1. Del

Simple aspects of complex functions

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Still preliminary version prone to errors. At least for some time it will not be changed (except may be for correction of stupid errors)

Changes:

A lot of minor and stupid errors corrected

Thank's to all that have contributed by finding errors!

(1.1) A *a domain* in the complex plane Ω is an open non-empty and connected subset of $\mathbb{C}$. Recall that a subset A of $\mathbb{C}$ (or any topological space for that matter) is said to be *connected* if it is not the union of two disjoint open sets. Equivalently one may require that A not be the union of two disjoint closed sets. The set A is *pathwise connected* if any two of its points can be joined by a continuous path, clearly a pathwise connected set is connected, but for general topological spaces the converse dos not hold; but luckily, it holds true for open subsets of the complex plane; so an open subset Ω of $\mathbb{C}$ is connected if and only if it is pathwise connected.

(1.2) The union of two connected sets is connected provided the two sets are not disjoint. Hence any point in A is contained in a maximal connected set. These maximal sets called a connected component of A, and they form a partition of A—they are pairwise disjoint and their union equals the whole space. Connected components are always closed subsets, but not necessarily open. An everyday example being the rationals $\mathbb{Q}$ with the topology inherited from the reals. As every non-empty open interval contains real numbers, the connected components of $\mathbb{Q}$ are just all the points. One says that $\mathbb{Q}$ is *totally disconnected*.

The *path-component* of a point z consists of all the point in the set A that can be joined to z by a continuous path. The different path-components form, just like the connected components, a partition of the space.

Problem 1.1.

a) Show that a path-wise topological space is connected.

b) A space is called *locally pathwise connected* if every point admits a neigbourhood basis consisting of open and path-wise connected sets; equivalently for every point p and every open set U containing p, one may find an open and path-wise connected set contained in U and containing p. Show that if a space is locally pathwise connected, it is connected if and only if it is pathwise connected.

✸

Domains can be very complicated and their geometric complexity and subtleties form now and again significant parts of the theory— or at least, are the reasons behind long and tortuous proofs of statements seeming obvious in simple situations one often has in mind—like slightly and nicely deformed disk with a whole or two. So a few example are in place:

Example 1.1. If Z is any closed subset of the real axis not being the whole axis. Then clearly $\mathbb{C} \setminus Z$ is open and connected (one can pass from the upper to the lower half plane by sneaking through $\mathbb{R} \setminus \mathbb{Z}$) Two specific examples of interesting closed sets Z can be $\{\, 1/n \mid n \in \mathbb{N} \,\} \cup 0$ and the Cantor set $\mathfrak{c}$. ❄

Example 1.2. For each rational number p/q in reduced form, let $L_{p/q}$ be the (closed) line segment of length $1/q$ emanating from the origin forming the angle $2\pi p/q$ with the positive real axis; *i.e.*, the points of $L_{p/q}$ are of the form $te^{2\pi pi/q}$ with $0 \leq t \leq 1/q$. Let $L = \bigcup_{p/q} L_{p/q}$. Then L is closed. This is not completely obvious (so prove it!). It hinges on the fact that only finitely many of the segments $L_{p/q}$ appear in the vicinity of a point z different from the origin. The complement U of L is therefore open, and it is connected (the ray from the origin through a point in U has just the origin in common with L, and z can be connected to points outside the unit disk, and as L is contained in the closed unit disk, this suffices) so it is a domain. The set U is not simply connected but has the homotopy type of a circle. ❄

Example 1.3. This example is a variant of the previous example; the origin and the point at infinity are just exchanged via $z \to 1/z$. Here it comes: Let $L_{p/q}$ consist of the points $te^{2\pi ip/q}$ with t real and $|t| > q$, and let U be the complement of $\bigcup_{p/q} L_{p/q}$. On shows that U is open as in the previous example. The line segment joining the origin to a point z in U is contained in U, and this shows that U is connected; in fact, it even shows that U is contractible. ❄

Problem 1.2. Let U be the complement of the product $\mathfrak{c} \times \mathfrak{c}$ in the open unit square $(0, 1) \times (0, 1)$. Show that U is a domain. ✸

Derivatives and the Cauchy-Riemann equations

In this section Ω will be a domain and f will be a complex valued function defined in Ω. The function f has two components, the real-valued functions $u(z) = \operatorname{Re} f(z)$, called *the real part* of f, and $v(z) = \operatorname{Im} f(z)$, *the imaginary part* of f. With this notation one writes $f = u + iv$.

The complex variable z is of course of the form $z = x + iy$ with x and y real, so any function $f(z)$ may as well be regarded as a function of the the two real variables x and y. All results about real functions of (some regularity class) from Ω to $\mathbb{R}^2$ apply to complex functions—but imposing the condition of holomorphy (that is, differentiability in the complex sense) on a function f makes it very special indeed, its properties will by far be stronger than those of general C^∞-function (or even real analytic functions).

(1.3) We adopt the convention of indicating partial derivatives by the use of subscripts, like *e.g.,* u_x, u_y. Taking a partial derivative is of course a differential operator and as such it will now and again be denoted by ∂_* with $*$ an appropriate subscript;*e.g.,* u_x will be denoted $\partial_x u$ and u_y by $\partial_y u$.

Clearly one has $f_x = u_x + iv_x$ and $f_y = u_y + iv_y$, or in terms of differential operators $\partial_x = \partial_x u + i\partial_x v$ and $\partial_y f = \partial_y u + i\partial_y v$. It turns out to be very convenient to use the differential operators ∂_z and $\partial_{\bar{z}}$ defined as

$$\partial_z = (\partial_x - i\partial_y)/2 \qquad \partial_{\bar{z}} = (\partial_x + i\partial_y)/2.$$

One verifies easily that $\partial_z\partial_{\bar{z}} = \partial_{\bar{z}}\partial_z$ at least when applied to functions for which ∂_x and ∂_y commute;*e.g.,* function being C^1. Another important formula, valid whenever ∂_x and ∂_y commute, is

$$4\partial_z\partial_{\bar{z}} = \Delta$$

where Δ is the Laplacian operator $\Delta = \partial_x^2 + \partial_y^2$; indeed, one finds

$$(\partial_x - i\partial_y)(\partial_x + i\partial_y) = \partial_x^2 + i\partial_x\partial_y - i\partial_y\partial_x - i^2\partial_y^2 = \partial_x^2 + \partial_y^2.$$

EXAMPLE 1.4. As a simple illustration let us compute $\partial_z z$ and $\partial_{\bar{z}} z$. One finds $\partial_z z = (\partial_x(x+iy) - i\partial_y(x+iy))/2 = (1 - i\cdot i)/2 = z$ and similarly $\partial_{\bar{z}} z = (\partial_x(x+iy) + i\partial_y(x+iy))/2 = (1 + i\cdot i)/2 = 0$. ❄

PROBLEM 1.3. Show that ∂_z and $\partial_{\bar{z}}$ satisfy Leibnitz' rule for products. ✷

The constituting definition — differentiability

The concept of holomorphy, that we are about to introduce, is constituting for the course, everything we shall do will hover about holomorphic functions, so the definitions in this paragraph are therefore the most important ones.

The notion we shall introduce is that of a *differentiable function in in the complex sense*, or *$\mathbb{C}$-differentiable* for short, and their derivatives. As f is a function of two real variables as well, there is also the notion of f being differentiable as such. In that case we shall call f differentiable in the real sense, or $\