_V]{\simeq} & V & \hookrightarrow & Y, \end{array}$$

where z_V is a local coordinate centered at the image point y of x valid in the vicinity V of y. On the open $z_U(U)$ set in $\mathbb{C}$ the map f materializes as a function $\tilde f$ holomorphic in $z_U(U)$, and the map $f^*\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ becomes the map $\mathbb{C}\{z_V\} \to \mathbb{C}\{z_U\}$ that sends z_V to $\tilde f(z_U)$.

We have the basis dz_V for $\mathfrak{m}_y/\mathfrak{m}_y^2$. and writing $\tilde f(z) = \tilde f'(0)z + z^2 g(z)$, we see that dz_V is sent to $\tilde f'(0)dz_U$ since the term $z^2g(z)$ belongs to $\mathfrak{m}_x^2$.

Lemma 5.2 *In local coordinates the derivative $D_x f$ sends the basis element $\hat{dz_U}$ to $f'(0)\hat{dz_V}$.*

5.2.2 Local appearance of holomorphic maps

The first step of understanding a map is to understand its local behavior, so also with holomorphic maps. The fist result in that direction is a version of *the inverse function* theorem formulated in our setting.

Proposition 5.5 *Let $f\colon X \to Y$ be a holomorphic map between two Riemann surfaces and let $x \in X$ be a point. Assume that the derivative $D_x f$ does not vanish. Then there exists an open neighbourhood U about x such that $f|_U\colon U \to f(U)$ is an isomorphism (i.e., biholomorphic).*

PROOF: The usual set up of coordinates round x and $y = f(x)$ is

$$\begin{array}{ccccc} z_U(U) & \xleftarrow[z_U]{\simeq} & U & \hookrightarrow & X \\ {\scriptstyle \tilde f}\downarrow & & \downarrow{\scriptstyle f|_U} & & \downarrow{\scriptstyle f} \\ z_V(V) & \xleftarrow[z_V]{\simeq} & V & \hookrightarrow & Y \end{array}$$

where $\tilde f$ is the representative of f in the local coordinates. By the lemma in the previous paragraph, $D_x f$ is just multiplication by $\tilde f'(0)$ in the basis dz_U and dz_V, hence $\tilde f'(0) \neq 0$, and from the earlier theory we know that thence $\tilde f$ is biholomorphic in a vicinity of 0, and shrinking U if necessary, the restriction $f|_U$ will be biholomorphic. ❑

Points where the derivative vanishes are said to be *ramification points* or *branch points* of the map f, and of course, it is *unramified* or *unbranched* at points where the derivative does not vanish. So, one may formulate the previous proposition by saying that a function is (locally) biholomorphic near points where it is unramified.

(5.1) Near a ramification point there is a local model for the behavior of f, depending on a number $\text{ind}_x f$ called the *ramification index*; which is closely related to the vanishing multiplicity we know from before.

Proposition 5.6 *Let $x \in X$ be a point and let $f\colon X \to Y$ be a holomorphic map. Then there exist coordinate patches (U, z_U) and (V, z_V) around x and $f(x)$ respectively, with $f(U) \subseteq V$ such that $z_V \circ f \circ z_U^{-1}(z) = z^n$.*

In short the result says that locally and after appropriate changes of coordinates both near x and near y, the map f is given as the n-power map $z \to z^n$. But of course, behind this is the formally precise but rather clumsy formulation of the proposition.

The integer n does not depend on the chosen coordinate, and is *ramification index* hinted at, and is denote by $\text{ind}_x f$.

PROOF: Again we start with a standard set up with the patches centered at x and $f(x)$, that is $z_U(x) = 0$ as well as $z_V(f(x)) = 0$. See diagram (5.4) below. By xxx is part 1, there is a holomorphic function g in U such that $\tilde{f} = g^n$ with $g(0) = 0$ and $g'(0) \neq 0$. By shrinking U we may assume that g is biholomorphic in U, and therefore can be use as a coordinate! Hence we introduce the new patch $(U, g \circ z_U)$. For w lying in this patch, we find $\tilde{f}_1 = \tilde{f} \circ g^{-1}(w) = g(g^{-1}(w))^n = w^n$ and are through.

$$\begin{array}{ccccccc} g(z_U(U)) & \xleftarrow[g]{\simeq} & z_U(U) & \xleftarrow[z_U]{\simeq} & U & \hookrightarrow & X \\ & \searrow^{\tilde{f}_1} & \downarrow \tilde{f} & & \downarrow f|_U & & \downarrow f \\ & & z_V(V) & \xleftarrow[z_V]{\simeq} & V & \hookrightarrow & Y \end{array} \qquad (5.4)$$

❑

PROBLEM 5.4. Show that $\tan\colon \mathbb{C} \to \widehat{\mathbb{C}}$ is unramified, but not surjective.Hence it is not a cover. Show that the image is $\widehat{\mathbb{C}} \setminus \{\pm i\}$, and show that $\tan\colon \mathbb{C} \to \widehat{\mathbb{C}}\{\pm i\}$ is a covering. ✷

PROBLEM 5.5. Find the ramification points of the map $f(z) = \frac{1}{2}(z + z^{-1})$. ✷

PROBLEM 5.6. Find the ramification points and the ramification indices of the $f(z) = z^n + z^{-m}$, n and m two natural numbers. ✷

PROBLEM 5.7. Show that a holomorphic map between two compact Riemann surfaces is either constant or surjective. Show that if the map is not constant, the fibres are all finite. ✷

5.3 Some quotient surfaces

This section starts with two examples. The second one is important, elliptic curves being a central theme in several branches of mathematics. We end the section with a general quotient construction valid for a wide class of very nice actions.

In all these cases the quotient map serve as a parametrisation of the quotient surface X/G, except that points on X/G correspond to many values of the parameter—it is the task of the group to keep account of the different values. This makes it particularity easy to find coordinates, locally they are just the parameter values.

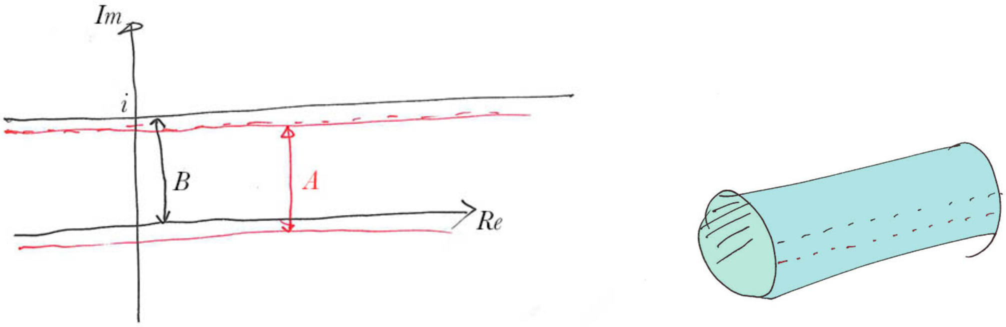
Example 5.1. The cylinder. On way of giving the cylinder an analytic structure is to consider it as the quotient of the plane by the action of the group generated by by the map $z \to z + i$. The topology on X is the quotient topology, the weakest topology making the quotient map $\pi\colon \mathbb{C} \to X$ continuous.

We shall put an analytic structure on X and this is an illustration of how the hocus-pocus with atlas and charts work, we shall do this in extreme detail. We shall specify an atlas with two charts. One is the infinite strip A between the real axis and the horizontal line $\operatorname{Im} z = 1$, or rather the image $\pi(A)$ in X. The quotient map π is a homeomorphism from A to $\pi(A)$, and the coordinate function on $\pi(A)$ is the inverse of this, we denote it by π_A^{-1}. That is the coordinate of $\pi(z)$ is z. The patch $\pi(A)$ covers most of the cylinder except the "seam", the image of the two boundary lines.

The second patch is *mutatis mutandis* constructed in the same way but from the different strip B between the horizontal lines $\operatorname{Im} z = 1 - t$ and $\operatorname{Im} z = -t$ where t is any real number between zero and one. The coordinate patch is the image $\pi(B)$ and the coordinate π_B^{-1}.

What happens then on the intersection $\pi(A) \cap \pi(B)$? What is the transition function? Is it holomorphic? First of all in A the inverse image $\pi_A^{-1}(\pi(A) \cap \pi(B))$ of the intersection is A with the line $\operatorname{Im} z = 1 - t$ removed since points on this line are not equivalent under the action to points in B.

So $\pi_A^{-1}(\pi(A) \cap \pi(B))$ has to components. The one where $0 \leq \operatorname{Im} z < 1 - t$ lies in B as well, and hence the transition function $\pi_B^{-1} \circ \pi$ is the identity. The other one, where $1 - t < \operatorname{Im} z < 1$, the composition $\pi_B^{-1} \circ \pi$ equals the translation $z \mapsto z - i$. In both cases the, the transition function is holomorphic and our two charts are analytically compatible. They constitute an analytic atlas and give the cylinder a complex structure.

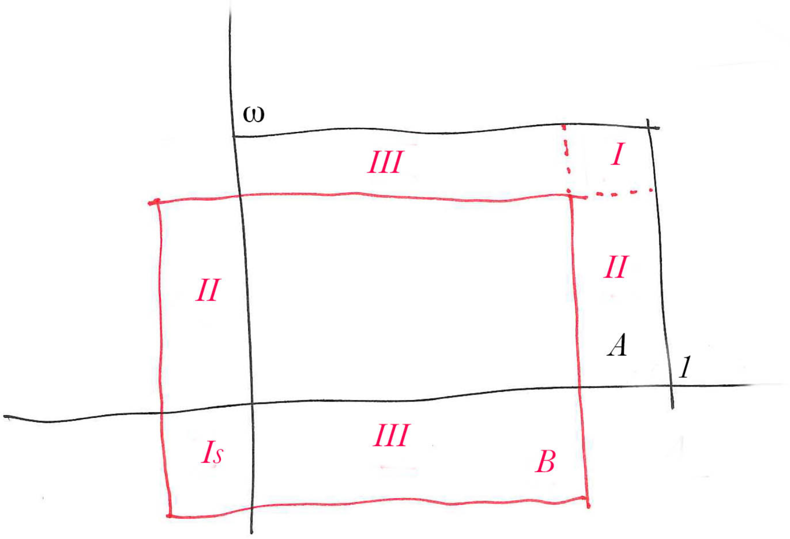


Figur 5.1: *The cylinder and the two coordinate patches.*

In fact, the cylinder is biholomorphic to the punctured plane $\mathbb{C}^*$. The biholomorphism is induced by the exponential function $e(z) = e^{2\pi z}$, that take values in $\mathbb{C}^*$. Clearly $e(z+i) = e(z)$, so e invariant under the group action, and therefore by the properties of the quotient space X, induces a continuous map $\tilde{e}\colon X \to \mathbb{C}^*$. It is easy to check using elementary properties of the exponential function (hence a task for zealous students) that $\tilde{e}$ is a homeomorphism. The only thing left, is to check that it is holomorphic, and this indeed comes for free: On the charts A and B the functions is be definition equal to $e^{2\pi z}$! The coordinate of a point $\pi(z)$ belonging to $\pi(A)$ (or $\pi(B)$) is z! ❄

Example 5.2. The next example is of the same flavour as the first, but the group action is more complicated—there will be two periods instead of just one –and the examples infinitely more interesting.

The Riemann surfaces will be compact with underlying topological space what topologists call a *torus*, a space homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$, which in bakeries is known as a doughnut. This is a genuine new surface—it is not biholomorphic to any open subset of the good old Riemann sphere $\widehat{\mathbb{C}}$— and it is known as an "elliptic curve". These spaces entered the world of mathematics at a time when to compute the circumference of an ellipse (Very important question just after the discovery that the planets move in ellipses!) was the cutting edge of science, and the length-computation ended up with integrals related to bi-periodic functions, and as we shall see, bi-periodic functions lie behind the group action defining this Riemann surface.



Figur 5.2: *The atlas of the torus.*

Let Λ be the lattice $\Lambda = \{ n_1\omega + n_2 \mid n_1, n_2 \in \mathbb{Z} \}$ (in the figure we have for simplicity drawn ω as purely imaginary). It is an additive subgroup of the complex numbers $\mathbb{C}$, and we can form the quotient group $\mathbb{C}/\Lambda$. This is also a topological space when equipped with the quotient topology, and it is homeomorphic to the product $\mathbb{S}^1 \times \mathbb{S}^1$. We let $\pi\colon \mathbb{C} \to \mathbb{C}/\Lambda$ be the quotient map, it is an open map.

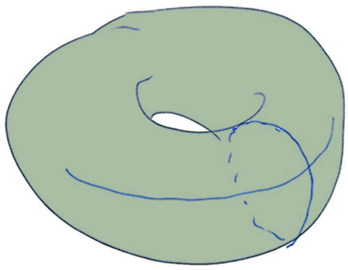
We intend to define an analytic structure on $\mathbb{C}/\Lambda$ in an analogous way as with the cylinder, by giving an atlas with two charts. The first is $A = \{ s + t\omega \mid 0 < s, t < 1 \}$, or rather the subset $\pi(A)$ of the torus. As no two points in A are congruent mod Λ, the set A maps invectively, and π being open, homeomorphically onto the open set $\pi(A)$ in $\mathbb{C}/\Lambda$. The coordinate of point $\pi(z)$ in $\pi(A)$ is simply z. The second chart is a small perturbation B of A, say $A = \{ s + t\omega \mid -\epsilon < s, t < 1 - \epsilon \}$, the image $\pi(B)$ is open and the coordinate of a point $p(z)$ is still z, but this time it must be chosen to lie B.

On the intersection of the two patches, the transition function is holomorphic. A quick (but incomplete) argument goes like this: Take a z in A whose image also lies in $\pi(B)$. Map it down to the torus and lift it back to a point w in B. Both z and the lift w lie in the same fibre of π, so w is a translate of z. Hence the transition functions are just translates, and we are tempted to say: which are holomorphic!

However, this is faulty since the difference $w - z$ can depend on z, and in fact it does!. One must assure oneself that this difference behaves holomorphically as a function of z. Luckily, the differences turn out to locally constant, *i.e.,* constant on the connected components of the intersection, and that will settle the case.

Contemplating figure 5.2 above, you easily convince yourself that this is true. The intersection manifests itself in A with four connected component, marked I, II, III

and IV in the figure, and the different translations are the as follows: The identity on IV, the map $z \to z - \omega$ on III, the map $z \to z - 1$ on II and finally the map $z \to z - i - 1$ in I. ❄



†

Problem 5.8. The quotient $\mathbb{C}/\Lambda$ is a group. Show that both the addition and the inversion maps are holomorphic. ✷

(5.1) Assume that G is a group acting holomorphically on a Riemann surface X. This means that all the action maps maps $x \mapsto g(x)$ are holomorphic, and of course the familiar axioms for an action must hold. If gh denotes the product of the two elements g and h in G, it holds true that $gh(x) = g(h(x))$, and $e(x) = x$ for all x where $e \in G$ is the unit. The set $Gx = \{ g(x) \mid g \in G \}$ is called the *orbit* of x, and the set $G(x) = \{ g \in G \mid g(x) = x \}$ of group elements that leave the point x fixed is called the *isotropy group* or *the stabiliser* of x.

The quotient X/G is as usual equipped with he quotient topology, a set in X/G being open if and only of its inverse image in X is open. This is equivalent to the quotient map $\pi \colon X \to X/G$ being open and continuos.

We concentrate on a class of particular nice actions called *free and proper*. They have following two properties.

- ☐ For any pair of points x and x' in X not in the same orbit, there are neighbourhoods U and U' of respectively x and x' with $U \cap gU'$ for all g.
- ☐ About every point $x \in X$ there is a neighbourhood disjoint from all its non-trivial translates; that is, there is an open U_x with $x \in U_x$ such that $gU_x \cap U_x = \emptyset$ for all $g \neq e$.

The first condition guarantees that the quotient X/G is a Hausdorff space. Indeed, if y and y' are two points in X/G, lift them to points x and and x' in X, and choose neighbourhoods U and U' as in the condition. Then $\pi(U)$ and $\pi U'$ are disjoint, if not there would be a point in U lying in the orbit a point in U', which is precisely what the condition excludes. And both $\pi(U)$ and $\pi(U')$ are open and one contains y and the other one y' so they are disjoint open neighbourhoods of respectively y and y'.

We proceed to define an analytic atlas on X/G. To begin with we chose one on X whose charts are (U, z_U) satisfy $gU \cap U\emptyset$ when $g \neq e$ (convince yourself that such an atlas may be found). The images $V = \pi(U)$ are open, and $\pi|_U$ are homeomorphisms onto U. The open patches of the atlas on X/G are the images V, and the coordinate functions w_V are given as let $w_V = z_U \circ \pi|_U^{-1}$. They take values in $z_U(U)$. We plan to show that this is an analytic atlas.

To this end let (V, w_V) and $(V', w_{V'})$ be two patches of the newly defined atlas on X/G. Our task is to show they are analytically compatible.

The part of $\pi^{-1}(V \cap V')$ lying in U is equal to the union of the different open sets $U \cap g(U')$ as g runs through G. These sets are open and pairwise disjoint since the sets $g(U')$ are , and therefore they form an open partition of $\pi^{-1}(V \cap V') \cap U$.

Now, there is only one partition of a locally connected set consisting of open and connected sets, namely the partition into connected components. The sets $U \cap g(U')$ are not necessarily connected, but it follows that they are unions of connected components of $\pi^{-1}(V \cap V') \cap U$.

It suffices to see that the transition function are holomorphic on each connected component of $z_U(\pi^{-1}(V \cap V') \cap U)$. But g^{-1} of course map maps $U \cap g(U')$ into the connected component $g^{-1}(U) \cap U'$ of $\pi^{-1}(V \cap V') \cap V'$, and the hence the transition function equals the restriction of $z_{U'} \circ g^{-1} \circ z_U$ on $z_U(U \cap g(U'))$.

Problem 5.9. Let a be a positive real number and let η_a de defined by $\eta_a(z) = az$. The clearly η_a takes the upper half plane $\mathbb{H}$ into itself. Let G be the subgroup of $\mathrm{Aut}(\mathbb{H})$ generated by η_a. The aim of the exercise is to show that G acts on $\mathbb{H}$ in a proper and free manner, and that the resulting quotient $\mathbb{H}/G$ is biholomorphic to an annulus:

a) Show that $\liminf_{n \neq 0} |a^n - 1| (a^n + 1)^{-1} > 0$.

b) Let $z_0 \in \mathbb{H}$ and choose an ϵ with $0 < \epsilon < \liminf_{n \neq 0} |a^n - 1| (a^n + 1)^{-1} |z_0|$. Let U be the disk $|z - zo| < \epsilon$. Show that the disks $a^n U$ all are disjoint from U when $n \neq 0$. Conclude that the action is proper and free.

c) Show that the quotient $\mathbb{H}/G$ is a Riemann surface. Show that the function

$$f(z) = \exp(2\pi i \log z / \log a)$$

is invariant under the action of G and induces an isomorphism between $\mathbb{H}/G$ and the annulus $A = \{\, z \mid r < |z| < 1 \,\}$ where $r = \exp(-2\pi^2/\log a)$.

✷

5.4 Covering maps

Coverings play a prominent role in topology, and they have similar important role in theory of Riemann surfaces. May be they even have a more central place there due to the Uniformisation theorem. This fabulous theorem classifies all the simply connected Riemann surfaces up to biholomorphic equivalency, and amazingly, there are only equivalence classes them, namely the class of the complex plane $\mathbb{C}$, of the unit disk[4] $\mathbb{D}$ and of the Riemann sphere $\widehat{\mathbb{C}}$.

As we shall see, every Riemann surface has a universal cover which is a Riemann surface with a holomorphic covering map. Combining this with the Uniformisation theorem, one obtains the strong statement that any Riemann surface is biholomorphic to a free quotient of one of three on the list! This naturally has lead to an intense study of the subgroups of the automorphism groups of the three. Neither the plane nor the sphere have that many quotient, and most of the Riemann surfaces are quotients of the disk. The corresponding subgroups of $\text{Aut}(\mathbb{D})$ form an extremely rich class of groups and can be very complicated.

It is also fascinating that the three classes of simply connected Riemann surfaces correspond to the three different versions of non-Euclidean geometry. The plane with the good old euclidean metric is a model for the good old geometry of Euclid and the other greeks, and the sphere naturally is a model for the spherical geometry. We already used the spherical metric when proving the Picard theorems. The renown french polymath Henri Poincaré put a complete metric on the disk, making it a model for the hyperbolic geometry, and naturally, that metric is called the *hyperbolic metric*.

(5.1) A *covering* map, or a *cover*, is a continuos map p between to topological spaces X and Y which fulfils the following requirement. Every point $y \in Y$ has an open neighbourhood U such that the inverse image decomposes as $p^{-1}(U) = \bigcup_\alpha U_\alpha$ where the U_α's are pairwise disjoint and are such that $p_{U,\alpha} = p|_{U_\alpha}$ is a homeomorphism between U_α and U. One says that the covering is *trivialized* over U; and in fact, it is trivial in the sense that there is an isomorphism $p^{-1}(U) \simeq U \times A$ such that p corresponds to the first projection, just send $u \in p^{-1}(U)$ to the pair $(p_{U,\alpha}(u), \alpha)$.

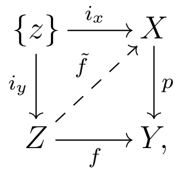
One usually assumes that Y is locally connected to have a nice theory. For us who only work with Riemann surfaces, this is not a restriction at all as points in a Riemann surface all have neighbourhoods being homeomorphic to disks. When the trivializing open set U is connected, the decomposition of the inverse image $p^{-1}(U) = \bigcup_{\alpha\in A} U_\alpha$ coincides with the decomposition of $p^{-1}(U)$ into the union of its connected components, which sometimes is useful.

(5.2) Covering maps have several good properties. For instance, there is a strong lifting theorem. Maps from simply connected spaces into Y can be lifted to X, that is one has the following theorem which we do not prove.

Proposition 5.7 *Assume that $p\colon X \to Y$ is a covering and that $f\colon Z \to Y$ is a continuous map where Z is simply connected. If z is a point in Z and x one in X such that $p(x) = f(z)$, there exists a unique continuous map $\tilde{f}\colon Z \to X$ with $\tilde{f}(z) = x$ and $f = p \circ \tilde{f}$.*

[4] or any Riemann surface biholomorphic to it. The upper half plane $\mathbb{H}$ is a very popular model.

For diagrammaholics, the proposition may be formulated with the help of the following diagram:



where i_x and i_y are the inclusion maps. One should read the diagrammatic message in the following way: The solid errors are given such that the solid square commutes, and the silent statement of the diagramm is that one can fill in a dotted arrow which makes the two triangular parts of the diagramm commutative.

(5.3) Coverings are as we saw locally homeomorphic to a product of an open set and a discrete space. And when the base Y is connected, this discrete space must up to homeomorphisms be the same everywhere; that is, the cardinality is constant over connected components of Y. One has:

Proposition 5.8 *If Y is connected is and $p\colon X \to Y$ is a cover, then the cardinality of the fibres $p^{-1}(y)$ is the same everywhere on Y.*

Proof: Let W_B be the set where $p^{-1}(y)$ is bijective to some given set B. Since p is locally trivial, W_B is open, and the same argument show that the complement $Y \setminus W_B$ is open as well (well, if the fibre is not bijective to B, it lies in some other W_C). It follows that $W_B = Y$ since Y is connected. ❏

In case all the fibres of p are finite, this can be phrased in a slightly different manner. Sending y to $\#p^{-1}(y)$ is a locally constant function on Y because p is locally trivial, and locally constant functions with integral values are constant on connected sets. The open sets U_α are frequently called *the sheets* or *the branches* over U, and if there are n of them, one speaks about an *n-sheeted covering*.

Problem 5.10. Show that the exponential map $\exp\colon \mathbb{C} \to \mathbb{C}^*$ is a covering. Let $a \in \mathbb{C}^*$ describe the largest disk over which exp is trivial. ✷

Problem 5.11. Let $f(z) = \frac{1}{2}(z + z^{-1})$. Consider f as a map from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. Show that f induces a unbranched double covering (synonymous with a 2-sheeted covering) from $\hat{\mathbb{C}} \setminus \{\pm 1\}$ to $\hat{\mathbb{C}} \setminus \{\pm 1, 0\}$. ✷

Problem 5.12. The tangent function $\tan z$ takes values in $\hat{\mathbb{C}} \setminus \{\pm i\}$. Show that $\tan\colon \mathbb{C} \to \hat{\mathbb{C}} \setminus \{\pm i\}$ is a covering. Be explicit about trivializing opens. Hint: It might be usefull that $\arctan z = (2i)^{-1} \log(1 + iz)(1 - iz)^{-1}$. ✷

Problem 5.13. Show that a holomorphic covering between Riemann surfaces then has a derivative which vanishes nowhere. Is the converse true? ✷

(5.4) A *universal covering* of a topological space X is a covering $p\colon Y \to X$ such that the space Y is simply connected, recall that tis means hat Y is path connecetd and that $\pi_1(Y) = 0$. It is not difficult to see that such universal coverings are unique up to a homeomorphism respecting the covering maps. That is, if $p'\colon Y' \to X$ is another one, there is a homeomorphism $\phi\colon Y' \to Y$ with $p' = p \circ \phi$; or diagrammatically presented, there is a commutative diagram

$$\begin{array}{ccc} Y' & \xrightarrow{\phi} & Y \\ & {\scriptstyle p'}\searrow \quad \swarrow{\scriptstyle p} & \\ & X. & \end{array}$$

Not all topological spaces have a universal covering. The condition to have one is rather long (close to a breathing exercise): The pace X must be connected, locally path connected and semi-locally simply connected. But don't panic, Riemann surface all satisfies these conditions, as every point has a neighbourhood homeomorphic to a disk.

PROBLEM 5.14. Let $A = \mathbb{C} \setminus \{\, 1/n \mid n \in \mathbb{N}\,\}$. Show A is not open and that that $0 \in A$. Show that any neighbourhood of 0 in A has loops that are not null-homotopic in A. Show that A does not have a universal covering. ✷

PROBLEM 5.15. Let $: Y \to X$ be a universal cover. let $\mathrm{Aut}_X(Y)$ be the set of homeomorphisms $\phi\colon Y \to Y$ such that $p \circ \phi = p$, that is, the homeomorphism making the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\phi} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & X, & \end{array}$$

commutative. Show that $\mathrm{Aut}_X(Y)$ is a group under composition. Fix a point $x \in X$. Show that ϕ by restriction induces a self-map of the fibre $p^{-1}(x)$. Show that if this self-map is the identity, then $\phi = \mathrm{id}_Y$. Show that $\mathrm{Aut}_X(Y)$ is a naturally isomorphic to a subgroup of the symmetric group $\mathrm{Sym}(\phi^{-1}(x))$. HINT: Use the lifting theorem (theorem 5.7 on page 158). ✷

PROBLEM 5.16. Show that the action of $\mathrm{Aut}_X(Y)$ is free and proper in the sense as in xxx. Show that it acts transitively on each fibre. ✷

5.4.1 Coverings of Riemann surfaces

Assume that X is a Riemann surface and that $p\colon Y \to X$ is a covering where Y for the moment is just a Hausdorff topological space. The analytic structure on Y is easily transported to Y in a canonical way so that the projection p becomes holomorphic. This is a very important result though it is almost trivial to prove.

Proposition 5.9 *Assume that X is a Riemann surface and that $p\colon Y \to X$ is a covering. Then there is unique analytic structure on Y such that p is holomorphic. In particular every Riemann surface has a universal cover that is a Riemann surface and the projection is holomorphic.*

Proof: Take any atlas $\mathcal{U}$ over X whose coordinate patches (U, z_U) are such that the opens U all trivialize p; that is, the inverse image $p^{-1}(U)$ decomposes in a disjoint union $\bigcup_{\alpha\in A} U_\alpha$ with each $\pi_{U,\alpha} : U_\alpha \to U$ being a homeomorphism. The atlas on X we search for, consists of all the U_α's for all the U's in $\mathcal{U}$ with the obvious choice of $z_U \circ p_{U,\alpha}$ for coordinate functions, and it turns out to be an analytic atlas. Indeed, on $U_\alpha \cap V_\beta$ one has

$$(z_U \circ p_{U,\alpha}) \circ (z_V \circ p_{V,\beta})^{-1} = z_U \circ p_{U,\alpha} \circ p_{V,\beta}^{-1} \circ z_V^{-1} = z_U \circ z_V^{-1}$$

since both $p_{U,\alpha}$ and $p_{V,\beta}$ are restrictions of same map p to $U_\alpha \cap U_\beta$.

It is obvious that the projection map p is holomorphic, contemplate the diagram below for a few seconds and you will be convinced:

$$\begin{array}{ccc} U_\alpha & \longrightarrow & z_U(U) \\ {\scriptstyle p_{U,\alpha}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\ U & \xrightarrow[z_U]{} & z_U(U) \end{array}$$

❑

(5.1) Recall that if Z is any simply connected space a map $Z \to X$ can be lifted to a map $Z \to \tilde{X}$ which is unique once the image of one point in Z is given. When Z is another Riemann surface and f is holomorphic, the lift will be holomorphic as well. We even have slightly stronger statement:

Proposition 5.10 *Assume that $p\colon Y \to X$ is a covering between Riemann surfaces and that $f\colon Z \to Y$ is a continuous map such that $p \circ f$ is holomorphic, then f is holomorphic.*

Proof: Again the hart of the matter is to choose an atlas compatible with the given data. Start with an atlas on Y whose coordinate neighbourhoods trivialize the covering p. For each U and each $z \in f^{-}1(U)$ there is patch V on Z centered at z and contained in $f^{-1}(V)$. And as Z i locally connected we can find such V's that are connected. Then $\tilde{f}(V)$ is contained in one of the $U_{U,\alpha}$'s, and one has $f|_V = p_{U,\alpha} \circ \tilde{f}|_V$. As $p_{U,\alpha}$ is biholomorphic in U_α this gives $\tilde{f} = f|_V \circ p_{U,\alpha}^{-1}$ implying that $\tilde{f}$ is holomorphic in V, and hence in Z since the V's cover Z. ❑

Problem 5.17. Check that in proposition 5.10 above, it suffices to assume that p be a local homeomorphism. ✷

Problem 5.18. Let Λ be a lattice in $\mathbb{C}$. Show that the projection map $\mathbb{C} \to \mathbb{C}/\Lambda$ is a universal cover for the elliptic curve $\mathbb{C}/\Lambda$. ✷

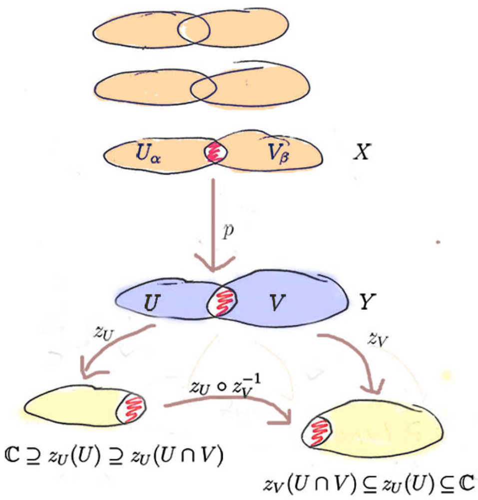
Problem 5.19. Let Λ be a lattice. A function is Λ-periodic if $f(z+\omega) = f(z)$ for all $\omega \in \Lambda$ and all $z \in \mathbb{C}$. Show that any holomorphic Λ-periodic function is constant. ✷

Example 5.3. We continue to explore the world of elliptic curves. In this example we study the holomorphic maps between two elliptic curves $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$, and shall show that they are essentially linear, that is induced by linear function $z \to az + b$ from $\mathbb{C} \to C$.

Let the $f\colon \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ be holomorphic. The salient point is that $p'\colon \mathbb{C} \to \mathbb{C}/\Lambda'$ is the universal cover of $\mathbb{C}/\Lambda'$, so that any holomorphic map from a simply connected Riemann surfaces into $\mathbb{C}/\Lambda'$ lifts to a holomorphic map into $\mathbb{C}$ by proposition 5.10. We apply this to the map $f \circ p$ and obtains a holomorphic function $F\colon \mathbb{C} \to \mathbb{C}$ that fits into the commutative diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{F} & \mathbb{C} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ \mathbb{C}/\Lambda & \xrightarrow[f]{} & \mathbb{C}/\Lambda'. \end{array}$$

Fix for a moment a member ω of the lattice Λ and consider the difference $F(z+\omega) - F(z)$. As a function of z it takes values in the discrete subset Λ' of $\mathbb{C}$. It is obviously continuous (even holomorphic), and hence it must be constant. Taking derivatives shows that $F'(z+\omega) = F'(z)$, so that the derivative is Λ-periodic, and from problem 5.19 we conclude that $F'(z)$ is constant. Hence $F(z) = az + b$. ❄



Figur 5.3: *Charts on a covering surface.*

5.5 Proper maps

Recall that a proper map between two topological spaces is a continuous map whose inverse images of compact sets are compact. A continuous map whose source space is compact, is automatically proper, and of course. Notice that the target space can be decisive for the map being proper or not; for instance, homeomorphisms are proper, but open embeddings[5] are usually not.

(5.1) Any proper, holomorphic maps between Riemann surfaces must have finite fibres. The fibres are discrete by proposition 5.4 on page 149 and as f is proper, they are compact as well.

Proper maps are always closed whether holomorphic or not. To see this, let b be a point in the closure of the image $f(A)$ of a closed a set $A \subseteq X$, and let $\{a_\nu\}$ be a sequence in A such that $\{f(a_\nu)\}$ converges to b. The subset $B = \{ f(a_n) \mid n \in \mathbb{N} \} \cup \{b\}$ of Y is compact. Hence the inverse $f^{-1}(B)$ is also compact because f is assumed to be proper. As $\{a_n\} \subseteq f^{-1}(B) \cap A$, there is a subsequence of $\{a_\nu\}$ converging to a point a in A, and by continuity, $f(a) = b$. We have thus proven

Proposition 5.11 *A proper, holomorphic map between two Riemann surfaces is closed and have finite fibres.*

[5] An open embedding is a map whose image is open and which is homeomorpic onto its immage.

Problem 5.20. Give an example of a smooth map between Riemann surfaces whose fibres are not all finite. Give an example of an open imbedding that is not proper. Give an example of an open imbedding (of topological spaces) that is proper, but not a homeomorphism. ✷

Problem 5.21. Show that the composition of two proper maps is proper. ✷

Problem 5.22. Assume that $f\colon X \to Y$ is proper and that $A \subseteq X$ is a closed, discrete set. Show that $f(A)$ is discrete. ✷

Problem 5.23. If $f\colon X \to Y$ is proper and $A \subseteq Y$ is closed, show that the restriction $f|_{X\setminus f^{-1}(A)}\colon X \setminus f^{-1}(A) \to Y \setminus A$ is proper. ✷

(5.2) Every covering map is a local homeomorphism by definition, but the converse is not true. A cheap example being an open immersion; that is, the inclusion map of an open set U in a space X. If U is not a component of X any point in the boundary of U will not have a trivializing neigbourhood. If you want a surjective example, there is an equally cheap one. Take any covering with more than two points in the fibres and remove one point from one of the fibres.

If the map in addition to being a covering also is a proper map, it will be a covering:

Lemma 5.3 *A proper, local homeomorphism $f\colon X \to Y$ is a covering map.*

Proof: Take any point $y \in Y$. The fibre $f^{-1}(y)$ is finite because f is proper. Round each point x in the fibre there is an open U_x which f maps homeomorphically onto an open V_x in Y. By shrinking these sets we may assume they are pairwise disjoint, *i.e.*, replace U_x with $U_x \setminus \bigcup_{x'\neq x} U_{x'}$ and notice that $x \notin U_{x'}$ if $x' \neq x$ since f is injective on $U_{x'}$.

The finite intersection $V = \bigcap_{x\in f^{-1}(y)} V_x$ is an open set containing y, and clearly the different sets $f^{-1}(V) \cap U_x$ for $x \in f^{-1}(y)$ are open, disjoint sets mapping homeomorphically onto V. ❑

5.5.1 The degree of a proper holomrphic maps

This section is about proper maps between Riemann surfaces and the cardinality of their fibres. Their fibres are finite, and case the map is a cover, all fibres have the same number of points as saw in prop xxx above. The theme of this paragraph is to extend this result to maps having branch points, however the branch points counted with a multiplicity which turns out to be equal to the ramification index $\operatorname{ind}_x f$.

Proposition 5.12 *Let $: X \to Y$ be a proper,holomorphic map between two Riemann surfaces. Then the number $\sum_{f(x)=y} \operatorname{ind}_x f$ is independent of the point $y \in Y$ and is called the degree of f. If f is not branched in any point in $f^{-1}(y)$, it holds true that $\#f^{-1}(y) = \deg f$.*

So let $f\colon X \to Y$ be a proper map. The points in X where the derivative $D_xF\colon T_{X,x} \to T_{Y,f(x)}$ vanishes are isolated points; indeed, locally in charts (U, z_U) of X and (V, z_V) on Y the function f is represented by the holomorphic function $\tilde{f} = z_V \circ f \circ z_U$, and the derivative $\tilde{f}'$ represents D_xf for $x \in U$. We know that $\tilde{f}'$ is holomorphic and hence has isolated zeros.

Hence the set $B = \{\, x \mid D_xf \,\}$ is a closed, discrete set in X called the *branch locus* or *ramification locus* of f. The image $f(B_f)$ is closed and discrete as well, our map f being proper, and on the open set $W = Xf^{-1}(f(B_f))$ the map f is unramified. Hence is a local homeomorphism there and since the restriction $f|_W\colon W \to Y \setminus f(B_f)$ is proper, it is covering by lemma 5.3 above.

Proposition 5.13 *Let* $: X \to Y$ *be a proper,holomorphic map between two Riemann surfaces. Then the number* $\sum_{f(x)=y} \operatorname{ind}_x f$ *is independent of the point* $y \in Y$ *and is called the* degree *of* f. *If* f *is not branched in any point in* $f^{-1}(y)$, *it holds true that* $\# f^{-1}(y) = \deg f$.

Proof: Let B be the branch locus of f and put $W = f^{-1}(f(B))$. Then $f|_W\colon W \to Y \setminus f(B)$ is a covering map. Moreover $f(B)$ being a discrete set, the complement $Y \setminus f(B)$ is connected, and by 5.8 on page 159 the number of points in the fibres $f^{-1}(y)$ is the same for all $y \in Y \setminus f(B)$.

So we pass to examining the situation round a fibre containing branch points. Let $f^{-1}(y) = \{x_1, \ldots, x_r\}$ and let $n_i = \operatorname{ind}_{x_i} f$, of course some of these can be one. By the local description of branch points (proposition 5.6 on page 152) we can find coordinate patches U_i with coordinate z_i round each x_i and V_i round y such that in the patch U_i one has $f(z_i) = z_i^{n_i}$.

Shrinking the U_i if necessary, they can be assumed to disjoint, and replacing V with the intersection $\bigcap_i V_i$, we can assume that $V = f(U_i)$ for all i. With this in place the inverse image $f^{-1}(V)$ decomposes as the union $\bigcup_i U_i$. Now, there are points y' in V such that the map f is unbranched over y', thence $\# f^{-1}(y')$ decomposes as $\sum_i \#(f^{-1}(y') \cap U_i)$. Clearly this sum equals $\sum_i n_i$, indeed, f is represented as $f(z_i) = z_i^{n_i}$ on the patches U_i and equations $z_i^{n_i} = \epsilon$ has n_i solutions. On the other hand all unbranched fibres have the same number of points, so we are through. ❑

Problem 5.24. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two proper holomorphic maps between Riemann surfaces. Show that the composition $g \circ f$ is proper and that one has $\deg g \circ f = \deg f \deg g$. ✷

Problem 5.25. Let $\Lambda \subseteq \mathbb{C}$ be a lattice. Show that for each integer n the map $z \to nz$ induces a proper map $[n]\colon \mathbb{C}/L \to \mathbb{C}/\Lambda$. Show that $[n]$ is unramified and determine its degree. Hint: Compute the derivative of $[n]$. ✷

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Contents

6. Del

Forms and integration

version $-\infty+\epsilon$ — Tuesday, November 1, 2016 12:12:30 PM
Very, very preliminary and incomplete version prone to mistakes and misprints! More under way.
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6.0.1 Differential forms

Let X be Riemann surface and let $V \subseteq X$ be an open set. By a 1-form on V we mean a function

$$\omega\colon U \to \bigcup_{a\in V} T_a^*,$$

where as usual T_a^* stands for the cotangent space of X at the point a.

When $U \subseteq V$ is a patch with coordinate z, the coordinate function z induces the cotangents dz and $d\overline{z}$ in every one of the cotangent spaces T_a^* with $a \in U$ and they form a basis for all theswe spaces; Loosely speaking they constitute a uniform or common basis for all the cotangent spaces in U. Therefore one may write

$$\omega|_U = pdz + qd\overline{z}$$

where p and q are functions in U. As its stands they are just set theoretical functions, but as the theory develops we shall impose different conditions on them. For instance, if they are smooth one says that the 1-form is smooth, and if both p and q are of class C^r, the form is of class C^r. In the other end of scale, they can be just Borel measurable in that case the form said to be measurable.

In an analogous way one introduces a 2-form as function

$$\omega\colon U \to \bigwedge^2 T_a^*.$$

Locally, in a patch (U, z), there is the basis $dz \wedge d\overline{z}$,so in the patch one has

$$\omega|_U = f dz \wedge d\overline{z}.$$

where p is a function, which of course may subjected to different kinds of regularity conditions. Now, suppose that (V, w) is another patch related to (U, z) with $w = w(z)$ as transition function on the intersection $U \cap V$. Then $dw = \partial_z w\, dz$ and $d\overline{w} = \partial_{\overline{z}}\overline{w}\, d\overline{z}$ In V the form ω has an expression $\omega|_V = g d\overline{w} \wedge dw$ and since the two expressions for ω must agree on the intersection, the following relation holds on $U \cap V$:

$$f = g \partial_z w\, \partial_{\overline{z}}\overline{w}. \tag{6.1}$$

PaaSnittet

AbsoluttVerdiTpForm (6.1) There is a slightly subtle observation we shall need later on when we define integrals. In short, every 2-form has an "absolute value"; a feature that is specific for forms of top degree on oriented manifolds. Since it holds that $\partial_{\overline{z}}\overline{w} = \overline{\partial_z w}$ the transition factor in (6.1) satisfies

$$\partial_z w\, \partial_{\overline{z}}\overline{w} = |\partial_z w|^2$$

and hence it is real and positive. This means that taking absolute values in (6.1) gives

$$|f| = |g|\, \partial_z w\, \partial_{\overline{z}}\overline{w}.$$

Hence there is real 2-form on X which we denote $|\omega|$ and which is shaped like

$$|\omega| = -\frac{i}{2}\,|f|\, d\overline{z} \wedge dz = |f|\, dx \wedge dy$$

on a patch where $\omega = f d\overline{z} \wedge dz$.

JacobianTwoForms

(6.2) The transition factor appearing in (6.1) above is nothing but the Jacobian determinant of the transition function effectuating the coordinate change. To see this, let $w = w(z)$ and name the real and imaginary parts of z as $z = x + iy$ and of w as $w = u + iv$. Then $\partial_z w = u_x + iv_x$ and $\partial_{\overline{z}}\overline{w} = u_x - iv_x$ from which we deduce the equalities

$$\partial_z w\, \partial_{\overline{z}}\overline{w} = u_x^2 + v_x^2 = u_x v_y - u_y v_x,$$

having the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ in mind.

(6.3) For the sake of a unified terminology, one frequently calls a function a 0-form, so we have 0-, 1- and 2-forms. On manifolds of higher dimension there are forms of higher degree, which we are not concerned about in this course.

6.0.2 Hodge decomposition

Dealing with analytic structures has certain advantages over just dealing with differentiable manifolds. There are several important structure that only materializes for analytic manifolds. One such structure is the *Hodge decomposition* of the vector space of forms.

Let E^1_X denote the vector space of 1-forms on X (of some regularity, but for the moment we are intentionally vague on precise requirement) The Hodge decomposition is a canonically decomposition of E^1_X in a direct sum

$$E^1_X = E^{1,0}_X \oplus E^{0,1}_X.$$

The forms belonging to the factors will be described locally, patch by patch, and it is quit remarkable that this gives a global decomposition. The subspace $E^{1,0}_X$ consists of forms which in the patches (U, z) only involve dz; that is, they have the shape $p\,dz$. This is independent of the coordinate used since if $z = z(w)$ is a holomorphic change of coordinates, one has $p\,dz = p\,\partial_w z\,dw$. Likewise, the space $E^{0,1}_X$ consists of forms locally of shape $q\,d\overline{z}$, and a similar argument as given for $E^{1,0}_X$ shows that neither this condition depends on the coordinate used; indeed, it holds that $d\overline{z} = \partial_{\overline{w}}\overline{z}\mathrm{d}\overline{w}$.

(6.1) There are two important operations one can perform on 1-forms. One is the usual complex conjugation, *i.e.*, if $\omega = p\,dz + q\,d\overline{z}$, the conjugate form $\overline{\omega}$ is given as

$$\overline{\omega} = \overline{q}\,dz + \overline{p}\,d\overline{z}.$$

Obviously complex conjugation interchanges the two summands $E^{1,0}_X$ ands $E^{0,1}_X$.

One says that a form ω is *real* if $\overline{\omega} = \omega$ and it is *imaginary* in case $\overline{\omega} = -\omega$. Inspecting the form ω in a chart, one finds that it is real when and only when it has the shape $\omega = pdz + \overline{p}d\overline{z}$. In the real coordinates given by $z = x + iy$ a real form ω is expressed as $\omega = 2udx + 2vdy$, where u and v are respectively the real and the imaginary part of p. Similarly, an imaginary form ω has the shape $\omega = pdz - \overline{p}d\overline{z}$ in a patch, and thus looks like $\omega = 2i(vdx + udy)$ in the real coordinates.

(6.2) The other operation is the so called *Hodge $*$-operation*. In contrast to the complex conjugation, it preserves the Hodge decomposition. The two spaces $E^{1,0}_X$ and $E^{0,1}_X$ are the eigenspaces of $*$; the former corresponds to the eigenvalue $-i$ and the latter to eigenvalue i. In local coordinates one thus has HodgeDualDefinisj

$$*(p\,dz + q\,d\overline{z}) = -ip\,dz + iq\,d\overline{z}. \tag{6.2}$$

This might appear fortuitous, but in expressed in real coordinates it is more transparent from where it comes. If the form is given as $\omega = p\,dx + q\,dy$, a small computation shows that thew Hodge-dual equals $-q\,dx + p\,dy$, and this is the familiar conjugate differential from complex analysis and theory of harmonic functions. The $*$-operation is not involutive, but satisfies $**\omega = -\omega$. hodge*

6.0.3 Exterior derivations

At the same time as we introduced the cotangent space, we introduced the differential df of a function f, which necessarily must be of class C^1. In a patch (U, z) it takes the form $df = \partial_z f\, dz + \partial_{\overline{z}} f\, d\overline{z}$. One may think about it as an avatar of the gradient of f we know from calculus courses.

In the setting of differential forms a function is considered a form of degree zero, and there is a construct for forms of higher degree similar to what we just did for functions. The exterior derivative of an n-form is an $n+1$-form, so in our setting, where no non-zero 3-forms exists, the exterior derivative of a 2-form is forced to be zero, and we need only care about the derivative of 1-forms.

(6.1) We shall work 1-forms on an open subset U of a Riemann surface X, so let ω be one, and assume that ω is of class C^1; that is, it is the coefficient functions in every patch are continuously differentiable. Its *exterior derivative* $d\omega$ is a 2-form that locally, in a patch where $\omega = pdz + qd\overline{z}$, by the formula

$$d\omega = (\partial_{\overline{z}} p - \partial_z q) d\overline{z} \wedge dz. \tag{6.3}$$

extder

As usual when giving a defining a form patch by patch, it must verified that the form does not depend on the patch. Expressing the form in another coordinate and applying the defining formula 6.3 must lead to the same form. So assume that $z = z(w)$ is a holomorphic change of coordinates. Then one has $dz = \partial_w z\, dw$ and $d\overline{z} = \partial_{\overline{w}} \overline{z}\, d\overline{w}$, and in the new coordinate the expression for ω becomes

$$\omega = p\, \partial_w z\, dw + q\, \partial_{\overline{w}} z\, d\overline{w},$$

and applying the recipe 6.3 to it one arrives at the expression

$$d\omega = \big(\partial_{\overline{w}}(p\, \partial_w z) - \partial_w(q\, \partial_{\overline{w}} z)\big)\, d\overline{w} \wedge dw. \tag{6.4}$$

ExtDerv2 Substituting $dz = \partial_w z\, dw$ and $d\overline{z} = \partial_{\overline{w}} \overline{z}\, d\overline{w}$ in the formula (6.3) one obtains the identity

$$d\omega = (\partial_{\overline{z}} p - \partial_z q) \partial_{\overline{w}} \overline{z}\, \partial_w z\, d\overline{w} \wedge dw,$$

whose right side is identical to the right side of equation (6.4) in view of the equalities $\partial_{\overline{w}}(p\, \partial_w z) = \partial_{\overline{w}} p\, \partial_w z = \partial_{\overline{z}} p\, \partial_{\overline{w}} \overline{z}\, \partial_w z$ and $\partial_w(q\, \partial_{\overline{w}} \overline{z}) = \partial_w q\, \partial_{\overline{w}} \overline{z} = \partial_z q\, \partial_w z\, \partial_{\overline{w}} z$. Notice that we draw on the coordinate change being holomorphic. This implies that $\partial_{\overline{w}} \partial_w z = \partial_w \partial_{\overline{w}} \overline{z} = 0$ since z depends holomorphically on w whereas $\overline{z}$ is anti-holomorphic in w.

(6.2) The exterior derivative of an exact form df vanishes. Indeed locally one may express df as $df = \partial_z f\, dz + \partial_{\overline{z}} f\, d\overline{z}$ and one finds

$$d^2 f = (\partial_{\overline{z}} \partial_z f - \partial_z \partial_{\overline{z}} f) d\overline{z} \wedge dz = 0,$$

as $\partial_{\overline{z}}\partial_z f = \partial_z\partial_{\overline{z}} f$. The formula below, which one may think of as an analogue to Leibnitz' rule for the derivative of product, is sometimes useful when dealing with forms. It is straightforward to verify it, a verification that is left to the zealous student.

$$d(f\omega) = df \wedge \omega + f d\omega. \tag{6.5}$$

usefulformel

(6.3) The exterior derivative decomposes as a sum $d = \partial + \overline{\partial}$ where $\overline{\partial}$ is the restriction of d to the Hodge factor $E_X^{1,0}$ and ∂ up to sign the restrictions to $E_X^{0,1}$, so they are given locally as

$$\overline{\partial}(p\,dz + q\,d\overline{z}) = \partial_{\overline{z}} p d\overline{z} \wedge dz \qquad \partial(p\,dz + q\,d\overline{z}) = -\partial_z q d\overline{z} \wedge dz.$$

(6.4) To have a unified notation, we shall henceforth denote the derivative of a function—that is of a zero form— with ∂f in stead of $\partial_z f\,dz$ and with $\overline{\partial} f$ in stead of $\partial_{\overline{z}} f\,d\overline{z}$. Thence one has the decomposition $d = \partial + \overline{\partial}$ for zero forms as well. The following formulas are straightforward to verify by simple calculations

$$d^2 = \partial^2 = \overline{\partial}^2 = 0,$$

and

$$\partial\overline{\partial} = -\overline{\partial}\partial$$

(6.5) The operator $\partial\overline{\partial}$ is up to a constant factor a global version of the Laplacian. Indeed, locally in a chart one computes

$$2\partial(\overline{\partial} f) = 2\partial(\partial_{\overline{z}} f\,d\overline{z}) = 2\partial_z\partial_{\overline{z}} f\,dz \wedge d\overline{z} = -i\Delta f dx \wedge dy \tag{6.6}$$

since (we recall) LaplacianDBarD

$$4\partial_z\partial_{\overline{z}} = (\partial_x - i\partial_y)(\partial_x + i\partial_y) = (\partial_x^2 + \partial_y^2) = \Delta$$

and $d\overline{z} \wedge dz = (dx - idy) \wedge (dx + idy) = 2idx \wedge dy$. Another formula in this direction whose flavour is more that of real forms is the following

$$d(*df) = \Delta f dx \wedge dy \tag{6.7}$$

To establish is, write $df = f_x dx + f_y dy$, Then $*df = -f_y dx + f_x dy$, and hence one finds Harmonicd*d
$d(*df) = -f_{yy} dy \wedge dx + f_{xx} dx \wedge dy = \Delta f dx \wedge dy$.

(6.6) These two derivations are maps

$$\overline{\partial}\colon E_X^{1,0} \to E_X^{1,1} \quad \text{and} \quad \partial\colon E_X^{0,1} \to E_X^{1,1}.$$

an

$$\overline{\partial}\colon E_X^{0} \to E_X^{1,0} \quad \text{and} \quad \partial\colon E_X^{0} \to E_X^{0,1}.$$

6.0.4 Holomorphic and harmonic forms

Our main interest are the holomorphic forms on a Riemann surface, in some sense they are the integrands in the integrals we are interested in. And under favourable circumstances, they can be integrated to give us holomorphic functions on the surface.

(6.1) A 1-form is called *holomorphic* if it locally in patches (U, z) has the shape $p\,dz$ where p is a holomorphic function of z. Every holomorphic 1-form is closed. Indeed, in a patch one has $d(f dz) = \partial_{\bar z} f\, d\bar z \wedge dz$, and so $d(f dz)$ vanishes if and only if $\partial_{\bar z} f = 0$; that is, if and only if f is holomorphic. This argument shows as well that a closed form is holomorphic if and only if $\overline{\partial}\omega = 0$, which also is equivalent to the two equations $\partial\omega = \overline{\partial}\omega = 0$ since $d = \partial + \overline{\partial}$.

EXAMPLE 6.1. There are no holomorphic one-forms on the Riemann sphere. Indeed, assume that ω was one. Om the patch $U_0 = \hat{\mathbb{C}} \setminus \{\infty\}$ one would have $\omega|_{U_0} = f(z)dz$ where f is an entire function, and om the patch $U_\infty = \hat{\mathbb{C}} \setminus \{0\}$ the form ω would have the shape $\omega|_{U_\infty} = g(w)dw$ where g is an entire function of w. On the intersection $U_0 \cap U_\infty$ one has $dw = -z^{-2}dz$, and hence

$$f(z)dz = -g(z^{-1})z^{-2}dz,$$

and therefore one has

$$z^2 f(z) = -g(z^{-1})$$

in $\mathbb{C} \setminus \{0\}$. This is clearly impossible; it would imply that $g(z^{-1})$ has a removable singularity at 0 with the value zero. But it is regular at $\{\infty\}$ as well, so by Liouville it would be constant and equal to zero. ❄

HarmonicFormsII (6.2) The one-form ω is said to be *harmonic* if locally it is the differential of a *harmonic function*; that is, the surface X has an atlas so that for every patch U one has $\omega|_U = df$ for a harmonic function in U. In view of equation 6.7 above a harmonic form ω is at the same time both closed and co-closed; that is, it fulfils the two conditions $d\omega = d{*}\omega = 0$.

With some very mild integrability conditions on the form ω the converse also holds. This is a deep theorem that we certainly come back to; indeed, it is the hub of this chapter. The proof hinges on a famous theorem of Hermann Weyl, the so called "Weyl's lemma".

Locally in a patch (U, z) there is an expression $\omega = pdz + qd\bar z$ for ω, and by a small computation one arrives at the following identity

$$d{*}\omega = i(\partial_{\bar z} p + \partial_z q) d\bar z \wedge dz$$

for $d{*}\omega$. Combining this with the expression $d\omega = (\partial_{\bar z} p - \partial_z q) d\bar z \wedge dz$ for the derivative $d\omega$, one sees that ω being closed and co-closed is equivalent to $\partial_{\bar z} p = \partial_z q = 0$. In other words, it holds that the functions p and q are respectively holomorphic and anti-holomorphic. In terms of the Hodge decomposition of $\omega = \alpha + \beta$ of ω, the $(1, 0)$-part α of ω is holomorphic and the $(1, 0)$-part β is anti-holomorphic.

(6.3) Assume that $\omega = df$ is an exact form. Then locally $df = pdz + qd\bar{z}$ with $p = \partial_z f$ and $q = \partial_{\bar{z}} f$. From the discussion above we deduce the formula

$$d(*df) = 2i\partial_{\bar{z}}\partial_z f d\bar{z} \wedge dz = 2^{-1}\Delta f dx \wedge dy,$$

and conclude that df is a harmonic form if and only if both the real and the imaginary part of f are (real) harmonic functions; in short, if f is harmonic.

The converse of this holds locally. One has

ClosedCoClsedHar

Proposition 6.1 *Let ω be a 1-form of class C^1. Assume it is closed and co-closed. Then ω is locally of the shape df for f a harmonic function; that is, it is harmonic.*

Proof: Since ω is a closed form it is locally exact by the Poincaré lemma (lemma 6.1 on page 179 below), and so for patches (U, z) of an atlas on X, one has $\omega|_U = df$ where f is a C^2 function in U. Now ω is co-closed so $d(*df) = 0$, but this is exactly the condition $\Delta f = 0$, so f is harmonic. ❑

6.1 Integration

The late danish mathematician Birger Iversen said once that in terms of junk food a 1-form feeds spaghetti and a 2-form feeds on pizza (and he added, a 3-form feeds on hamburgers). There will be no hamburgers on our menu, but as a solace for the lovers of exotic junk food we shall resort to a quadruple integral in the course of proving the so called Weyl's lemma about harmonic functions.

In more serious terms, this means that a 1-form can be integrated along a path and a 2-form over a surface. The two are treated somehow differently in this text. Although it is feasible to integrate 1-forms along non-compact (read infinite) paths, we concentrate on compact paths, *i.e.*, those parametrized over finite intervals. They will be sufficient for our needs.

Concerning surface integrals, however, we shall frequently be using improper integrals. The Riemann surfaces we study are not necessarily compact and it is paramount to be able to integrate 2-forms over the whole surface. Of course just as with traditional improper integrals, not all forms have a finite integral so we need a concept of integrable forms. At a few but crucial moments we use forms that are not continuous in an essential way, but surely, they will be locally integrable.

6.1.1 Line integrals

Recall that a path γ in X is a piecewise continuosuly differentiable map $\gamma\colon I \to X$ where I is an interval $[a, b]$. In what follows we shall give menaing to the integrtal $\int_\gamma \omega$ where Ω is a 1-form on X.

There are two steps in the definition, the first being the case when the path γ is entirely contained in a coordinate patch U with coordinate z. We are then in the familiar situation with a line integral of a form in an open disk in the complex plane, and writing the form ω as $\omega = pdz + qd\bar{z}$ in U, the integral is given as

$$\int_\gamma \omega = \int_a^b (p(\gamma(t))\gamma'(t) + q(\gamma(t))\overline{\gamma(t)}')dt.$$

(Strictly speaking the integration takes place along the path $z \circ \gamma$ in $\zeta(U)$). Any change of coordinate in D brings along a corresponding reparametrization of the path, and a straightforward application of the formula for the change of variable in an integral, shows that the integral does not depend on the choice of coordinate.

In the second step, where γ can be any path piecewise of class C^1 in X, one chooses a finite open covering D_i of the image $\gamma(I)$ by coordinate disks which can be done since $\gamma(I)$ is compact. Applying Lebesgue's lemma one finds a partition $\{t_i\}$ of the interval I such that every one of the subintervals $[t_{i-1}, t_i]$ is contained in the inverse image $\gamma^{-1}(D_j)$ of one of the D_j's. Let the restriction of γ to $[t_{i-1}, t_i]$ be denoted by γ_i. The integral of ω along γ_i is well defined by what we did in step one, and of course, we put

$$\int_\gamma \omega = \sum_i \int_{\gamma_i} \omega.$$

It remains to be checked that the integral neither depends on the choice of covering nor on the choice of partition. This is small exercise involving a common refinement of the two partitions, whose details are left to the zealous students to fill in.

(6.1) The integral of an exact form df is just the difference of the values f takes the end points of γ. In particular the intregral does not depend on the path γ as longs the end points are fixed. If f is a function of class C^1 one has

$$\int_\gamma df = f(b) - f(a),$$

where γ is a path from a to b.

We have seen several instances of the converse being true when doing function theory in the complex plane, and this holds true also for Riemann surfaces, and the proof is the same of course with the necessary adjustments to notation and wording.

KlassikExact

Theorem 6.1 *Let ω be a continuous 1-form on X. Assume that the integrals of ω around closed paths all vanish. Then ω is exact, i.e., there is a function f of class C^1 with $df = \omega$.*

PROOF: Let x_0 be a fixed point in X, and letx be any point in X and γ_x any path from x_0 to x. If γ_x' is another one, the composite $\gamma_x' \cdot \gamma_x^{-1}$ is a loop at x_0 and by the hypothesis that integrals of ω around loops vanish, it holds true that

$$0 = \int_{\gamma_x' \cdot \gamma_x^{-1}} \omega = \int_{\gamma_x'} \omega - \int_{\gamma_x} \omega,$$

and the integral of ω along a path leading from x_0 to x has the same value whatever the path is. Hence putting

$$f(x) = \int_{\gamma_x} \omega$$

gives us a well defined function on X. A variant of a familiar argument shows that $df = \omega$. Indeed, let D be a coordinate disk centered at x, and let h be a complex number with that $x + h \in D$. Choose a path γ_x joining x_0 to x and let l be the line segment from x to $x + h$, then the composite $l \cdot \gamma_x$ is a path from x_0 to $x + h$. We find using the parametrisation $x + th$ with $0 \leq t \leq 1$ of l that

$$f(x+h) - f(x) = \int_l \omega = h \int_0^1 p(x+th)dt + \overline{h} \int_0^1 q(x+th)dt.$$

Since p and q are continuous, the two integrals tend respectively to $p(x)$ and $q(x)$ as h tends to zero. Letting h approch zero through real values gives $\partial_x f = p + q$, and when h approches zero trough imaginary values one finds $\partial_y f = ip - iq$. In view of the equalities $2\partial_z = \partial_x - i\partial_y$ and $2\partial_{\overline{z}} = \partial_x + i\partial_y$ this implies that $\partial_z f = p$ and $\partial_{\overline{z}} f = q$. ❑

The integrals $\int_\gamma \omega$ of ω around closed loops γ are traditionally called the *periods* of the form ω, hence the name "theorem of vanishing periods" for the theorem. They played prominent role when computing integrals where high tech (and they still do even if the center of mass of the theory has shifted somehow); the periods of ω determine all the integrals $\int_\gamma omega$ where γ is any path, and gives a grip on the ambiguity of integrals $\int_{\gamma_x} \omega$ where γ_x joins a base point x_0 to x.

(6.2) The following corollary is a special case of general principle which one normally contributes to Henri Poincaré, hence it is frequently called *the Poincaré lemma.*

Corollary 6.1 *A closed* 1*-form* ω *is locally exact.* PoincareLemma

Proof: It suffices to prove that ω is exact over any disk D in X. One may define a function f in D by integrating ω along the ray joining the origin of D to x. On a triangle with corners 0, x and $x + h$, Stokes' theorem holds and since ω is closed, it follows that

$$f(x+h) - f(x) = \int_l \omega.$$

The rest of the proof is word for word the same as the last part of the proof of **??** above. ❑

(6.3) The following is a fundamental result. It is a variant for surfaces of one of the main result in the calculus of forms on manifolds. The proof is *mutatis mutandis* the same as the one we gave of 1.16 on page 53, just a few obvious changes in the notation are necessary.

Theorem 6.2 *Let γ and γ' be two homotopic paths in X. Assume that either there is a homotopy between them fixing the end points, or that both paths are closed. Let ω be a continuous closed 1-form in X. Then it holds true that* HopotopiInvarInteg

$$\int_\gamma \omega = \int_{\gamma'} \omega.$$

In particular, integrals of closed forms along null-homotopic loops vanish. If you are suspicious about integrating along constant paths, you can argue by dividing a closed loop in two $\gamma = \gamma \cdot \gamma'$. Then $\gamma \sim \gamma'$, and

$$\int_\gamma \omega = \int_\gamma \omega + \int_{\gamma'} \omega = 0$$

since by the theorem

$$-\int_\gamma \omega = \int_{\gamma^{-1}} \omega = \int_{\gamma'} \omega.$$

On a simply connected Riemann surface all loops are null-homotopic and the following corollary holds true:

Corollary 6.2 *Every closed 1-form on a simply connected Riemann surface is exact.*

In partr icular this applies to the universal covering $\tilde{X}$ of a Riemann surfaace X.

6.1.2 A classical view

This connects up to the classical view on Riemann surfaces associated to so called multivalued functions.

Let Ω be a domain in the complex plane, and let $\mathcal{U}$ be a covering of Ω by open sets. Assume given a holomorphic function f_U for each $U \in \mathcal{U}$; this is what the old-timers called a "function element". Assume further that on the intersections $U \cap V$ of very pair of opens U and V form $\mathcal{U}$, the derivatives of the functions f_U and f_V coincide. This amounts to the differences $f_U - f_V$ all being constant. So the functions do not glue together to make a function in Ω, but their derivatives patch up to a global holomorphic 1-form ω in Ω!

Now comes the salient point. The pullback to $\tilde{\Omega}$ is exact! So there is a function, unique up to an additive constant, such that $dF = pi^*\omega$. In opens $U' = \pi^{-1}(U) \subseteq \tilde{\Omega}$ where π is biholomorphic, one has $f_U \circ \pi|_{U'} = F|_{U'}$ and F is in some sense a global realization of the function elements $\{f_U\}$.

6.2 Paths and homotopy

Every integral has two parts which are equally important. There is an integrand and a path of integration. In many text books on Riemann surfaces the integrands —may be rightfully—receive most of the attention and the paths often come in the background. They are however important and they pose some subtle issues not to complicated to resolve, but some times requier some fiddling work.

We treat mostly paths and homotopy and touch the concept of chainsm for two reasons. The students all have a mandatory course in general topoly the the basics of paths and homotopy are tretaed, but not everyone has a course in algebraic topology where homology os done. Covering spaces and the universal coverings are fundamental concepts in the theory and of course they they are based on paths and homotopy.

6.2.1 Normal forms

The elements of $\pi_1(X)$ are homotopy classes $[\gamma]$ of continuous paths, but continuous paths can be extremely complicated. The Peano curve for instance, fills up a square and the Osgood curve is a Jordan curve with a positive area. But the world is so well shaped, that in every homotopy class there are well behaved paths. Every class has smooth and regular representatives, and can be factored in a product of classes having simple representatives, that is paths without self-crossings.

(6.1) We start making paths piecewise "linear" in the following sense:

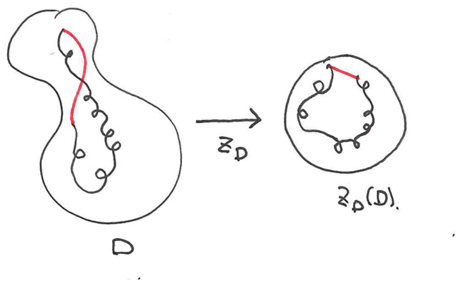
Let (D, z) be a coordinated disk, and let $L' \subseteq z(D)$ be a line segment. The inverse image $L = z^{-1}(L')$ is called a *linear segment* in D. Any path γ contained in D is homotopic to a linear segment having shearing end points a and b with γ. Indeed, the genuine disk $z(D)$ in the complex plane is convex and hence $F(s,t) = sz_D \circ \gamma + (1-s)L'$ is a homotopy in $z(D)$ fixing end points between the image of γ and the line segment joining $z(a)$ to $z(b)$. Bringing things back to the Riemann surface gives us the homotopy $z^{-1} \circ F(s,t)$ from γ to L in D.

A path is *piecewise linear* if it can be decomposed into a sequence of linear segments. Notice that we do not require these segments to be disjoint, so there can be *multiple point* on the path, *i.e.,* points through which pass several of the linear segments. If only two linear segments pass by the point we call it a *double point*.

Lemma 6.1 *Any γ is homotopic to a piecewise linear path whose only multiple points are double points.* Lineare veier

Proof: Let γ be a continuos path with the parameter running in the interval I. Pick a finite covering of the image $\gamma(I)$ by disks and let $\{t_j\}$ be a partition of I such that the image of each subinterval $[t_{i-1}, t_i]$ lies in D_j for some j. By what we just said, the restriction of γ to $[t_{i-1}, t_i]$ is homotopic in D to a linear segment. Performing these homotopies for each subinterval in succession closes the argument for the first part of the statement.

In a disk, if more than two line segments pass through the center, just move the end points of the segments slightly, and the multiple point resolves into a bunch of double points. There is, however, a rate of exchange; a point by which the path passes n times is worth $n(n-1)/2$ double points. ❑



Problem 6.1. In a disk any path is null-homotopic. Explain why the above proof does not show that any path on X is null-homotopic. ✱

Problem 6.2. Prove the statement that an n-tuple point can be deformed into $n(n-1)/2$-double points. ✱

Smoothing out the corners where two linear segments meet is just a matter of some elementary manipulation with smooth functions. One replaces a small part of the corner by a smooth bend matching the two linear segments to the second order. We leave the details to the ever-zealous students as a sequence of exercises (problems 6.5–6.7 below).

(6.2) Recall that one says that a smooth path γ is *regular* if the derivative of γ never vanishes. When a path is without multiple points—that is $\gamma\colon I \to X$ is injective—one calls it *simple* and if additionally it is closed, it is a *Jordan path*. A regular path has at any point a well defined tangent vector pointing in the forward direction. Being smooth and regular is a property local in the parameter interval, so it is not excluded that smooth regular curves have self-intersections, just as the piecewise linear paths can have. However, the self-intersections can be reduced to double points by a homotopy as the next lemma shows. The proof consists of smoothing the corners of the piecewise linear paths one finds by lemma 6.1 above.

GlattRegHomotp **Lemma 6.2** *Any path in X is homotopic to a smooth, regular path whose self intersections are just double points.*

(6.3) Bad examles Smooth paths can behave rather badly. They can have infinitely many self intersections, (a path can for instance come back on itself even if it is smooth), and there can be points through which the path passes infinitely often. Lemma 6.2 shows that pathologies of this type can be removed by a homotopy, so they will not bother us much in the future, but it is worthwhile having in the back of the mind that even the smooth world can be exotically complicated.

Example 6.2. Let $f(t)$ be a smooth real function of a real variable t all whose derivatives vanish at the origin. A path γ in $\mathbb{C}$ with parameter interval $[-1/\pi, 1/\pi]$ is given by

$$\gamma(t) = \begin{cases} (t^2, f(t)\sin 1/t) & \text{when } t \in [0, 1/\pi] \\ (t^2, 0) & \text{when } [-1/\pi, 0] \end{cases}$$

Then γ is a closed smooth path and for any natural number n the parameter values $1/n\pi$ and $-1/n\pi$ give the same point in $\mathbb{C}$. ❄

Example 6.3. Let $f(t)$ be as in the previous example. Define a path $\gamma(t)$ by giving it in polar coordinates as $\gamma(t) = (f(t)\sin 1/t)e^{it}$, with $t \in [-1/\pi, 1/\pi]$. Then γ is smooth and passes infinitely often by the origin. ❄

Problem 6.3. Show that there is a smooth path going to and fro along the interval $[0, 1]$ infinitely often. ✷

(6.4) Double points can not always be removed by a homotopy. The simples example being a figure eight in the complex plane with to points removed, one inside each bend. The figure eight F is parametrized as you write it, one bend traversed clockwise and the other counterclockwise. This figure eight is what the topologist call a deformation retract of X implying that $\pi_1(F) = \pi_1(X)$, and this group is free on two generators, *i.e.,* isomorphic to $\mathbb{Z} * \mathbb{Z}$. It is a result in topology that a deformation retract stays a deformation retract in a homotopy, so the figure eight can not be homotopic to a simple closed curve since the latter is topologically a circle with fundamental group equal to $\mathbb{Z}$.

However any homotopy class can be factored as the product of finitely many homotopy classes each one containing a smooth and regular Jordan path.

Lemma 6.3 *Any loop γ is homotopic to a finite product $\gamma_1 \ldots \gamma_2$ of loops that are smooth, regular and simple.*

Proof: The proof goes by induction on the total number of self crossings the path has. We define with reparametrisation the path such that the initial point of the path is a multiple point, and such that the parameter interval is $[0, 1]$.

Let s be the time when γ first comes back to $\gamma(0)$; that is, the first parameter value for which $\gamma(s) = \gamma(0)$. We denote by γ' the first loop of γ; that is the path obtained by confining the parameter to $[0, s]$. This clearly a simple path.

Let and γ'' be the rest of ,path with the parameter running in $[s, 1]$. Then we have the factorization $\gamma = \gamma' \cdot \gamma_1$ and the path γ' is closed since $\gamma'(s) = \gamma(0) = \gamma(1) = \gamma''(1)$. The path γ'' has one crossing less than γ and by induction we can factorize $[\gamma'']$ as in the lemma, and after shaping up the first loop γ' by a smoothing and a regularization, we are through. ❑

6.2.2 Tubular neighbourhoods or bands

Every closed, smooth, regular and simple path γ has a *tubular neighbourhood*. On a surface this is a narrow band surrounding the path—on a three dimensional manifold it would be a tube, hence the name. Since X is orientable the band decomposes into two connected components when the path γ is removed; there is one part to the left of the path and one to the right (no orientable surfaces contains Möbius bands).

To construct a band B surrounding γ we choose a partition $\{t_i\} = \{t_0, \ldots, t_r\}$ of the parameter interval I with γ mapping the subintervals $[t_{i-1}, t_i]$ into patches, and as usual we let $\gamma_i = \gamma|_{[t_{i-1},t_i]}$. In the patches the paths γ_i have non-vanishing tangent vectors $T_i(t)$ and hence non-vanishing normals $N_i(t)$ pointing to the left. Where the paths γ_i and γ_{i+1} meet—that is, at the point with parameter among the t_i-s—there are two normals, one defined using γ_i and one using γ_{i+1}. Since the tangent at $\gamma(t_i)$ is non-zero, they have the same direction, but *A priori* there is no reason they should have the same length. However, multiplicating the normals by appropriate smooth functions one can make them agree.

In the patch where γ_i lives, the band B_i consists of the points with $\gamma_i(t) + uN_i(t)$ and with $|u| < \epsilon$, and since the the two normals N_i and N_{i+1} coincide at the points t_i separating the subintervals, they match up to a band B. The boundary ∂B has two components, both are closed, regular and smooth. If they are oriented in the canonical way with B to the left, one is freely homotopic to γ and the other one to $-\gamma$.

Problem 6.4. Convince yourself that this works, remember that the path is closed. Hint: Let $N_1(t)$ and $N_2(T)$ be parallel vector fields on the regular curve γ and $\gamma(t_0)$ a point. Then there is a positive smooth function κ with $1-\kappa$ supported in a prescribed small interval round t_0 such that $N_1 = \kappa(t)N_2$. Finish off by induction on i, with special care since $\gamma(t_r) = t(\gamma_0)$. ✱

Smooth1 **Problem 6.5.** Given two positive numbers d_1 and d_2 with $d_2 > d_2$. Given four real numbers a_1, a_2 and b_1, b_2. Show that there exists an increasing functions u of class C^∞ with $u(d_i) = a_i$, $u'(d_i) = b_i$ and $u^{(j)}(d_i) = 0$ for $j \geq 2$ and $i = 1, 2$. ✱

Smooth2 **Problem 6.6.** Let γ be a path in a disk D centered at the origin in $\mathbb{C}$. Assume that γ is parametrized by $[-1, 1]$ and consists of two different line segments meeting at the origin. Show that γ is homotopic to a regular, smooth path δ with $\delta(t) = \gamma(t)$ for $|t| \geq \epsilon$ for any preassigned positive number ϵ. ✱

smooth3 **Problem 6.7.** Prove lemma 6.2. ✱

Problem 6.8. Let X be the elliptic curve $\mathbb{C}/\Lambda$ where Λ is a square lattice generated by 1 and i, which also is called the lemniscate lattice. Let $\pi\colon \mathbb{C} \to X$ be the parametrization (*i.e.,* the quotient map).

a) Show that the image in X of the line parametrized as $(t, \alpha t)$ is a closed curve if and only if α is a rational number. Let $\gamma_1(t) = \pi(t, 0)$ and $\gamma_2(t) = \pi(0, t)$.

b) Show that the image of the line $(t, \alpha t)$ is homotopic to the path $\gamma_1^p \circ \gamma_2^q$ when $a = p/q$ is the reduced representation of α as the quotient of two natural numbers.

c) Show that any closed curve on X is homotopic to one of the curves $\gamma_1^p \circ \gamma_2^q$.

✱

6.2.3 De Rahm theory

In 1931 the Swiss mathematician George De Rahm proved a fundamental theorem that connects the spaces smooth, real differential forms on a manifold with the cohomology groups of the manifold with coefficients in $\mathbb{R}$, and there is a version involving complex forms and cohomology with complex coefficients that follows immediately from the real case. Several proofs are around, and you can find many good references. A nice introduction to general the De Rahm theory is in

www1.mat.uniroma1.it/people/piazza/deRham-thm.pdf

(6.1) To give a taste of the bigger theorem, we shall formulate De Rahm's theorem for 1-forms on Riemann surfaces but merely sketch a proof for the easy part of it. This is in fact a fundamental and important result being the connection between the purely topological invariant $\pi_1(X)$ of the Riemann surface X and the analytical invariant $H^1_{DR}(X)$. More precisely, it is the dual group $\operatorname{Hom}(\pi_1(X), \mathbb{C})$ of groups homomorphisms from the fundamental group into the complex numbers that relates to the De Rahm $H^1_{DR}(X)$. By a handful of theorems in algebraic topology, this dual group is isomorphic to the cohomology group $H^1(X, \mathbb{C})$, but for us the relation with $\pi_1(X)$ is quit satisfactory.

The compact, connected and oriented smooth manifolds of dimension two are completely classified, and these are exactly the underlying smooth manifolds of the compact Riemann surfaces. Up to diffeomorphism there is one such manifold X_g for each natural number number g—the famous genus.

The fundamental groups $\pi_1(X_g)$ are all well known. They are not very complicated groups (although infinite and non-abelian), but we do not dive into a closer description. For us the important thing is that $\operatorname{Hom}_{\mathbb{C}}(\pi_1(X_g), \mathbb{C})$ is a vector space of complex dimension $2g$.

We are going to see that the Hodge-decomposition of E^1 induces a decomposition of $H^1_{DR}(H)$ into two spaces $H^{1,0}$ and $H^{0,1}$ both of dimension g.

(6.2) The De Rahm group $H^1_{DR}(X)$ we are most concerned about is defined as the middle cohomology of the complex below (called the De Rahm complex, by the way) where E^i_X stands for the vector space of smooth, complex i-forms:

$$0 \longrightarrow E^0 \xrightarrow{d} E^1 \xrightarrow{d} E^2 \longrightarrow 0$$

That is $H^1_{DR}(X) = \operatorname{Ker} d / \operatorname{Im} d$, or in words the vector space of closed smooth and complex forms modulo the subspace of exact smooth and complex forms. Of course one puts $H^0_{DR}(X) = \operatorname{Ker} d_0$ and $H^2_{DR}(X) = \operatorname{Coker} d_2$. Since our Riemann surfaces are connected by convension, the group $H^0_{DR}(X)$ reduces to $\mathbb{C}$; indeed if $df = 0$ the function f must be constant. The space $H^2_{DR}(X)$ consists of all 2-forms modulo the exact ones. This a more suble space, which we may be come back to.

(6.3) The main theorem in the previous paragraph shows that the integral $\int_\gamma \omega$ is constant on the homotopy class $c = [\gamma]$ containing γ. Of course this statement must be taken with a small grain of salt only being meaningful for the representatives of $[\gamma]$ being pointwise C^1, but lemma 6.1 on 181 save us.

The isomorphism in De Rahm's theorem comes from the most natural paring

$$\pi_1(X) \times H^1_{DR}(X) \to \mathbb{C},$$

namely the one defined by integration a form agains a path:

$$([\gamma], [\omega]) \mapsto \int_\gamma \omega.$$

Certainly one must verify that the integral $\int_\gamma \omega$ does not depend on the chosen representatives γ and ω. We proved in 6.2 that the integral $\int_\gamma \omega$ does only depend on the homotopy class of γ when ω is a closed form, and by it is a much simpler result that integrals of exact forms round loops vanish. So, indeed, the pairing is well defined.

The pairing induces a map

$$\Phi\colon H^1_{DR}(X) \to \operatorname{Hom}_{\mathbb{C}}(\pi_1(X), \mathbb{C})$$

which sends a class $[\omega]$ to the map sending $[\gamma]$ to the integral $\int_\gamma \omega$; that is $\Phi([\omega])([\gamma]) = \int_\gamma \omega$. Proposition **??** on page **??** tells us that if this map is identically zero, the form ω is exact, in other words the class $[\omega]$ vanishes. Hence Φ is injective, and we have proved half (confessedly, by far the easiest half) of the theorem:

Theorem 6.3 *The map Φ is an isomorphism $H^1_{DR}(X) \simeq \operatorname{Hom}_{\mathbb{C}}(\pi_1(X), \mathbb{C})$.*

6.2.4 Surface integrals

The aim of this section is to define the integral over X of a 2-form ω, generalizing the old acquaintances from calculus, the surface integrals. The Riemann surfaces X we are interested in are not all compact (and for the moment not even second countable) so we shall include improper integrals in the definition. This opens the way to L^2-spaces of forms, but the prise to pay is a definition with some nooks and the corners and some laborious checking.

Just as for line integrals there are two steps. In the initial step, which is the easy one, we define integrability and the integral of 2-forms supported in a coordinate patch. In the second step we resort to partitions of unity to extend the definition to 2-forms with some very mild restrictions on their support. The restrictions are kind of artificial and rooted in that the topology of X is not *a priori* second countable — we simply assume that the supports of the forms are second countable.

(6.1) We begin with the easy case that the two-form ω is supported in a coordinate patch U with coordinate z. So we identify U and $z(U)$ and assume that U is an open subset of the complex plane. And as usual, we let let $z = x + iy$. In the plane open set U the 2-form ω is expressed as $\omega|_U = f d\bar{z} \wedge dz = 2if dx \wedge dy$, and we say that ω is integrable over U if the function f is integrable, that is f the function is measurable and the Lebesgue integral $\int_U |f|\,dxdy$ is finite. In case ω is integrable, we define the integral of ω as

$$\int_X \omega = 2i \int_U f\,dxdy. \tag{6.8}$$

By the paragraph (6.2) on page 172 this condition is independent of the coordinate we use, and by paragraph (6.1) on the same page, the integral in (6.8) has the same value whatever change of coordinate we make. Hence the definition is legitimate. DefIntDisk

(6.2) In the second step, we loosen the hypothesis and do not assume that ω has support in a patch. We say that the 2-form ω is *integrable* if there is a countable family of coordinate patches $\{D_i\}$ and a partition of unity $\{\eta_i\}$ subordinate to that family such that the following three conditions are fulfilled IntegrabelBetingels

- The form ω is supported in the union $\bigcup_i D_i$.
- Each $\eta_i\omega$ is integrable in D_i.
- The series $\sum_i \int_{D_i} |\eta_i\omega|$ is convergent.

And then, if ω is integrable, we define

$$\int_X \omega = \sum_i \int_{D_i} \eta_i\omega. \tag{6.9}$$

Of course, it is necessary to establish that the notion of integrability and the definiton of the integral do not depend on the choices made, *i.e.*, the choice of the family of patches and of the partition of unity. To that end, assume that $\{\epsilon_j\}$ is a second partition of unity subordinate to a family $\{D'_j\}$ of coordinate patches fulfilling the three conditions above. Our task is to establish the following: DefIntegral

Lemma 6.4 *With the two sets of data given above, the 2-form ω is integrable with respect to $\{\epsilon_i\}$ and $\{D_i\}$ if and only if it is integrable with respect to $\{\epsilon_j\}$ and $\{D'_j\}$. In case it is, one has* Lemma0DefInt

$$\sum_i \int_X \eta_i\omega = \sum_j \int_X \epsilon_j\omega.$$

The proof will rely on two further lemmas that follow.

Lemma 6.5 *The forms $\epsilon_j\eta_i\omega$ are integrable over $D_j \cap D'_i$ and one has the equality $\int_{D_i} \eta_i\omega = \sum_j \int_{D'_j\cap D_i} \epsilon_j\eta_i\omega$.* Lemma1DefInt

Proof: This is basically a consequence of Lebesgue's dominated convergence theorem. Let $\omega = f d\bar{z} \wedge dz$ in the patch D_i, and then $|\omega| = |f|\, dx \wedge dy$ there. By hypothesis $\eta_i\omega$ is integrable on D_i meaning that $\eta_i\omega$ is meseaurable and $|\eta_i f|$ has a finite integral over D_i. One has $|\epsilon_j \eta_i f| \leq \sum_{j<m} |\epsilon_j \eta_i f| \leq |\eta_i f|$ since $\sum_{j<m} \epsilon_j \leq \sum_j \epsilon_j = 1$. Hence $\epsilon_j \eta_i f$ is integrable and by Lebesgue's dominated convergence theorem one deduces, using that $\sum_j \epsilon_j = 1$, the equality

$$\sum_j \int_{D'_j \cap D_i} \epsilon_j \eta_i f = \int_{D'_j \cap D_i} \sum_j \epsilon_j \eta_i f = \int_{D_i} \eta_i f,$$

which is just the statement of the lemma, taking into account the definition in step one of integrals over patches. ❑

Lemma2DefInt

Lemma 6.6 *The double series $\sum_{i,j} \int_{D'_j \cap D_i} |\epsilon_j \eta_i \omega|$ converges.*

Proof: Let M and N be two arbitrary natural numbers. Using that $|\epsilon_i \eta_i \omega| = \epsilon_j \eta_i |\omega|$ one has the following self explanatory sequence of equalities and inequalities

$$\sum_{i<N, j<M} \int_X |\eta_i \epsilon_j \omega| = \sum_{i<N} \int_X \sum_{j<M} \eta_i \epsilon_j |\omega| \leq \sum_{i<N} \int_X \sum_j \eta_i \epsilon_j |\omega| =$$
$$= \sum_{i<N} \int_X \eta_j |\omega| < \sum_i \int_X \eta_j |\omega| .$$

❑

We proceed to finish the proof of lemma 6.4. By lemma 6.6 the double series

$$\sum_{i,j} \int_X \epsilon_i \eta_j \omega$$

converges absolutely and the terms can be rearranged at will. In particular we have

$$\sum_{i,j} \int_X \epsilon_j \eta_i \omega = \sum_i \sum_j \int_X \epsilon_j \eta_i \omega = \sum_i \int_X \eta_i \omega$$

where the last equality was proven in lemma 6.5. A analogous formula with the roles of η_i and ϵ_j interchanged holds true by symmetry , and we conclude that

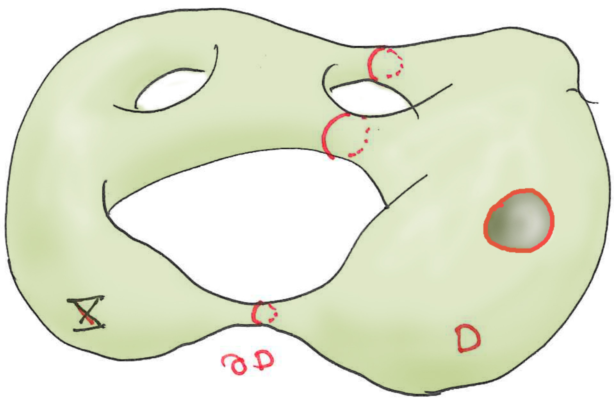
$$\sum_i \int_X \eta_i \omega = \sum_j \int_X \epsilon_j \omega,$$

which is what we intended to establish.

6.2.5 Stokes' formula

In Green's theorem, which is Stokes' in the plane, there is an issue of the orientation of the boundary; especially when the boundary disintegrates into several components that may nested this can be slightly subtle. Normally it is solved by saying that the the boundary must be traversed with the domain D lying to the port, as a sailor would say; that is, it lies to the left when you look in the direction of the forward tangent.

When generalizing to a Riemann surface this issue persist, and it resolved in the same way. The point being that Riemann surfaces are canonically oriented and this orientation induces a canonical orientation of the boundary. The procedure is the same as in the plane: Keep the domain to the port. However, on a Riemann surface it can be challenging to keep track of the different boundary components and their orientations.



Figur 6.1: *A domain on a Riemann surface with boundary.*

(6.1) We come to the formulation of Stoke's theorem; we only need the special case for 1-forms with compact support. The theorem is part of the area of mathematics called "calculus on differentiable manifolds" and a proof may be found in most text books covering that area—but since we, contrary to most text books, work with surfaces that are not *a priori* second countable, we briskly indicated the salient points of the proof.

Theorem 6.4 (Stoke's theorem) *Let D be region in X with a piecewise smooth boundary ∂D. Let ω be a 1-form with compact support and of class C^1. Then the following equality holds true:*

$$\int_D d\omega = \int_{\partial D} \omega.$$

The boundary ∂D which appears is a possibly infinite chain, but the support of ω being compact there are only finitely many non-zero terms to the left. The components of the boundary ∂D are given the orientation they inherit from the canonical one on X, *i.e.*, they have the region D on their left. Notice that we do not assume that D is connected nor relatively compact. Whether D is open or closed or neither is not an issue, neither the boundary nor the integral depends on such conditions.

PROOF: The tactics are to reduce the theorem to Green's theorem by use of a partition of unity. As the support K of ω is compact, it has a finite covering $\{D_i\}$ by disks which has a partition of unity η_i subordinate to it. Then it holds true that

$$\int_{\partial D} \omega = \sum_i \int_{\partial D} \eta_i \omega \quad \text{and} \quad \int_D d\omega = \sum_i \int_D d(\eta_i \omega).$$

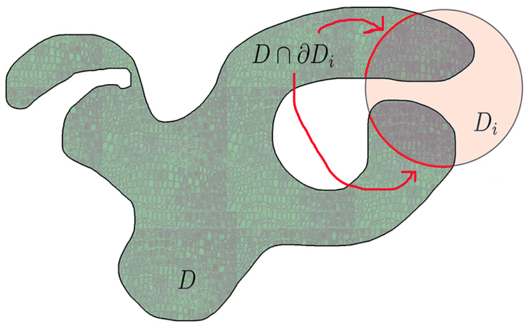
Corresponding terms in the two sums satisfy

$$\int_{\partial D} \eta_i \omega = \int_{\partial(D \cap D_i)} \eta_i \omega = \int_{D \cap D_i} d(\eta_i \omega) = \int_D d(\eta_i \omega),$$

where the equality in the middle comes from Green's theorem in the plane. The salient point is that boundary $\partial(D \cap D_i)$ can be split into two parts, one being $\partial D \cap D_i$ and the other being contained in ∂D_i. On the latter the form $\eta_i \omega$ vanishes and we have

$$\int_{\partial(D \cap D_i)} \eta_i \omega = \int_{\partial D} \eta_i \omega.$$

❑



Figur 6.2: *Randen til området i en disk.*

(6.2) We shall frequently use two corollaries of Stokes' theorem. The first is what one could call a *partial integration* formula. If ω is a 1-form and f a function, both of class C^1, one has the equality

$$d(f\omega) = df \wedge \omega + f\, d\omega. \tag{6.10}$$

When f and ω are integrable we may integrate over D to obtain the formula

$$\int_D d(f\omega) = \int_D df \wedge \omega + \int_D f\, d\omega,$$

PartialIntegration1 and when Stokes's theorem is applicable, this in turn leads to the formula

$$\int_{\partial D} f\omega = \int_D df \wedge \omega + \int_D f\, d\omega. \tag{6.11}$$

PartialIntegrasiom

(6.3) The second corollary is fundamental and often in use. It says that integrals over X of derivatives of forms with compact support vanish. Comparing the statement in the lemma with the one variable analogue can be instructive. If a smooth and real function f on $\mathbb{R}$ has compact support then $\int_{\mathbb{R}} f'(x)dx = f(a) - f(b)$ where a and b lies on either side of the support, and consequently the integral vanishes.

Corollary 6.3 *If ω is a 1-form of class C^1 with compact support, one has $\int_X d\omega = 0$.* KompaktStotteFor

Proof: Let γ be a closed, smooth and regular path in X and let B be a tubular neigbourhood of γ. Then X decomposes in two parts: The band B and its complement $B^c = X \setminus B$. After a short moment of reflection, one realizes that if $\partial B = \gamma_1 - \gamma_2$, where γ_1 and γ_2 are the two boudary components of the band, then $\partial B^c = \gamma_2 - \gamma_1$; to put it simply, B and its complement B^c are on opposite sides of ∂B. Hence by Stokes' formula, $\int_X \omega = \int_B \omega + \int_{B^c} \omega = 0$. ❑

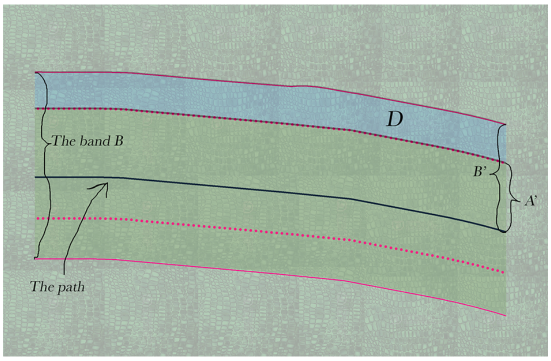
Figur 6.3: *A green Riemann surface with a red band*

6.2.6 The class of a path

In modern geometry a common technic is to associate to "subgeometric objects" a cohomology class in a cohomology theory (preferably your favorite one). This vague statement is made precise in our context. We want to associate to any closed, smooth path on X a class in the De Rahm cohomology group $H^1_{DR}(X)$. That is, to a closed path γ, we associate the class of a closed 1-form χ_γ, which in fact will be smooth and of compact support, and up to exact forms it should only depend on the homotopy class of γ.

(6.1) The starting point is to chose a band B round the path γ, and then shrinking the band a little to get a band in the band, that is a second and smaller band A lying within the first. The portions of A and B lying to the left of the curve are denoted by A' and B', and we put $D = B' \setminus A'$. It is a band lying some distance away and to the left of the path γ (D is bluish on the figure).

Given the bands above, we define a function f with compact support on X in the following manner. On A' it is constant and equals 1, in D it decreases in a smooth way to zero, and in the complement of B' it takes the constant value 0. Notice that f is not continuos. When γ is crossed from the left to the right f jumps from 1 to 0.



Figur 6.4: *The different bands surrounding the path* γ.

ClassOfClosePath (6.2) Then 1-form χ_γ associate to γ is defined by putting

$$\chi_\gamma = \begin{cases} 0 & \text{on } \gamma \\ df & \text{off } \gamma \end{cases}$$

The support of χ_γ is evidently contained in the blue band B' and χ_γ is of class C^∞. It is exact off γ but not in the entire X, and it a closed form. Evidently the form χ_γ depends on the several choices made, but the differences between forms arrising from various choices wil be exact, so the class $[\chi_\gamma]$ in the De Rahm group is well defined.

ClassOfPathIntegral **Proposition 6.2** *When α is a closed 1-form of class C^1, one has the equality*

$$\int_\gamma \alpha = -\int_X \alpha \wedge \chi_\gamma.$$

In particular the integral $\int_X \alpha \wedge \chi_\gamma$ does only depend on the free homotopy class of γ.

Proof: The support of χ_γ is contained in region D defined above, and the boundary of D has two components. We push them a small amount to obtain paths. The one farthest from γ is pushed slightly farther away into the region where f vanishes. And the other is moved slightly closer to γ, into the region where f equals one. The resulting paths are respectively named γ' γ'' and they are both homotopic to γ. We find using integration by parts, the insensitivity of the integral to homotopy and that fact that α is closed, the sequence of equalities which finishes the proof:

$$\int_{\gamma} \alpha = \int_{\gamma'} f\alpha - \int_{\gamma''} f\alpha = \int_X d(f\alpha) = \int_X df \wedge \alpha + \int_X f d\alpha = \int_X df \wedge \alpha.$$

❑

Finally, we have to see that χ_γ only depends on the homotopy class of γ:

Proposition 6.3 *If γ and γ' are two freely homotopic closed paths, it holds true that $[\chi_\gamma] = [\chi_{\gamma'}]$ in the De Rahm group $H^1_{DR}(X)$.*

Proof: It suffices to show that $\int_\delta \chi_\gamma = \int_\delta \chi_{\gamma'}$ for all closed paths δ, since then by proposition **??** on page **??** the difference $\chi_\gamma - \chi_{\gamma 1}$ is exact. One finds

$$\int_\delta (\chi - \chi') = \int_X \chi_\delta \wedge (\chi - \chi') = \int_\chi \chi_\delta - \int_{\chi'} \chi_\delta = 0,$$

using proposition 6.2 above and the insensitivity of the integral to homotopy. ❑

Problem 6.9. Show that the class $[\chi_\gamma]$ does not depend on the choices of the bands A and B and the function f. Hint: Given two sets of bands, use the largest B and narrowest A to find a common band for the two situations. If f' and f are two choices of functions, the difference $f - f'$ is smooth on the entire surface X. ✷ ChiUavhengig

Problem 6.10. Show the one dimensional analogue of proposition 6.2: Given a point $a \in \mathbb{R}$ define an integrable and positive function g_a such that $\int_{\mathbb{R}} f g_a = f(a)$ for all C^1 functions f. Hint: Partial integration. ✷

6.2.7 Intersection of paths

From a naive view point two closee paths on X can of course intersected, and the naive way of measuring how big the intersection is , is just to count the number of common points. To be able to prove theorems about the intersection, we want the "measure" or "the intersection product" to *e.g.,* be invariant under homotopy.

Given two closed paths γ_1 and γ_2. One say that they *intersect properly* at a point x if they both are regular near x and their tangents at the point are neither parallel nor antiparallel. They are in *general position* if this happens in every point of their intersection. Any pair of paths can be brought into general position by a homotopy. Indeed, they can both be brought into piecewise linear paths, and after one has been moved slightly, if necessary , they will have no common linear components and no common break points (points where linear components meet). If one insists on smoothing them, one can do that as before but making the modifications so close to the break points that smoothing out the corners does not affect the intersection points.

The tangents to the two curves intersecting propely in a point x are different and one defines the *local intersection multiplicity* $(\gamma_1, \gamma_2)_x$ at x as 1 or -1 according to the principal angle[1] between the tangents being positive or not. Then of course $(\gamma_1, \gamma_2) = -(\gamma_2, \gamma_1)$, since the angle from T_2 to T_1 is the negative of the one from T_1 to T_2.

Finally, one define the intersection product of the two paths γ_1 and γ_2 by summing up all the local contributions; that is, one puts

$$(\gamma_1, \gamma_2) = \sum_{x \in \gamma_1 \cap \gamma_2} (\gamma_1, \gamma_2)_x.$$

The idea in this paragraph is to express the intersection product as integral of forms, and in the way se that it is constant on homotopy classes (at least among the piecewise smooth members) and in executing that Stokes' theorem will be useful.

we want multiplisities at each intersecton so that sum is an invariant unde homotopy. If the two curves are reasonably place As an application of Stokes' we Let c_1 and c_2 be two homotopy classe and let γ_1 γ_2 be two paths representin the classes. We may assume that γ_1 both are

6.3 Quatdratic integrable forms

Recall that a 1-form α is measurable form if for any patch (U, z) the functions p and q appearing in the expression $\alpha|_U = pdz + qd\bar{z}$ are Lebesgue-measurable. Another measurable form β is equal to α almost everywhere if its component functions in every patch coincide almost every where with those of α, that is if locally $\beta = p'dz + q'd\bar{z}$ with $p = p'$ a.e and $q = q'$ a.e. A change of coordinates does not affect this, as p and p' (respectively q and q') pick up the same factor when the coordinate change.

(6.1) Recall the two operations $*\alpha$ and $\overline{\alpha}$ we introduced for 1-forms. The $*$-operation and the conjugation anticommute; that is it holds that $*(\overline{\alpha}) = -\overline{*\alpha}$. Indeed, in a patch where $\alpha = pdz + qd\grave{b}z$ one has

$$*(\overline{\alpha}) = *(\overline{p}d\overline{z} + \overline{q}dz) = -i\overline{p}d\overline{z} + i\overline{q}dz$$

and on the other hand it holds true that

$$\overline{*\alpha} = \overline{-ipdz + iqd\overline{z}} = i\overline{p}d\overline{z} - i\overline{q}dz = -\overline{\alpha}.$$

Furthermore one can move the star through the wedge, that is it holds true that:

$$*\alpha \wedge \beta = \alpha \wedge *\beta. \tag{6.12}$$

starWedge

DefNorm (6.2) In the previous section we introduced the notion of integrable 2-forms. Now the turn has come to 1-forms and we are about to explain what we mean by quadratically integrable 1-forms. So let ω be a 1-form on X and consider the integral

$$\|\omega\|^2 = \int_X \omega \wedge \overline{*\omega}. \tag{6.13}$$

In case this integral is finite one says that ω is *quadratically integrable* over X and DefNorm (6.13) serves as the definition of the norm $\|\omega\|$. That the integral is positive[2] unless ω vanishes almost everywhere is seen as follows. In a patch where $\omega = pdz + qd\bar{z}$ one has $\overline{*\omega} = i\bar{p}d\bar{z} - i\bar{q}dz$, and hence

$$\omega \wedge \overline{*\omega} = (pdz + qd) \wedge (i\bar{p}d\bar{z} - i\bar{q}dz) = \tag{6.14}$$

$$= -i(p\bar{p} + q\bar{q})d\bar{z} \wedge dz = 2(|p|^2 + |q|^2)dx \wedge dy, \tag{6.15}$$

and this is a non-negative expression that vanishes if and only if the form ω vanishes NrmPositiv almost everywhere. We also observe that p and q are quadratically integrable in the patch.

(6.3) It is a matter of easy computations to see that the parallelogram law holds. That is, one has the equality

$$\|\alpha + \beta\|^2 + \|\alpha - \beta\|^2 = 2\|\alpha\|^2 + 2\|\beta\|^2,$$

the corresponding relation between the integrand holds even before we integrate. A consequence is that the sum (and the difference) of two quadratically integrable 1-forms is integrable, and hence the quadratically integrable forms form a complex vector space that we shall denote $L^2(X)$.

(6.4) There is an inner product on the space $L^2(X)$ of quadratically integrable forms that induces the norm we just defined. It is given by the following integral

$$(\alpha, \beta) = \int_X \alpha \wedge \overline{*\beta},$$

which is finite once both the forms α and β are quadratically integrable. This follows for instance by integrating the two relations

$$2\,\mathrm{Re}\,\alpha \wedge \overline{*\beta} = (\alpha + \beta) \wedge \overline{*(\alpha + \beta)} - \alpha \wedge \overline{*\alpha} - \beta \wedge \overline{*\beta}$$

$$2\mathrm{Im}\,\alpha \wedge \overline{*\beta} = (\alpha + \beta) \wedge \overline{*(\alpha + i\beta)} - \alpha \wedge \overline{*\alpha} - \beta \wedge \overline{*\beta}.$$

[1] that is the one between $-\pi$ and π

[2] Needless to say, the norm $\|\omega\|$ is the positive square root of the integral in (6.13).

Switching the order of α and β results in conjugating the inner product, that is $\overline{(\alpha,\beta)} = (\beta,\alpha)$, which the following small calculation one has using that $\overline{*\alpha} = -*\overline{\alpha}$ shows:

$$\overline{\alpha \wedge *\overline{\beta}} = \overline{\alpha} \wedge *\beta = -\beta \wedge *\overline{\alpha} = \beta \wedge \overline{*\alpha}.$$

In paragraph (6.2) we showed that product is positive definite and we have defined a genuine inner product.

Proposition 6.4 *The product (α,β) is a positive definite complex inner product on the space L^2X of quadratically intergrable forms. Hence $L^2(X)$ is pre-Hilbert space. One has $(*\alpha, *\beta) = (\alpha,\beta)$.*

PROOF: The only thing that is not already shown is the formula $(\alpha,\beta) = (*\alpha, *\beta)$. To that end we offer the following computation

$$\int_X *\alpha \wedge \overline{**\beta} = \int_X *\alpha \wedge \overline{-\beta} = \int_X \alpha \wedge *(\overline{-\beta}) = \int_X \alpha \wedge *\overline{\beta}.$$

❑

(6.5) The space $L^2(X)$ of quadratic integrable 1-forms is in general not a complete vector space. Our surfaces X are for the moment not even second countable, so we must live with not knowing whether $L^2(X)$ is complete or not, that is can not assume it is. However, this functions well with a little care.

Among the many situations when $L^2(X)$ is complete The easiest case to establish is when X is compact. For the sake of completeness (!!) we sketch a proof in that case. Any Cauchy sequence $\{\omega_n\}$ in $L^2(X)$ of global forms induces in a any patch U a Cauchy sequence in the space $L^2(U)$, and we know from real analysis that this space is complete. Hence we get quadratically integrable function ϕ_U to which the sequence $\{\omega_n|_U\}$ converges. The rest of the proof consists of checking that the $\omega|_U$ can be patched together to a global quadratically integrable form ω and that the original sequence converges to ω in L^2.

Proposition 6.5 *If X is compact, the space L^2X is complete, hence it is a Hilbert space.*

PROOF: We have the Cauchy sequence $\{\omega_k\}$ in $L^2(X)$; that is $\|\omega_n - \omega_m\|_X \to 0$ when $n,m \to \infty$. Let $\{U_i\}$ be a family of patches, finite since X is compact, and let $\{\eta_i\}$ be a partition of unity subordinate to $\{U_i\}$. Then given $\epsilon > 0$ we have

$$\|\omega_n - \omega_m\|_X = \sum_i \|\eta_i\omega_n - \eta_i\omega_m\|_{U_i}$$

On the other hand we have $\|\omega_n - \omega_m\|_{U_i} \leq \|\omega_n - \omega_m\|_X$ hence $\{\omega_n|_{U_i}\}$ is a Cauchy sequence in the space $L^2(U_i)$. This space is complete and the sequence $\{\omega_n|_{U_i}\}$ converges to a 2-form ω_U in $L^2(U_i)$.

On the intersections $U_i \cap U_j$ one obviously has that $\omega_{U_i} = \omega_{U_j}$ a.e, since they are L^2-limits of the same sequence. A small argument implies that they can be altered on a set of measure zero to patch together to a global 2-form ω. Indeed, if K_{ij} denotes the set in U_{ij} where they disagree, the finite union $K = \bigcup_{i,j} K_{ij}$ is a set of measure zero. Letting χ_K denote the characteristic function of K (the one that equals 1 on K and 0 off) one sees that the forms $\chi_K \omega_{U_i}$ agree on the intersections, and patch together to a global measurable (even locally integrable) form ω.

It remains to see that ω is quadratically integrable and that the sequence $\{\omega_n\}$ converges to ω in $L^2(X)$. We find

$$\|\omega - \omega_n\|_X = \sum_i \|\eta_i \omega - \eta_i \omega_n\|_X \leq \sum_i \|\omega - \omega_n\|_{U_i}.$$

Given a positive ϵ there is an N such that $\|\omega - \omega_n\|_{U_i} < \epsilon$ for $n > N$. *A priori* this N depends on i, but the covering being finite the largest work for all i. Hence for $n > N$ one has

$$\|\omega - \omega_n\|_X < r\epsilon$$

where r is the number of patches, which is a constant in the context, and we are through. ❏

6.4 A closer study of $L^2 X$

Weel technique, $E^1_\infty(X)$ denote the set of smooth 1-forms on X with compact support. There is an avatar of Stokes' theorem for such functions, namely

$$\int_X d(\eta\omega) = 0$$

for any C^1 form ω, indeed Stokes gives

(6.1) We introduce two closed subspaces E and E^* of $L^2(X)$. The first space E is the closure of the subspace consisting of the exact forms $d\eta$ where η runs through all smooth functions with compact support; that is E is the closure of the set $\{\, d\eta \mid \eta \in C_0^\infty(X)\,\}$ in $L^2(X)$. This means that any form ω in E is the L^2-limit of a sequence $\{d\eta_i\}$ of differentials of smooth functions η_i with compact support.

The other subspace E^* is analogously defined as the closure of the space whose elements have the shape $*d\eta$ for η smooth with compact support; that is, it is the closure of the set $\{\, *d\eta \mid \eta \in C_0^\infty(X)\,\}$. So to say, the space E^* is just the star of E. The spaces E and E^* are closed vector subspaces of $L^2(X)$.

Lemma 6.7 *The two subspaces E and E^* are orthogonal, in particular they have no non-zero common element.* SnittNull

PROOF: Let to begin with ϵ and η be two smooth functions whose supports are compact. We compute using partial integration and obtain

$$(d\epsilon, *d\eta) = -\int_X d\epsilon \wedge \overline{d\eta} = -\int_X d(\epsilon d\overline{\eta}) = 0,$$

where the last equality holds since $\epsilon d\overline{\eta}$ has compact support (proposition 6.3 on page 191). Assume now that $\alpha = \lim_i \epsilon_i$ and $\beta = \lim_j *\eta_j$ are elements in E and $*E$ respectively, so that the ϵ_i-s and the η_j-s all lie in $C_0^\infty(X)$. As the inner product is continuous in L^2-norm, we get from the above that

$$(\alpha, \beta) = (\lim_i \epsilon_i, \lim_j \eta_j) = \lim_{i,j}(\epsilon_i, *\eta_j) = 0.$$

❑

(6.2) The orthogonal complements of E and E^* are of basic interest. They consist of what one respectively calls *weakly closed* and *weakly co-closed* forms. The elements of $E^{*\perp}$ are by definition those integrable forms satisfying $(\omega, *d\eta) = 0$, whereas those in $E^\perp$ are characterized by the relation $(\omega, d\eta) = 0$; in both cases the equalities must remain valid for all $\eta \in C_0^\infty(X)$.

The reasons behind the names "weakly closed" and "weakly co-closed" become clear with the lemma below. It tells us that sufficiently regular forms in $E^{*\perp}$ are genuinely closed, and those in $E^\perp$ are genuinely co-closed:

WeaklyClosed

Lemma 6.8 *Assume that ω is a quadratically integrable C^1-form. Then ω belongs to $E^{*\perp}$ if and only if ω is closed, and it belongs to $E^\perp$ if and only if it is co-closed.*

PROOF: By partial integration the relation

$$(\omega, *d\eta) = -\int_X \omega \wedge d\overline{\eta} = \int_X d(\overline{\eta}\omega) - \int_X \overline{\eta} d\omega,$$

holds for all smooth functions η with compact support. The support of η being compact one has $\int_X d(\overline{\eta}\omega) = 0$ after corollary 6.3 on page 6.11. This yields

$$(\omega, *d\eta) = -\int_X \overline{\eta} d\omega.$$

The first part of the lemma now follows since the integral to the right vanishes for all $\eta \in C_0^\infty(X)$ if and only if $d\omega = 0$. The proof of the second part of the lemma is *mutatis mutandis* the same as of the first. ❑

(6.3) The orthogonal complement to the space spanned by smooth co-closed and compactly support forms are the *weakly exact* forms. If such a form is of class C^1 it is genuinely exact.

WeaklyExactC1 **Proposition 6.6** *A quadratically integrable* 1*-form* ω *of class* C^1 *is exact if and only if* $(\omega, \beta) = 0$ *for all smooth co-closed forms* β *with compact support.*

Proof: Assume $(\alpha, \beta) = 0$ for all co-closed, smooth and compactly supported forms β. To see that ω is exact, it suffices by the theorem of vanishing periods (theorem 6.1 on page 178) to see that $\int_\gamma \omega = 0$ for every loop γ. In paragraph (6.2) on page 192 we constructed the De Rahm class of loops γ. They are represented by a real, smooth and closed forms χ_γ with compact support, and their constituting property is that

$$\int_\gamma \alpha = \int \alpha \wedge \chi_\gamma$$

for all closed C^1-forms α. We deduce from this, using that $*\chi_\gamma$ is co-closed, the following

$$0 = (\omega, *\chi_\gamma) = \int_X \omega \wedge \overline{*(*\chi_\gamma)} = -\int_X \omega \wedge \chi_\gamma = -\int_\gamma \omega,$$

and we are done.

The other implications follows easily by use of partial integration. Indeed,assume ω to be exact and let $\omega = df$ where f is a C^2-function on X. For any 1-form β being smooth, co-closed and of compact support, we find

$$(df, \beta) = \int_X df \wedge \overline{*\beta} = \int_X d(f\,\overline{(*\beta)}) - \int_X f\, d\overline{(*\beta)} = \int_X d(f\,\overline{*\beta}) = 0.$$

where we use that integrals over X of closed forms of compact support vanish (corollary 6.3 on page 191) and that $*\beta$ is closed. ❏

Problem 6.11. Show the "co-version" of proposition 6.6 above. That is ω is co-exact if and only if $(\omega, \beta) = 0$ for all closed β, smooth and of compact support. ✷

The space of harmonic forms

By far the most interesting subspace of $L^2(X)$ is the subspace $H = E^\perp \cap E^{*\perp}$. By trivial and elementary linear algebra one sees that $(E \oplus E^*)^\perp = E^\perp \cap E^{*\perp}$ This has several consequences. First all, there is a direct sum decomposition

$$L^2 = E \oplus E^* \oplus H.$$

Secondly, the technical lemma below shows that the space H is the subspace of $L^2(X)$ consisting of the quadratic integrable forms one calls *weakly harmonic*, and combining this with the miraculous Weyl's lemma one concludes that the forms in H are genuinely harmonic, and this is the main theorem of the present section. Such a result is serious bootstrapping; we start out by forms that are merely measurable with finite integrals, and end up concluding that they in fact are smooth.

(6.4) A TECHNICAL LEMMA The technical lemma is formulated in a real setting, which possibly makes it a little more transparent. The setting is local and computations take place in a patch U in X with coordinate $z = x + iy$. We are given a form ω expressed as $\omega = pdx + qdy$ which is integrable in U, and we shall make use of test-functions η that are smooth and compactly supported in U. The point of the lemma is to express the integral of p and q against the Laplacian $\Delta\eta$ of η in terms of the inner product on $L^2(X)$ and thus preparing the ground for applications of Weyl's lemma.

TekniskLemma **Lemma 6.9** *Let η be a smooth and real function in a patch (U, z) whose support is compact. Then one has the two equalities*

$$(\omega, d\eta_x - *d\eta_y) = \int_U p\Delta\eta dx \wedge dy,$$
$$(\omega, *d\eta_x + d\eta_y) = \int_U q\Delta\eta dx \wedge dy.$$

PROOF: This is a matter of some simple computations. One has

$$d\eta_x = \eta_{xx}dx + \eta_{xy}dy \quad \text{and} \quad * \, d\eta_y = -\eta_{yy}dx + \eta_{xy}dy$$

Hence the equality

$$d\eta_x - *d\eta_y = (\eta_{xx} + \eta_{yy})dx = \Delta\eta dx,$$

which yields the first equation in the lemma:

$$(\omega, d\eta_x - *d\eta_y) = \int_X \omega \wedge *(\Delta\eta dx) = \int_U p\Delta\eta dx \wedge dy.$$

To show the second equation one applies the first to the form $-(*\omega) = qdx - pdy$ observing that

$$(-(*\omega), d\eta_x - *d\eta_y) = (\omega, *d\eta_x + d\eta_y)$$

since $(*\alpha, *\beta) = (\alpha, \beta)$. ❑

(6.5) The impact of this technical lemma is that the Hodge-components of a form α in H are weakly harmonic; in other words, if $\alpha = pdx + qdy$ in a patch (U, z), the two component functions p and q are weakly harmonic. Indeed, for any smooth η with compact support in U the two left integrals in lemma 6.9 vanish since α lies in both $E^\perp$ and $E^{*\perp}$. Consequently the right integrals vanish also, and this is just the definition of p and q being weakly harmonic.

By Weyl's lemma xxx on page xxx, it follows that p and q are harmonic functions, and we are more than half way in the proof of the following:

Theorem 6.5 *Assume that ω is a quadratically integrable form. Then ω is harmonic if and only if $\omega \in H$.*

PROOF: Assume that ω lies in H. We found out just before the theorem that the local component functions of ω in any patch are harmonic. In particular, the form ω is smooth. By lemma 6.8 it is therefore both closed and co-closed and hence is a harmonic form, as follows from proposition 6.1 on page 177.

To establish the implication the other way let ω be a harmonic form. By the observation in the paragraph about harmonic forms (paragraph 6.2 on page 176) the form ω is at the same time closed and co-closed, and being smooth it therefore belongs to both $E^{\perp}$ and $E^{*\perp}$ by lemma 6.8. ❑

6.4.1 Relation with the De Rahm group

The main theorem about harmonic forms has several severe consequences and is really the hub of the theory. For instance, it establishes a very close relation between the space of harmonic forms and the De Rahm group.

Proposition 6.7 *Any class $[\omega]$ in the De Rahm group $H^1_{DR}(X)$ is represented by a harmonic form. In case X is compact, the harmonic representative of a class is unique, hence H and $H^1_{DR}(X)$ are isomorphic vector spaces.*

The proposition says that the canonical map $H \to H^1_{DR}(X)$ sending a form α to its class $[\alpha]$ always is surjective. In particular if the De Rahm group $H^1_{DR}(X)$ is non-zero, on can conclude that there are harmonic forms on X. In the compact case the canonical map is even an isomorphism.

For instance this implies that the classes of closed paths, *i.e.,* classes of the shape $[\chi_\gamma]$ are represented by harmonic forms α_γ, which in the compact case is uniquely defined by the free homotopy class of $[\gamma]$. In the non-compact case the map is not injective, *e.g.,* the derivative of any holomorphic function will lie in the kernel.

PROOF: Pick any closed ω of class C^1 (closed formes forcibly are). As $L^2(X) = E \oplus E^* \oplus H$ the space of weakly closed forms $E^{*\perp}$ obviously satisfies $E^{*\perp} = E \oplus H$. The form ω being closed is weakly closed and lies in $E^{*\perp}$. It can therefore be decomposed as a sum

$$\omega = \beta + \alpha$$

with $\beta \in E$ and $\alpha \in H$. Harmonic forms are closed so it holds that $0 = d\omega = d\beta + d\alpha = d\beta$. Hence β is closed, and being the difference between two forms of class C^1 it is C^1 as well.

Now, let γ be any loop in X. It has a closed form χ_γ associated with it and, β being closed and C^1, the first equality below holds true (by proposition 6.2 on page 192)

$$\int_\gamma \beta = (\beta, *\chi_\gamma) = 0,$$

and the second holds since χ_γ is co-close and hence lies in $E^{\perp}$. By the theorem of vanishing periods, the form β is exact, and it follows that $[\omega] = [\alpha]$.

If the Riemann surface X is compact, it has no globally defined harmonic functions on it, so harmonic forms can not be exact. In other words, the canonical map $H \to H^1_{DR}(X)$ is injective. ❑

PROBLEM 6.12. Assume that γ is loop in X that is *non-separating* meaning that the complement $X \setminus \gamma$ is connected. Show that there is harmonic form α on X such that $\int_\gamma \alpha = 1$. ✷

6.5 Existence of harmonic functions

It is of course a fundamental result that on any Riemann surface X there are non-constant meromorphic function. The study of a Riemann surface is for most of its parts based on understanding the meromorphic functions that live on it. Finding a meromorphic function on a Riemann surface is not a trivial matter, and the result is specific for Riemann surfaces, that is for analytic manifolds of complex dimension one. Already in dimension two there are examples of manifolds without non-constant meromorphic functions. Finally, Riemann surfaces turn out to have lots of meromorphic functions, but to begin with, we will be happy to just find for one!

auxFu

(6.1) AN AUXILIARY FUNCTION We shall need the function $h(z) = z^{-n} + \overline{z}^n$ which is harmonic for $z \neq 0$ being the sum of a holomorphic and an anti-holomorphic function. It has the property that the angular part of its conjugate differential $*dh$ vanishes on the unit circle, so that by Stokes' theorem we obtain:

$$\int_{\mathbb{D}} d(\eta *dh) = \int_{\partial\mathbb{C}} \eta *dh = 0, \tag{6.16}$$

AuxOne

for any η of class C^1 around the unit disk. A little computation yields that

$$*dh = niz^{-n}\frac{dz}{z} + ni\overline{z}^n\frac{d\overline{z}}{\overline{z}}, \tag{6.17}$$

AngularPart

and on the unit circle, where $z = e^{it}$, we find $dz/z = idt$ and $d\overline{z}/\overline{z} = -idt$, and (6.17) reduces to the equality

$$*dh|_{\partial\mathbb{D}} = -n(z^{-n} - \overline{z}^n)dt = 0$$

since $z^{-1} = \overline{z}$ on $\mathbb{D}$.

The function we shall use is an avatar of the function h being h made smooth in a smaller disk D' about the origin. Let $D'' \subseteq D'$ be another small disk and choose a smooth function in $\mathbb{D}$ that vanishes on D'' and equals one in $D \subseteq \overline{D'}$. Then $g = \eta h$ is a smooth function in D which equals h in the annular region $D \setminus D'$.

(6.2) Let us agree to say that a complex harmonic function u in $X \setminus \{x\}$ has a *pole of order n* at the point x if there is a patch (U, z) centered at x such that for some non-zero complex constant a the function $u - az^{-n}$ on $U \setminus \{x\}$ can be extended to a harmonic function in U. Of course, from the point of view of singular behavior the real and the imaginary parts $\operatorname{Re} u$ and $\operatorname{Im} u$ resemble $\operatorname{Re} az^{-n}$ and $\operatorname{Im} az^{-n}$ near x. For instance if the real part of a harmonic function behaves like $(\alpha x - \beta y)(x^2 + y^2)^{-1}$ near a simple pole with $a = \alpha + i\beta$.

(6.3) The main theorem whose proof occupy the rest of this section is the following; it asserts that we always can find a harmonic function on X with just one pole where the singular behavior is prescribed:

Theorem 6.6 *Let X be a Riemann surface and let $x_0 \in X$ and let x_0 be a point. Let n be a natural number. Then there exists a harmonic function u in $X \setminus \{x_0\}$ having a pole of order n at x_0. Furthermore there is a neighbourhood U of x_0 such that* HarmFuFinnes

- $u - z^{-n}$ *is harmonic in* U,
- $\|du\|_{X \setminus U} < \infty$,
- $(du, d\eta) = (du, *d\eta) = 0$ *for all smooth η having compact support and vanishing in* U.

(6.4) The set up In this paragraph we describe set up and the main ingredients of the proof of theorem 6.6 above. The situation is as follows. We fix a point x_0 in the Riemann surfaces X and additionally we fix two disks D' and D both centered at x_0 and D' being the smaller. The open annulus $D \setminus \overline{D'}$ will be denoted by A.

Furthermore we are given a smooth function θ in a neighbourhood of $\overline{D}$, harmonic in the annulus A and havimg the property that the angular component of $*d\theta$ vanishes along the boundary ∂D of D so that (6.16) holds for $*d\theta$. Of course our favorite example of such a function is the function g we studied in paragraph (6.1); which can be transported to X once we choose a coordinate in D making D a disk of radius one.

(6.5) The main player and the first lemma The main player in the proof is the 1-form Θ on X defined by

$$\Theta = \begin{cases} d\theta & \text{in the closed disk } \overline{D} \\ 0 & \text{outside the closed disk } \overline{D} \textit{ i.e.}, \text{ in } X \setminus \overline{D} \end{cases}$$

It certainly not smooth having a discontinuity across the boundary ∂D, but it evidently lies in $L^2(X)$, since the norm $\|\Theta\|_X$ equals the integral $\int_D |\theta|^2$ which is finite. And clearly Θ is of compact support.

Now, the Hilbert-space $L^2(X)$ of quadratic integrable forms on X decomposes in the orthogonal direct sum $E \oplus E^\perp$ and consequently the form Θ nay be written as $\Theta = \alpha + \beta$ where $\alpha \in E$ and $\beta \in E^\perp$; recall that $E^\perp = E \oplus E^* \oplus H$. The main lemma in the proof of theorem 6.16 above is the following

Lemma 6.10 *The form α is harmonic off the smaller disk D', that is, it is harmonic in $X \setminus \overline{D'}$.*

Proof: Once we have established that α is smooth, it follows from lemma 6.8 that α is a closed form since α belongs to E and $E \oplus H = E^{*\perp}$, and it will be co-closed as well. Indeed, near points in the annulus A it holds true that $\alpha = d\theta - \beta$ and $d\theta$ being smooth β is smooth and both are co-closed since the function θ is harmonic in A and β lies in $E^{\perp}$. In the vicinity of points outside the disk D, one has $\alpha = -\beta$, so when α is smooth, β will be as well and hence both are co-closed by proposition 6.8 on page 6.8. It follows from proposition 6.1 on page 177 that α harmonic.

In the disk D we express α as $\alpha = pdx + qdy$. To prove that α is a smooth form we once more appeal to Weyl's lemma, and it will suffice to find a neighbourhoods round all points not belonging to D' where p and q are weakly harmonic. That is, we must exhibit neighbourhoods U such that

$$\int_U p\Delta\eta = \int q\Delta\eta = 0$$

for all smooth functions η supported in U, and we can as well require the neighbourhoods to be disks. To this end we shall make use of the technical lemma 6.9 on page 200, and check that $(\alpha, d\eta) = (\alpha, *d\eta) = 0$ for all smooth functions η compactly supported in U.

Since $E \subseteq E^{*\perp}$, the equality $(\alpha, *d\eta) = 0$ is for free. For the other equality, one observes that $\beta \in E^{\perp}$, so that $(\beta, d\eta) = 0$ for all $\eta \in C_0^\infty(X)$, hence $(\alpha, d\eta) = (\Theta, d\eta)$.

For any disk U in $X \setminus D'$ one may write $U = U_1 \cup U_2$ with $U_1 = U \cap A$ and $U_2 = U \cap X \setminus D$. We find, by partial integration and Stokes' theorem the equalities for η is supported in U

$$(\Theta, d\eta) = \int_U \Theta \wedge *d\eta = \int_{U_1} \Theta \wedge *d\eta = \int_{U_1} d(\eta *\Theta) = \int_{\partial U_1} \eta *d\theta = 0$$

where the last integral vanishes for the following reason: The boundary ∂U_1 has two components. One is part of ∂D, and there $*d\eta$ vanishes by hypothesis, and the other is disjoint from the support of η. Hence $\eta *d\theta$ vanishes there as well. ❑

(6.6) The second lemma The next lemma concerns the behavior of α near the basepoint x_0, that is the behavior in the smaller disk D', We shall appela to lemma 6.9. Let η be smooth of compact support in D. Using that $d\theta$ and $*d\eta_y$ are orthogonal we find

$$(d\theta, d\eta_x) = (d\theta, d\eta_x - *d\eta_y) = \int \theta_x \Delta\eta$$

$$(\alpha, d\eta_x) = \int p\Delta\eta$$

and since $\theta - \alpha = \beta \in E^{\perp}$ it holds true that

$$\int (p - \theta_x)\Delta\eta = 0$$

for all η. A reasoning, *mutatis mutandis* the same, shows that

$$\int (q - \theta_x)\Delta\eta = 0$$

for all η as well.

Hence by Weyl's lemma $\alpha - d\theta$ is smooth in D, and since θ is smooth, it follows that α is smooth in the entire Riemann surface X.

Lemma 6.11 *The form α is smooth and exact in X, and $\alpha - \Theta$ is harmonic in the disk D.* Lemma2ExHarm

Proof: We already established that α is smooth, and by a by now standard reasoning, $\alpha - \Theta$ is harmonic. It remains to see that α is exact. Now $\alpha \in E$, and by lemma 6.8 on page 6.8 all co-closed smooth forms are orthogonal to α. By proposition 6.6 on page 199 we conclude that α being smooth, is exact. ❑

(6.7) Proof of theorem 6.6 So far the function θ was not explicit, however we use a θ constructed with the help of the auxiliary function from paragraph 6.1.

The form α is exact, so let $\alpha = df$ where f is a smooth function on X which is harmonic in the complement of the smaller disk D'. Let u be the function defined by

$$u = \begin{cases} f - \theta + h & \text{in } D \setminus \{x_0\} \\ f & \text{in } X \setminus \overline{D'} \end{cases}$$

In the intersection of the two domains, that is the annular region A, the two definitions agree since $h = \theta$ there and consequently u is well defined everywhere away from x_0. It his clearly harmonic outside D', since f harmonic there, and in $D' \setminus \{x_0\}$ lemma 6.11 tella us that $f - \theta$ is harmonic *a priori* the function h is harmonic, and $u - z^{-n} = f - \theta - h - z^{-n} = f - \theta - \overline{z}^n$ is clearly harmonic in D'. This proves the first part of the theorem.

For the remaining two statements, take neighbourhood U to be D. outside D it holds that $du = \alpha$ which lies in $L^2(X)$. As α lies in E it is orthogonal to forms of type $*d\eta$ with η having compact support, and off D it holds true that $\alpha = -\beta$ which by choice lies in $E^{\perp}$.

6.6 Existence of meromorphic functions

The aim of this section is to show that every Riemann surfaces has a non-constant meromorphic function; that is

6.6.1 Recap on meromorphic forms

Recall that a 1-form on X is said to be *meromorphic* if there is an open dense set V such that ω is holomorphic on V, and if there is a covering of X by patches (U, z) such that $\omega = f(z)dz$ where f is meromorphic in U. The ω is properly defined away from the poles of the f's, and these form a discrete set P in X. We may assume that patches (U, z) are disks each containing just one pole at the origin.

(6.1) Assume now that ω is a given meromorphic form on X, and that (U, z) is a coordinate patch around a point x in X. Let (V, w) be another patch around x. In the intersection $U \cap V$ the relation between the coordinates z and w has the form $w = w(z)$ where w is biholomorphic. In the patch (V, w) one has an expression $\omega = g(w)dw$ and therefore $\omega = g(w(z))\partial_z w\, dz$ in $U \cap V$. Hence the identity

$$g(w(z))\partial_z w = f(z)$$

holds true in $U \cap V$ and $\partial_z w$ vanishes nowhere in $U \cap W$ the coordinate w depending biholomorphically on z.

One observes that since $\partial_z w$ is biholomorphic it holds true that $\mathrm{ord}_{w(x)} g(w) = \mathrm{ord}_{z(x)} f(z)$. Hence one can speak about *the order* $\mathrm{ord}_x \omega$ of the meromorphic differential ω at x, and therefore also *the divisor* (ω) of the meromorphic form ω. It is given as $(\omega) = \sum_{x \in X} \mathrm{ord}_x \omega$. This divisors is positive if and only if ω is holomorphic, and it is an easy exercise to check that $(f\omega) = (f) + (\omega)$ for any meromorphic function f on X.

(6.2) Given two non-zero meromorphic forms on the Riemann surfaces X. In some sense their "quotient" is a meningfull construct, and it is a meromorphic function on X. In a precise formulation; given two meromorphic forms ω_1 and ω_2 then there is unique meromorphic function f such that $\omega_1 = f\omega_2$, and f evidently merits the name "the quotient of ω_1 by ω_2".

Indeed, locally in a patch (U, z) the two forms satisfy relations like $\omega_i = f_i(z)dz$ where the f_i's are meromorphic functions none of which vanishes identically. In another coordinate patch (V, w) the forms are shaped like $\omega_i = g_i(w)dw$, and the transition relations on $U \cap V$ have the form

$$g_i = f_i \partial_z w.$$

Hence the quotients g_1/g_2 and f_1/f_2 coincide on the intersection $U \cap V$ and therefore can be patched together to give a meromorphic function on $U \cup V$. The quotient formed on the patches in an atlas in this way, fit together to give a global meromorphic function on X.

Example 6.4. On the Riemann sphere $\hat{\mathbb{C}}$ the form $\omega = dz$ is meromorphic. In the patch $\mathbb{C}^* = U_\infty \cap U_0$ the relation $z = w^{-1}$ holds, so that $dz = -w^{-2}dw$ there. Hence dz has a pole of order 2 at infinity. Every other meromorphic form on the Riemann sphere is shaped like $f(z)dz$ where f is any function meromorphic in $\hat{\mathbb{C}}$; that is, f is any rational function. ❄

6.6.2 Existence of meromorphic forms

The theorem xxx tells us that there are harmonic functions on X with a prescribed singular behavior at a given point in X. Recall the way to obtain a holomorphic function with a given real harmonic function u as real part in a domain Ω. One forms the conjugate differential $*du$ and tries to integrate it, and in case of success in writing $*du = dv$, the function $f = u + iv$ will be holomorphic.

We mimic this process *mutatis mutandis* to obtain a holomorphic 1-form from a harmonic one. If α is harmonic, it is patchwise presented as $\alpha = pdz + qd\overline{z}$ where p and $\overline{q}$ are holomorphic. An easy computation shows that $\alpha + i * \alpha = 2pdz$, and hence $\alpha + i * \alpha$ is holomorphic.

To get hold of meromorphic 1-forms, we start with the function u given us in theorem 6.6 that is harmonic in $X \setminus \{x\}$ and behves like z^{-n} near x. The differential du is a harmonic form in $X \setminus \{x\}$ and near x it can be expresses as $(-nz^{-n-1} + \phi(z))dz$ with ϕ harmonic in the vicinity of x. It follows that $du + i *du$ is holomorphic inan x. We have thus established the following theorem

ExistenceOfMeroF

Theorem 6.7 *Let X be a Riemann Surface, $x \in X$ a point and n a natural number. Then there is a meromorphic 1-form on X having a pole of order $n+1$ at x as its sole singularity.*

(6.1) Now, we are ready for finding a non-constant meromorphic function on X, and naturally, we shall exhibit it as the "quotient" of two meromorphic forms. To this end, pick two different points x_1 and x_2 on the Riemann surface. By theorem 6.7 above X affords to meromorphic forms ω_1 and ω_2 having a pole of order two respectively at x_1 and x_2 and having no singularities elsewhere. Then ω_1 being holomorphic at a point where ω_2 has a pole, is not a constant multiple of ω_2, and hence "quotient" f with $\omega_1 = f\omega_2$ is not constant. We thus proved

Theorem 6.8 *Let X be a Riemann surface. Then X has a non-constant meromorphic function. In other words, there is a non-constant holomorphic map $f\colon X \to \hat{\mathbb{C}}$*

Notice that even if we control the poles of the two forms completely, we have no control at all on their zeros. Hence the fibre of f over the point at ∞ (or over 0 for that matter) can contain several other points than x_1, and it usually does. Indeed, any zero of ω_2 that is not cancelled by a zero of ω_1 will be a pole of f.

In case X is compact it has a degree but the theorem says nothing of this degree. For most Riemann surface it is not two. Those Riemann surfaces being double covers of the Riemann sphere are called *hyperelliptic* and, expressed in a very vague way, they form a "thin" part of all Riemann surfaces at least if the genus is as 3 or more. Compact Riemann surfaces of genus g may be parametrized by a space of dimension $3g - 3$ and the hyperelliptic ones correspond to points in a subspace of dimension $2g - 1$.

(6.2) The first consequence of having a meromorphic function, is that the topology of X is second countable. One may even show that X is triangulable.

Proposition 6.8 *If X is a Riemann surface, then X is second countable*

Knowing that there is a non-constant holomorphic function $f\colon X \to \hat{\mathbb{C}}$, this is an immediate consequence of the proposition lemma that usuallu goes under name of the *Poincaré-Volterra* lemma—indeed a holomorphic map has discrete fibres (propo xxxx). The Poincaré-Volterra lemma was proved independently by Poincaré and Volterra in 1888, but it seems that the statement is due Cantor. One consequence of the proposition above is

Corollary 6.4 *Let X be a Riemann surface and let $\tilde{X}$ be a universal cover. Then the natural map $\tilde{X} \to X$ has countable fibres. The fundamental group $\pi_1(X)$ is coutable.*

Proof: In a second countable space discrete sets must be countable; indeed if $D \subseteq X$ is discrete, there is for each $x \in D$ an open set U from any basis with $U \cap D = \{x\}$. ❑

(6.3) The old-timers expressed this by saying that a holomorphic function takes "at most countably many values" in a point. This is a little like $\log z$ which we know is defined only up to multiples $2\pi i$. Frequently when a function is *e.g.,* defined by a differential equation or an algebraic equation in a domain Ω, there are several local solutions. On the universal cover $\tilde{\Omega}$ these patch together to a global solution $f\colon \tilde{\Omega} \to \mathbb{C}$, and the different local values at a point z correspond to the fibre over z of the natural map $\tilde{\Omega} \to \Omega$.

(6.4) Here comes the Poincaré-Volterra lemma; the proof is an exercise in general topology:

Proposition 6.9 *Assume that X is a connected topological manifold and that $f\colon X \to Y$ is a continuous map to a Hausdorff space Y whose fibres are discrete. If Y is second countable, then X is.*

Proof: The proof has two stages. First we define a particular basis $\mathcal{B}$ for the topology on X and secondly we show that this basis is countable.

In the first stage, we begin with a countable basis $\mathcal{U}$ for the topology on Y of open sets. For any U from $\mathcal{U}$ the different connected components of $f^{-1}(U)$ can be second countable or not, and we include those which are in $\mathcal{B}$. So $\mathcal{B}$ of all components of inverse images $f^{-1}(U)$ that are second countable. We claim that $\mathcal{B}$ is a basis for the topology of X. Notice that since X is a locally connected space, the components of the open sets $f^{-1}U$ are all open.

To this end, let $x \in X$ and let V a open neighbourhood of x. We must come up with a set B from $\mathcal{B}$ with $x \in B \subseteq V$. Using that X is a manifold, there is a relatively compact W neighbourhood of x homeomorphic to a ball in some euclidean space, in particular it is second countable, and since the fibre $f^{-1}(f(x))$ where x belongs, is discrete, we may choose W so that $W \cap f^{-1}(f(x)) = \{x\}$.

The image $f(\partial W)$ of the boundary of W is compact and closed and contains x. Hence there is an open set U from the basis $\mathcal{U}$ containing $f(x)$ which is disjoint from

$f(\partial W)$. Let B be the connected component of $f^{-1}(U)$ where x lies. We claim that $B \subseteq W$, and hence B is second countable and belongs to $\mathcal{B}$. Since $B \cap \partial W =$ it holds that

$$B = (B \cap W) \cup (B \cap X \setminus \overline{B})$$

The two sets appearing in the union are open and the union is clearly disjoint. Hence B, being connected, must equal one of the sets and the other is empty. As $x \in B \cap X$, it must hold that $B = B \cap X$, and therefore $B \subseteq W$.

The second stages starts with the elementary observation that only countably many of the connected components of $f^{-1}(U)$ can intersect a given B from $\mathcal{B}$.

Let $\mathcal{B}_n$ be the subset of $\mathbf{B}$ of those B's that can be connected with B_0 theorug a chain of sets from $\mathcal{B}$ of lebgth $n+1$: that is the sets for which there exsists a sequence $B_0, \ldots, B_n$ og sets from $\mathcal{B}$ with $B_n = B$ and $B_i \cap B_{i+1} \neq \emptyset$. Since X is connected it holds true that $\bigcup \mathcal{B}_n = \mathcal{B}$. Evidently the union $\bigcup_{B \in \mathcal{B}_n, n} B$ is open and connected and therefore meets every element B from $\mathcal{B}$. This implies that B meets some B' in some $\mathcal{B}_n$ hence B belongs to $\mathcal{B}_{n+1}$.

To see that $\mathcal{B}$ is countable it suffices to see that each $\mathcal{B}_n$ is, and this follows by induction: There are only countably many possible B's from $\mathcal{B}$ that meets a given A in B_n, and hence B_{n+1} is countable if $\mathcal{B}_n$ is. ❑

Example 6.5. Assume that X has a holomorphic 1-form that never vanishes. Show that there is meromorphic function f_2 and f_3 with a double and a triple pole at x as only singularities. Show that there is a non trivial cubic polynomial $P(x, y)$ such that $P(f_3, f_2) = 0$.

Problem 6.13. Let $D = \sum_x n_x x$ be a positive divisor; *i.e.*, $n_x \geq 0$ for all x and as usual, the n_x's form a locally finite familly. Let $L(D)$ be the space of meromrphic functions f with $(f) \geq -D$. Show that $\dim L(D) \leq degD + 1$. ✷

7. Del

The uniformization theorem

On the road to the uniformisation theorem, we exploit theorem **??** to find a meromorphic function f on a simply connected Riemann surface X having a simple pole as its sole singularity. This gives a holomorphic map $f\colon X \to \hat{C}$ whose fibre over the point ∞ is just one point and that point counts with multiplicity one. Hence when f is proper it must be an open embedding as follows since the fibres of proper maps all have the same number of points (counted with multiplicities). The image $f(X)$ is therefore a simply connected domain, so if is not the entire Riemann sphere $\hat{\mathbb{C}}$, and it is either biholomorphic to $\mathbb{C}$ or $\mathbb{D}$ by Riemann mapping theorem. But properness of the map, is quite sublet.

7.0.1 The point of departure

From what our study of harmonic functions in the previous chapter, we easyly deduce that any Riemann surface carries non-constant meromorphic functions with just one pole and that pole is simple, and this function is our point of departure, and in the end of the day f will turn out to be an open immersion, that is it will be a biholomorphic between X and $f(X)$, and $f(X)$ is an open subset of $\hat{\mathbb{C}}$ by the Open Mapping Theorem. This follows immediatly if we know that f is proper since fibres of proper maps have the same number of points when counted with the appropriate multiplicity, and the f one simple pole its fibre over ∞ has just one point. However this a rather long way to go that we start on here.

Proposition 7.1 *Let X be a simply connected Riemann surface and $p_0 \in X$ a point. Then there is a meromorphic function f on X having a simple pole at p_0 as sole singularity. For any open U containing p the Diriclet norm of f satisfies $\|df\|_{X\setminus U} < \infty$ and forthermre it holds true that $(df, d\eta) = (df, *d\eta) = 0$ for any smooth function η of compact support not meeting U.*

Proof: From the existence theorem of harmonic functions 6.6 on page 203 we obtain a function g that is harmonic in the set $X \setminus \{p_0\}$ and has a singular behavior like z^{-1} at p_0. Let $u = \operatorname{Re} g$. The differential $\omega = du + i *du$ is holomorphic in $X \setminus \{p_0\}$ and has the shape $-z^{-2}dz + \phi$ near p_0 where ϕ is holomorphic near p_0.

We claim that ω is exact in $X \setminus \{p_0\}$. Indeed, since X simply connected, either $X \setminus \{p_0\}$ is simply connected or its fundamental group is $\mathbb{Z}$ generated by a small circle c round p_0. But then $\int_c \omega = \int_c z^{-2}dz + \int_c \phi = 0$, so by the Theorem of Vanishing Periods one has $\omega = df$ for a holomorphic function on $X \setminus \{p_0\}$, and clearly the principal part of f at p_0 equals z^{-1}.

Notice that $f = u + iv$ where v is harmonic function such that $dv = *du$ away from p_0.

As to the statement about the norm, a standard calculation using properties of the inner product and that u is real one deduces that

$$(df, df) = 2(du, du)$$

and the statement about $\|df\|_U$ follows from the theorem 6.6. ❑

(7.1) One small observation is that if X is compact, the degree of f is one and hence f is an isomorphism between X and the Riemann sphere. So the only compact Riemann surface of genus zero is the Riemann sphere. Phrased in a slightly different manner, there is up to isomorphism only one analytic structure on the two-sphere.

7.0.2 Notation

The proof centers around the subsets of X where either the imaginary part u of f or the real part v of is bounded from one side, and it is convenient to introduce a notation for these sets. So for α a real number we let Let $Z_\alpha = u^{-1}(-\infty, \alpha] = \{\, p \mid u(p) \leq \alpha \,\}$, and $Z^\alpha = u^{-1}(-\infty, \alpha] = \{\, p \mid u(v) \geq \alpha \,\}$. The corresponding sets where imaginary part is bounded from either side are denoted by W_α and W^α.

An essential part of the proof is to control the asymptotic behavior of $f(p)$, that is when p is far away from the base point p_0. The presise menaing of this is as follows.

The main ingredient in the proof is the following property

Proposition 7.2 *The function $u(x)$ tends to zero when x tends to infinity in $X \setminus \overline{D}$. That is, for every $\epsilon > 0$, there is a compact set $K \subseteq X \setminus \overline{D}$ such that $|u(x)| < \epsilon$ when $x \notin K$ (but $x \in X \setminus \overline{D}$).*

First of all, it suffices to show that the restriction of f to the inverse image of any half plane is proper. Indeed, any compact set in $\hat{\mathbb{C}}$ is contained in a half plane, and by a benign coordinate shift in $\mathbb{C}$ we we can assume the half plane to be the upper half plane.

Lemma 7.1 *If $u = \operatorname{Re} f$ then for any $\epsilon > 0$ there is a compact set K with $x \in K$ such that $|u(z)| < \epsilon$ when $z \notin K$.*

(7.1) The lemma we prove in this paragraph is fairly general and is valid for any continuous function u on a topological space X (which will be a Riemann surface for us). We say that the function u tends to a value c when p tends to infinity if and only if for any positive ϵ given, there is compact K in X such that $|u(p)-c|<\epsilon$ whenever $p\notin K$. In a similar manner we say $u(p)$ teds to infinity, if there for any constant C one may find a compact K in X with $u(p)>C$ for all p not in K. We introduce some notation and for constants c and d we let $X_c=u^{-1}(-\infty,c]$ and $X^d=u^{-1}[d,\infty)$

Lemma 7.2 *Let u be any real function on the topological space X. Either $u(p)$ tends to a value c when p tends to infinity, or $u(p)$ tends to $\pm\infty$, or one may find $\alpha<\beta$ such that neither $X_\alpha=u^{-1}(-\infty,\alpha]$ nor $X^\beta=u^{-1}[\beta,\infty)$ is compact.*

Proof: Assume that neither of two first three possibilities occur. If all the sets $X_\alpha=u^{-1}(-\infty,\alpha]$ were compact $u(x)$ would evidently tend to infinity with x, and if all $X^\beta=u^{-1}[\beta,\infty)$ were compact $u(x)$ would tend towards $-\infty$. Hence there is at least one pair α,β with both sets X_α and X^β non-compact. If α and β are different we are through, so assume $\alpha=\beta$. Given $\epsilon>0$. If $X_{\alpha-\epsilon}$ or $X^{\alpha+\epsilon}$ is non-compact we are through; indeed, we may use one of the pairs $X_{\alpha-\epsilon},X^\alpha$ or $X_\alpha,X^{\alpha+\epsilon}$ according to the case. So we can assume that $K=X^{\alpha+\epsilon}\cup X_{\alpha-\epsilon}$ is compact. In its complement it holds that $|u(x)-\alpha|<\epsilon$, and hence u tends towards the constant α. ❑

(7.2) To apply this lemma we need to get rid of the cases that $u(p)$ tends to $\pm\infty$. We have our function u whose derivate is $d\alpha$ it is smooth and harmonic off a small but fixed disk D about the base point p_0. The conjugate differential $*d\alpha$ is closed and since we work on a simply connected surface it is exact. Hence u has a conjugate function v with $dv=*\alpha$. It is smooth and harmonic where u is, that is off D. The constant $C=\max(\sup_{p\in D}|u|,\sup_{p\in D}|v|)$ plays a role in what follows.

Lemma 7.3 *The function v does not tend to $\pm\infty$ in X^c.*

Proof: Assume that v tends to infinity in X^c. Let $d>C$, and let $W=u^{-1}[-\infty,c)\cap v^{-1}[d,\infty)$ and assume that W is compact. Since $d>C$ the disk D is lies in the exterior of W. The boundary ∂W has two components B_1 and B_2. One, say B_1, is part of level curve $v(p)=d$ and the other is part of the level curve $u(p)=c$. Stokes' formula gives

$$0=\int_W d(*dv)=\int_{\partial W}*dv=\int_{B_1}*dv+\int_{B_2}*dv$$

The boundary ∂W has two components B_1 and B_2. The component B_1 being part of the level curve $v(p)=c$ the form $*dv$ is as we know from calculus courses tangent to B_1 and one has

$$\int_{B_1}*dv>0.$$

On the other hand along the second component B_2 which is part of the level curve $u(p)=c$, the form $*dv=du$ is orthogonal to the tangent and the so the integral of $*dv$ along B_2 vanishes. Contradiction! A similar argument with the set $v^{-1}(-\infty,d]$ shows that v does not go to $-\infty$. ❑

Similar resultas hold for all compinations of the sets Z's, and the W's and with the role of u and v interchanged.

Lemma 7.4 *There are cinztants c_i so that $\psi_i = \eta_i + c_i$ with ψ_i converges to u in every coordinate patch.*

PROOF: Fix c_i so that ψ_i converges in $L^2(U)$ to u, the set of points x in X for which ψ_i convegers to u in L^2-norm over a coordinate beighbourhood is clearly open. But in fact it is closed as well, for if x i a boundary point there is a neighbourhood V of x and constants c'_i such that $\psi_i + c'_i$ tend to u over V. But V contains an open set over which ϕ_i tends to u as well, so c'_i tends to zero, and ψ_i tends to u in the entire neighbourhood V.

For any bunded, open and convex plane set Ω and any smooth ψ defined in a neighbourhood of Ω with $\int_\Omega \psi = 0$, one has teh following estimate

$$\|\psi\|_2 \leq C_\Omega \|d\psi\|_2$$

where C_Ω is a positive constant than only depends on the domain Ω.

Let $c_i = \int_K |\eta_i|^2 d\alpha \wedge *d\alpha\phi$ ❑

Lemma 7.5 *Let β be given. Then $v^{-1}(\beta, \infty)$ is a connected set.*

PROOF: Let H' a the connected component where p_0 lies, and assume that H is another connecetd component. Let $\xi(t)$ and $\eta(t)$ be two auxiliary functions both C^∞, bounded with bounded derivatives and are such that $\xi(t) > 0$ for $t > 0$ and $\xi(t) = 0$ when $t \leq 0$ and such that $\eta'(t) > 0$ for all t. Clraly $\xi(v-\beta)\eta(u)$ is a limit of smooth functions of compact support. Define the function h on X by $h(p) = \xi(v-\beta)\eta(u)$ for $p \in H$ and $h(p) = 0$ elswere. One finds in H by differentiating

$$\begin{aligned} h_x &= \eta'\xi u_x + \xi'\eta v_x = \eta'\xi u_x - \xi'\eta u_y \\ h_y &= \eta'\xi u_y + \xi'\eta v_y = \eta'\xi u_y + \xi'\eta u_x \end{aligned}$$

Off H and on ∂H all derivatives of h vanish (easy induction on the order of the derivative) so h is smooth and has support in H. Assume that K is a compact subset of H, for instance a path connecting two points. Then replace h by ηh where η is a smooth function of compact support that takes the value one on a relative compact open V neighbourhood of K and takes values between 0 and 1.

Then dh is of compact support and in side U we find

$$(du, dh) = \int_X (h_x u_x + h_y u_y) dx \wedge dy > 0$$

since

$$h_x u_x + h_y u_y = \eta\xi'(u_x^2 + u_y^2) = \eta\xi' |f'(x)|$$

But by xxxx, the $(du, dh) = 0$.

❑

7.0.3 The commutative fundamental groups

Proposition 7.3 *Assume that the imaginary part $v(p)$ tends to zero when p tends to infinity in Z_α for all α. Then the maps f_+ and f_- are proper and hence biholomorphic.*

Proof: We substantial point is that f_+ and f_- are proper. We know they are unramified, so once we have established that they are proper, they will be coverings. But H_+ and H_- are simply connected, so they are isomorphism.

To see that *e.g.*, f_+ is proper, let $K \subseteq H_+$ be a compact set. The imaginary part of points in K are bounded below by say β, and the real parts are bounded above by say α. Since $v(p)$ tends to zero in Z_α, there is a compact K_α such that for points p in $K_\alpha \cap Z_\alpha$ it holds that $v(p) < \beta$. But then $f^{-1}(K) \subseteq K_\alpha \cap Z_\alpha$ and consequently it is compact. ❑

Lemma 7.6 *One of the three cases occur:*

There is a constant c so that for all α the imaginary part $v(p)$ tend to c in Z_α or in Z^α or the real part $u(p)$ tends to c in W_α or W^α.

There exists $\alpha < \beta$ and $\gamma < \delta$ such that the four intersections

$$Z_\alpha \cap W_\gamma \quad Z_\alpha \cap W^\delta \quad Z^\beta \cap W_\gamma \quad Z^\beta \cap W^\delta$$

are non-compact

Lemma 7.7 *The imaginary part v does not tend to $\pm\infty$ in Z_α nor in Z^α. The real part u does not tend to infinity in W_α nor in W^α.*

Lemma 7.8 *One of the following thwo cases occure:*

- *There is a constant c so that for all α the imaginary part $v(p)$ tend to c in Z_α or in Z^α or the real part $u(p)$ tends to c in W_α or W^α.*
- *There exist real numbers $\alpha < \beta$ and $\gamma < \delta$ such that the four intersections*

  $$Z_\alpha \cap W_\gamma \quad Z_\alpha \cap W^\delta \quad Z^\beta \cap W_\gamma \quad Z^\beta \cap W^\delta$$

  are non-compact

7. Del

The uniformization theorem

version $-\infty + \epsilon$ — Friday, November 25, 2016 8:12:11 AM

Very, very preliminary and incomplete version prone to mistakes and misprints! More under way.

2016-11-30 13:04:29+01:00

There a many proofs of the Uniformisation theorem

On the road to the uniformisation theorem, we exploit theorem **??** to find a meromorphic function f on a simply connected Riemann surface X having a simple pole as its sole singularity. This gives a holomorphic map $f\colon X \to \hat{C}$ whose fibre over the point ∞ is just one point and that point counts with multiplicity one. Hence when f is proper it must be an open embedding as follows since the fibres of proper maps all have the same number of points (counted with multiplicities). The image $f(X)$ is therefore a simply connected domain, so if is not the entire Riemann sphere $\hat{\mathbb{C}}$, and it is either biholomorphic to $\mathbb{C}$ or $\mathbb{D}$ by Riemann mapping theorem. But properness of the map, is quite sublet.

7.0.1 The point of departure

From what our study of harmonic functions in the previous chapter, we easyly deduce that any Riemann surface carries non-constant meromorphic functions with just one pole and that pole is simple, and this function is our point of departure, and in the end of the day f will turn out to be an open immersion, that is, it will be a biholomorphic map between X and $f(X)$, and the image $f(X)$ is an open subset of $\hat{\mathbb{C}}$ by the Open Mapping Theorem. This follows immediatly if we know that f is proper since fibres of proper maps have the same number of points when these are counted with the appropriate multiplicity, and our function f having just one simple pole has a fibre over ∞ with just one point. However it is a rather long way to go to see that f is proper.

PointOfDep

Proposition 7.1 *Let X be a simply connected Riemann surface and $p_0 \in X$ a point. Then there is a meromorphic function f on X having a simple pole at p_0 as sole singularity. For any open U containing p the Diriclet norm of f satisfies $\|df\|_{X\setminus U} < \infty$ and forthermre it holds true that $(df, d\eta) = (df, *d\eta) = 0$ for any smooth function η of compact support not meeting U.*

Proof: From the Existence Theorem of harmonic functions 6.6 on page 203 we obtain a function g that is harmonic in the set $X\setminus\{p_0\}$ and has a singular behavior like z^{-1} in a vicinity of p_0. Let $\tilde{u} = \operatorname{Re} g$. The differential $\omega = d\tilde{u} + i * d\tilde{u}$ is holomorphic in $X\setminus\{p_0\}$ and has the shape $-z^{-2}dz + \phi$ near p_0 where ϕ is holomorphic near p_0.

We claim that ω is exact in $X\setminus\{p_0\}$. Indeed, since X simply connected, either $X\setminus\{p_0\}$ is simply connected or its fundamental group is $\mathbb{Z}$ generated by a small circle c round p_0. But then

$$\int_c \omega = \int_c z^{-2}dz + \int_c \phi = 0,$$

so by the Theorem of Vanishing Periods one has $\omega = df$ for a holomorphic function on $X\setminus\{p_0\}$, and clearly the principal part of f at p_0 equals z^{-1}.

Notice that $f = \tilde{u} + i\tilde{v}$ where $\tilde{v}$ is a harmonic function such that $d\tilde{v} = *d\tilde{u}$ away from p_0.

As to the statement about the norm, a standard calculation using properties of the inner product and that u is real one deduces that

$$(df, df) = 2(d\tilde{u}, d\tilde{u})$$

and the statement about $\|df\|_{X\setminus U}$ follows from the theorem 6.6. ❑

CompactCase

(7.1) One small observation is that if X is compact the map f is automatically proper and hence is a biholomorphic map between X and the Riemann sphere. The only compact Riemann surface of genus zero is therefore the Riemann sphere. Phrased in a slightly different manner, there is up to isomorphism only one analytic structure on the Riemann sphere:

ThewCompact case

Theorem 7.1 *If X is a compact simply connected Riemann surface, then X is biholomorphic to the Riemann sphere.*

This closes the case of compact, simply connected surfaces and henceforth we assume that X is not compact.

(7.2) As stated in the proposition 7.1 the function f enjoys the property that $\|df\|_{X\setminus U} < \infty$ for all open neigbourhood of the pole p_0.

Recall that $df \wedge \overline{*df} = -i\,|J(f)|\, d\bar{z} \wedge dz = |J(f)|\, dx \wedge dy$, where $J(f)$ denotes the Jacobian of f, locally given as $|f'(z)|^2$. If A is a region in X where f is injective the usual theorem about changing variables in the integral shows that $\int_A df \wedge \overline{*df} = \|df\|_A^2$

equals there area of the image $f(A)$. If f is not injective parts of $f(A)$ can be covered several times by f so the equality does not persist, however the inequality

$$\text{area of } (A) \leq \int_A df \wedge \overline{*df} = \|df\|_A^2$$

is generally true (of course A must reasonably nice, *e.g.*, a domain).

It follows that given $\epsilon > 0$, there is a compact K in X so that $\|df\|_K < \epsilon$. Indeed, by definition of the integral there is partition of unity η_i with η_i supported in a compact set K_i so that

$$\int_{X \setminus U} df \wedge \overline{*df} = \sum_i \int_{K_i} \eta_i df \wedge \overline{*df} = \sum_i \|\eta_i df\|_{K_i}^2 .$$

Since X is not compact the sum to the right is a genuine infinite sum, and that $\|df\|_{X \setminus U}$ is finite means precisely that this sum converges. Hence there is an N so that

$$\|df\|_{X \setminus U_N}^2 = \sum_{i > N} \|\eta_i df\|_{K_i}^2 < \epsilon$$

where $U_N = X \setminus \bigcup_{i \leq N} K_i$. We have thus established

Proposition 7.2 *Given an $\epsilon > 0$, there exists a compact set K such that for any domain A in X disjoint from K the image $f(A)$ has an area at most equal toϵ.*

7.0.2 Notation and conventions

The proof centers around the subsets of X where either the imaginary part u or the real part v of f is bounded from one side, and it is convenient to introduce a notation for these sets. So for c a real number we let $Z_c = u^{-1}(-\infty, c] = \{ p \mid u(p) \leq c \}$, and $Z^c = u^{-1}(-\infty, c] = \{ p \mid u(v) \geq c \}$. The corresponding sets where the imaginary part is bounded from either side are denoted by W_c and W^c. When $c' < c$ one obviously have $Z_{c'} \subseteq Z_c$ and $Z^c \subseteq Z^{c'}$ and the corresponding relations for W_c and W^c hold.

An essential part of the proof is to control the asymptotic behavior of $f(p)$, that is the over all size of $|f|$ when p is a point far away from the base point p_0. The precise meaning of this is as follows. In general, a continuos function ϕ on a topological X space is said to tend to a constant c when p tends to infinity if for any $\epsilon > 0$ one may find a compact set K such that $|\phi(p) - c| < \epsilon$ when $p \notin K$. We say that ϕ tends to ∞ if for any c there is a compact set K such that one has $\phi(p) > c$ for $p \notin K$, and finally, ϕ to $-\infty$ if for every c one has $\phi(p) < c$ for p not lying in a compact K.

The upper and lower half planes are denoted by H_+ and H_- respectively, and f_+ and f_- are the restrictions of f to respectively $f^{-1}(H_+)$ and $f^{-1}(H_-)$.

7.1 A connectedness theorem

This section is devoted to the proof of a basic connected result. The regions where either the real part or the imaginary parts are bounded—from above or from below—by a constant, are connected. This is a basic ingredient The proof of the uniformisation theorem we present is based on this result, but one nice corollary we include in this section, is that function f is unramified. Like in Hitchcock film, the suspense increases through out. We start with some words about a simple dipole, continue with a local study of f and finally prove the global connectedness theorem.

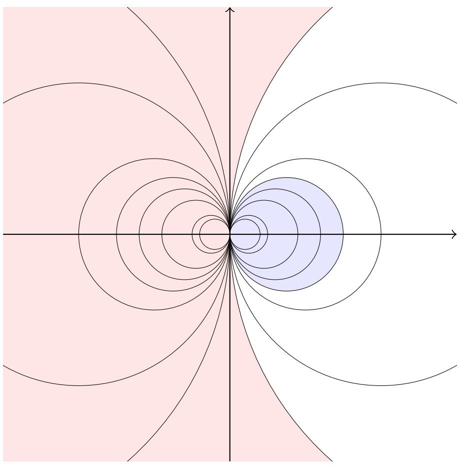
(7.1) THE SIMPLE DIPOLE It is worth while casting a glans on the the simple function $1/z$ and its behavior near the origin. One has

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2}$$

and the level sets of the real and imaginary parts are circles with centers one axis. The locus where the real part takes the value c, for instance, is given by the equation

$$x = c(x^2 + y^2),$$

which describes a circle through the origin with centre at $c^{-2}/2$. The regions $u(z) > c$ and $u(z) < c$ are respectively the interior and the exterior of this circle. We have depicted some level sets of the simple dipole z^{-1} in figure 7.2 where one of the regions of type $u > c$ coloured red and a region of the type $u < c$ coloured blue.



Figur 7.1: *Level sets of the real part of the simple dipole* z^{-1}.

Dipol

(7.2) THE PICTURE NEAR THE POLE This sounds like an entry in Fridjof Nansen's diary, but the substance is that locally near the pole of f, the configuration of the level sets is a slight deformation of that of the simple dipole.

Indeed, if w is the standard coordinate on the Riemann sphere near the north pole the relation $w = f(z)^{-1}$ holds in a patch (U, z) near p_0. Since the pole of f is simple, the derivative $\partial_z w$ does not vanish at p_0, and hence f induces a biholomorphic map between U and the polar neighbourhood $V = f(U)$ in $\hat{\mathbb{C}}$, and in V one has $f(z) = w^{-1}$.

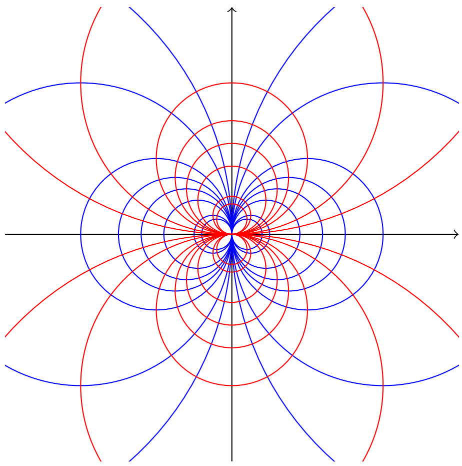
The point is that w^{-1} is a simple dipole at the north pole (a simple polar dipole, one is tempted to say), and hence locally the the "dipole" $f(z)$ is biholomorphic to a simple dipole. For instance, we immediately get the following lemma:

LocalConn

Lemma 7.1 *Given a real number c. The four sets* $u^{-1}(-\infty, c)$, $u^{-1}(c, \infty)$, $v^{-1}(-\infty, c)$ *and* $v^{-1}(c, \infty)$ *are locally connected near* p_0.

Proof: The statement is true for the corresponind sets in the simple dipole picture. ❑

Another conclusion one can draw is that if c and d are real numbers whose absolute values are sufficiently large, the level sets $v(p) = c$ and $u(p) = d$ have components that are simple closed paths contained in the neigbourhood U. Indeed, the level sets of a simple dipole will be contained in a given open neighbourhood once the levels are sufficenetly large in absolute value.



Figur 7.2: *Level sets of the simple dipole* z^{-1}.

(7.3) The conectedness theorem We are now ready for the fundamental connectedness result in this part of the story. It says that the sets from lemma 7.1 are not only locally connected near p_0, but in fact they are connected.

Connecte

Proposition 7.3 *Let c be a given real number. The four loci* $u^{-1}(-\infty, c)$, $u^{-1}(c, \infty)$, $v^{-1}(-\infty, c)$ *and* $v^{-1}(c, \infty)$ *are connected.*

Notice that the statement is about open loci where u or v are bounded. Of course the corresponding closed sets will also be connected, but since the closure of two different components of the open loci could have a common point, the statement in the proposition is stronger.

Proof: We shall carry out the proof for the set $v^{-1}(c, \infty)$, the proofs of other three cases being *mutatis mutandis* the same.

Assume that $v^{-1}(c, \infty)$ is not connected. Thence, as we know it is locally connected near p_0, $v^{-1}(c, \infty)$ must have at least one connected component whose closure does not contain p_0. Pick your favorite one and call it H.

We need two auxiliary real functions $a(t)$ and $b(t)$ of one real variable. They are both smooth and bounded and have bounded derivatives, and additionally, they satisfy the following properties: When $t > 0$ it holds that $a(t) > 0$ and $a(t) = 0$ when $t \leq 0$ and the function b is positive everywhere. Notice that a being smooth, all its derivatives vanish at the origin.

The main character of the piece is a function h that is defined on the Riemann surface X by the assignment

$$\begin{cases} a(v-c)b(u) & \text{in } H \\ 0 & \text{off } H. \end{cases}$$

As we soon shall see, the function h is smooth, its differential dh is quadratically integrable and dh is the L^2-limit of differentials of smooth functions of compact support. Hence dh belongs to the space E from the previous chapter.

Computing derivatives of h in a small patch one finds— using that u and v are harmonic conjugates—in the part of the patch lying within H that

$$\begin{aligned} h_x =& b'au_x + a'bv_x = b'au_x - a'bu_y \\ h_y =& b'au_y + a'bv_y = b'au_y + a'bu_x \end{aligned}$$

On the boundary of H both a and a' vanish, so the two partials vanish there as well. This shows that h is of class C^1, but in fact, an easy induction implies that the all higher derivatives vanish as well. Therefore h is smooth.

The differential dh is the limit of forms of the shape $d\eta_i$ with η_i smooth of compact support. Indeed, as H does not have the basepoint p_0 in its closure, the form du is a limit of forms $d\eta_i$. Performing the construction above with η_i in place of u, *i.e.,* letting $h_i = a(\operatorname{Re}\eta_i - c)b(\operatorname{Im}\eta_i)$, it is easy to see that dh_i tends to dh in L^2-norm using that a and b are bounded with bounded derivatives. It follows by xxx that

$$(du, dh) = 0 \tag{7.1}$$

NullingAvIndreProd

On the other hand, in a patch U in H one finds that

$$h_x u_x + h_y u_y = ba'(u_x^2 + u_y^2)$$

which is positive almost everywhere in U, both a' and b being positive there and $u_x^2 + u_y^2$ only vanishing at the critical points of f which form a discrete set. Hence one has

$$(du, dh) = \int_U du \wedge *dh = \int_U (h_x u_x + h_y u_y) dx \wedge dy > 0.$$

This being true for all patches in H it follows that

$$(du, dh) = \int_X du \wedge *dh > 0$$

which contradicts (7.1). ❏

(7.4) THE MAP f IS UNRAMIFIED As announced, the following is a direct consequence of the connectedness theorem:

Proposition 7.4 *The holomorphic map $f\colon X \to \hat{\mathbb{C}}$ is unramified; i.e., its derivative never vanishes.*

PROOF: At the point p_0 this since the pole is simple, or if you want, it follows from the formula (7.2) in exercise 7.3 below as well. So assume that p is point different from p_0 where the derivative vanishes. After changing f by an additive constant we may assume that $f(p) = 0$.

After proposition xxx there is a patch (U, z), centered at p, where f has the shape $z \mapsto z^n$ with $n \geq 2$. For simplicity we assume that $n = 2$, leaving the general case to the zealous students

The set $u^{-1}(0, \infty)$ is no more locally connected at p, it consists of the parts of the first and third open quadrant lying in the patch. Choose a point z_1 in one of these. Then $-z_1$ lies in the other, and since $u^{-1}(0, -\infty)$ is connected by the Connectedness theorem there is a path entirely contained in $u^{-1}(0, \infty)$ connecting the two points. This path can be closed up by adding the line segment between z_1 and $-z_1$. So in the end of the day, we have a closed path γ_1 in $u^{-1}(0, \infty)$, shaped like a line at the origin.

In a similar fashion we produce a closed path γ_2 in $u^{-1}(-\infty, 0)$. The two paths intersect at the origin, transversally since they they are shaped like different line segments near the origin, and the local intersection number is therefore ± 1, the sign depending the orientations. Now, the origin is their only common point, the sets $u^{-1}(0, \infty) > 0$ and $u^{-1}(-\infty, 0) > 0$ being disjoint. So for the global intersection number we have $\gamma_1 \cdot \gamma_2 = \pm 1$ which is impossible as all intersection numbers on a simply connected surface vanish. ❑

PROBLEM 7.1. Show that lemma 7.1 is not true for $f(z) = z^{-n}$. ✸

PROBLEM 7.2. Given real numbers c and c' with $c' < c$. Show that the two sets $u^{-1}(c', c)$ and $v^{-1}(c', c)$ are locally connected near p_0. ✸

PROBLEM 7.3. Assue that $f(z) = z^{-1} + \phi(z)$ with ϕ holomorphic near 0. Show the equality DerivExerc

$$(f(z)^{-1})' = \frac{1 - z^2\phi'(z)}{(1 + z\phi(z))^2}. \tag{7.2}$$

✸ DeriveetAtPnull

PROBLEM 7.4. Given real numbers $c' < c$. Show that the sets $u^{-1}(c', c)$ and $v^{-1}(c', c)$ are connected. HINT: Find appropriate auxiliary functions. ✸

7.1.1 Bounding the real and the imaginary part

We continue the praxis letting (U, z) be a patch near the pole p_0 where f is biholomorphic so the local configuration of the level sets in U is that of a simple dipole.

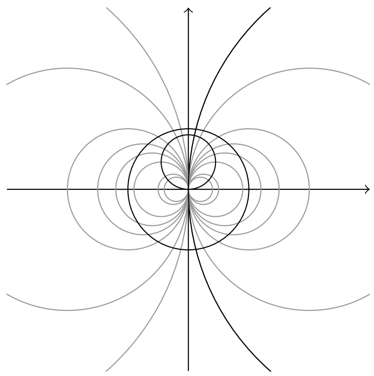
Proposition 7.5 *Assume that Z^c is not compact, then the imaginary part v does not tend to infinity in Z^c.*

The corresponding statements for v on Z_c and for u on W^c and W^c follows by applying the proposition to the functions $\pm f$ and $\pm if$.

Proof: Assume that v tends to infinity in Z^c.

To begin with we choose d positive and so big that the level set $v = c$ has a component entirely contained in the neighbourhood U—it is shaped like a small circle passing through p_0, and we denote by D the disc it bounds. Locally near the pole the set $W_d \cap U$ is the complement of D. Let A be a disc centered at 0 containing D and contained in U. The situation is sketched in figur 7.3 below.

Now the set $W = Z^c \cap W_d$ is compact since v tends to infinity in Z^c, and hence the set $W' = W \setminus A$ is compact as well.



Figur 7.3: *The situation at the pole.*

LocalPictNeraPoe In the set W' the functions u and v are conjugate harmonic functions, in particular since v is harmonic one has

$$0 = \int_{W'} d(*dv) = \int_{\partial W'} *dv. \tag{7.3}$$

MainIntegral The contradiction we shall arrive at, is that the boundary integral to the right must be strictly positive. To evaluate that integral we observe that the boundary $\partial W'$ can be split into three parts.

The first part, which we call B_1, is contained in the level set $v = d$. The disk D where $v > d$ in U being entirely enclosed in A, the boundary part B_1 does not meet the neigbourhood U, but since $Z^c \cap W_d$ is compact and Z^c not, it must be non-empty. Indeed, were it empty, the intersection $Z^c \cap W_d$ would be both open and closed in Z^c, and Z^c being connected it would follow that $Z^c = Z^c \cap W_d$. This is impossible since Z^c is assumed to be non-compact. Along the path B_1, which is a part of the level set $v = d$ and oriented according to the rule from Stoke's theorem, the form $*dv$ is parallel to the tangent. Hence the integral $\int_{B_1} *dv$ is strictly positive:

$$\int_{B_1} *dv > 0.$$

The second part B_2 is contained in the level set $u = c$. The functions u and v are conjugate harmonic functions in W so it holds that $*dv = du$. Now from calculus we know that the from du vanishes along the level set $u = c$ (in calculus-lingo du is the gradient and the gradient is normal to level curves) Hence the integral $\int_{B_1} *dv$ vanishes:

$$\int_{B_2} *dv = 0.$$

The third and last part B_3 is completely contained in the patch U, and is formed by the sector of the circle ∂A where $u > c$. It is has two end points—say p_1 and p_2—both lying on the level set $u = c$, we find

$$\int_{B_3} du = u(p_1) - u(p_2) = c - c = 0.$$

And there we are. We have a contradiction, The integral in (7.3) is both zero and strictly positive! (7.3) is both zero and strictly positive! ❏

Decay at infinity

Having the physics in mind and regarding f as an electric dipole on our surface, it is pretty clear that the field induced must decay to zero at infinity. The effect of the dipole is minimal at large distances. And indeed, this is true in the mathematical setting as well, except for a constant, which physically is just reflects the choice of zero for the units.

Proposition 7.6 *There is a number a so that the imaginary part $v(p)$ tends to a in Z_c for every c.* TendStoZero

Replacing f by $f - a$ we may as well assume that v tends to zero in Z_c. The rest of the is devoted the proof of this proposition, but first shall see that it implies the Uniformisation theorem:

(7.1) Proposition 7.6 implies the theorem. Indeed, f_+ is proper, for if K is compact subset of the upper half plane, then the real part is bounded above, say by c. The imaginary part of the points in K is bounded away from zero, say by $\operatorname{Im} z > \epsilon$ when z lies in K. Since v tends to zero in Z_c, t follows that $f_+^{-1}(K)$ is contained in a compact set, hence compact being closed.

It follows that f_+ being unramified by xxx, is biholomorphic to $f_+^{-1}(H_+)$ and H_+, ditto, f_- is a biholomorphic from $f_-^{-1}(H_-)$ and H_-. If $f(p_1) = f(p_2) = z$, clearly z is real, and there is a neighbourhood D of z and disjoint neighbourhoods D_i of p_i mapping biholomorphically to D. Since $f_-^{-1}(H_-) \cup f_+^{-1}(H_+)$ is dense in X, parts of D_1 and D_2 both map to $D \cap H_+$ contradicting that f_+ is injective.

First of all, it suffices to show that the restriction of f to

KonstantExists

Lemma 7.2 *Let u be any continuous real function on a topological space X. Either $u(p)$ tends to a value c when p tends to infinity, or $u(p)$ tends to $\pm\infty$, or one may find $\alpha < \beta$ such that neither $X_\alpha = u^{-1}(-\infty, \alpha]$ nor $X^\beta = u^{-1}[\beta, \infty)$ is compact.*

PROOF: Assume that neither of two first three possibilities occur. If all the sets $X_\alpha = u^{-1}(-\infty, \alpha]$ were compact $u(x)$ would evidently tend to infinity with x, and if all $X^\beta = u^{-1}[\beta, \infty)$ were compact $u(x)$ would tend towards $-\infty$. Hence there is at least one pair α, β with both sets X_α and X^β non-compact. If α and β are different we are through, so assume $\alpha = \beta$. Given $\epsilon > 0$. If $X_{\alpha-\epsilon}$ or $X^{\alpha+\epsilon}$ is non-compact we are through; indeed, we may use one of the pairs $X_{\alpha-\epsilon}, X^\alpha$ or $X_\alpha, X^{\alpha+\epsilon}$ according to the case. So we can assume that $K = X^{\alpha+\epsilon} \cup X_{\alpha-\epsilon}$ is compact. In its complement it holds that $|u(x) - \alpha| < \epsilon$, and hence u tends towards the constant α. ❑

Lemma 7.3 *One of the following statemets holds:*

- *There real numbers $\alpha < \beta$ and $\gamma < \delta$ such that $W^\gamma \cap Z_\alpha$, $W^\delta \cap Z_\alpha$, $W^\gamma \cap Z^\beta$ and $W^\delta \cap Z^\beta$ are non-compact.*
- *The imaginary part v or the real part u tends to a constant either in Z_α for all α such that Z_α is non compact, or in Z^α for all α such that Z_α is non-compact.*

PROOF: Assume the contrary to the second statement. Then u does not tend to constant in X itself, so in view of xxx, lemma 7.2 above then gives us an α and a β with $\alpha < \beta$ such that neiter Z_α nor z^β is compact. Then we apply the lemma 7.2 to v on Z_α and Z^β and conlude that there exist two pairs $\gamma < \delta$ and $\gamma' < \delta'$ of real numbers with $W_\gamma \cap Z_\alpha$, $W^\delta \cap Z_\alpha$, $W_{\gamma'} \cap Z^\beta$ and $W^{\delta'} \cap Z^\beta$ are all non-compact sets. Renaming the smaller of the numbers γ and γ' as γ and the bigger of δ and δ' to δ, we are trough. ❑

Finally, let ϵ be a numbef less that $(\beta - \alpha)(\delta - \gamma)$ given and find a cmpact such that the $\|XK\|df < \epsilon$. Pick points x_1, x_2, x_3 and x_4 in the sets $W^\gamma \cap Z_\alpha$, $W^\delta \cap Z_\alpha$, $W^\gamma \cap Z^\beta$ and $W^\delta \cap Z^\beta$.

Since the Z's and the W's are connecetd, we can joint the points such tha the enuong closed curve has an image enclosing the rectangle $[\alpha, \beta] \times [\gamma, \delta]$, and together with xxxx this contradicts xxxx.

Lemma 7.4 *Assume that there exists α such that $Z_\alpha \cap W_0$ is compact for all $\beta > \alpha$ one has $Z^\beta \cap W_0$ compact. Then f_- is proper.*

7.1.2 Abelian fundamental group

The fundamental group $\pi_1(X)$ acts on the universal cover $\tilde{X}$ of a Riemann surface X. The action is free and proper meaning that for any $x \in \tilde{X}$ there is a neigbourhood U such that $gU \cap U = \emptyset$ for non-trivial g from $\pi_1(X)$.

Hence it is great interest to study subgroups of the automorphism groups $\mathrm{Aut}(\tilde{X})$ in the three cases.

(7.1) The Riemann sphere The autumorphism group of the sphere is $\mathrm{Aut}(\hat{\mathbb{C}}) = \mathrm{PGl}(2, \mathbb{C})$ that is functions of the form $\psi(z) = (az+b)(cz+d)$. They can be represented by matrices whoce determinant is one, so lives in the exact sequence

$$1 \longrightarrow \{\pm I\} \longrightarrow \mathrm{Gl}(2, \mathbb{C}) \longrightarrow \mathrm{Aut}(\hat{\mathbb{C}}) \longrightarrow 1$$

The fixed points of y is determined by $\psi(z) = z$ which is a quadratic equation. It has one or two solustions, dependig on the discrimant. The discriminant equals $(\mathrm{tr}\, A)^2 - 4$ when A is normalised so that $\det A = 1$.

The action is homogenous, even three-point homogenous. Given z_1,z_2 and z_3 different points all in the finite plane the following fractional linear map sends the triple $\xi = (z_1, z_2, z_3)$ to the triple $(0, \infty, 1)$:

$$\psi_\xi(z) = \frac{z - z_1}{z - z_2}\frac{z_3 - z_1}{z_3 - z_2}$$

The no non-trivial element act freely on the sphere and we have

Proposition 7.7 *The only Riemann surface that have $\hat{\mathbb{C}}$ as univeral cover is $\hat{\mathbb{C}}$ itself.*

(7.2) The finite plane C In this case the automorphisms are all of the form $az + b$. Hence the group $\mathrm{Aut}(\mathbb{C})$ sits in the exact sequence

$$0 \longrightarrow T \longrightarrow \mathrm{Aut}(\mathbb{C}) \longrightarrow R \longrightarrow 0$$

where T is the subgroup of traanslations, that is $\tau_b(z) = z + b$ and R is the subgroup of the so called *dilations*, *i.e.*, automorphisms of the form $\rho_a(z) = az$. The gooup R acts by conjugation on T by the formula $\rho_a \circ \tau_b \circ \rho_a^{-1} = \tau_{ab}$, and $\mathrm{Aut}(\mathbb{C})$ is the *semidirect* product of the two.

If $a \neq 1$, the automorphism $z \mapsto az + b$ has a fixed point, since the equation $az + b = z$ in that case has aolution. One deduces that if G acts greely on $\mathbb{C}$ then all elements in G are pure translations. Obviously two translations commute, hence G is abelian. We shall see that it has at most two generators:

Proposition 7.8 *Assume that $G\,\mathrm{Aut}(\mathbb{C})$ acts freely. Then G is free abelain on at most two geerators.*

PROOF: Let $\Lambda \subseteq G$ be a lattice group $\tau_\lambda(z) = z + n_1\omega_1 + n_2\omega_2$. Assume that there is an alement $g(z) = z + c$ in γ not in Λ. Simultanions Dirichlet approximation gives us to any given natural number k three integers N, p_1 and p_2 with

$$\left|c - p_i N^{-1}\right| < k^{-1}N^{-1}$$

hence for any $\epsilon > 0$ we can find N and $\lambda \in \Lambda \subseteq G$ such that

$$\left|g^N(0) - \lambda(0)\right| = |Nc - p_1\omega_1 - p_2\omega 2| < k^{-1} \leq \epsilon$$

This is impossible since the action is proper. ❑

So there are just two cases eithet $\pi_1(X)$ is free abelian of rank one, generated by $z \mapsto z + b$ say with $b \neq 0$, or $\pi_1 X$ is free abelina of rank two. In the latter case elements are of the form $z \mapsto z + n_1\omega_1 + n_2\omega_2$.

Proposition 7.9 *Assume that X is a Riemann surface whose univeral cover is biholomrphic to $\mathbb{C}$. Then*

- *X is biholomorphic yo $\mathbb{C}$;*
- *$\pi_1(X) \simeq \mathbb{Z}$ and X is biholomorphic to $\mathbb{C}^*$.*
- *$\pi_1(X) \simeq \mathbb{Z} \oplus \mathbb{Z}$ and X is biholomorphic to an elliptic curve $\mathbb{C}/\Lambda$ where Λ is a lattice in $\mathbb{C}$.*

In first case X is biholomorphic to $\mathbb{C}^*$ (also known under the alias "the holomorphic cylinder"). The function $\Phi(z) = \exp(2\pi i b^{-1} z)$ is clearly invariant under G, its derivative is nowhere vanishing, and $\Phi(z) = \Phi(z')$ if and only if z and z' are equivanet under G.

Proposition 7.10 *If f and g commute then f permutes the fixed points of g and g permutes the fixed points of f.*

PROOF: If $g(x) = x$ one has $f(g(x)) = g(f(x)) = g(x)$ ❑

Hence if f has a unique fixed point x, thehn g has x as only fixed point as well.

Lemma 7.5 *If f anfg g commutes, they either shear the fixed points or both are of order two and one swaps the fixed points of the other.*

PROOF: Assume f has 0 and ∞ as fixed points, and g has 1. Then sends 0 to ∞ and ∞ to 0, *i.e.*, og the shape bz^{-1}. But 1 is fixed so $b = 1$. Hence $f(z) = -z$. ❑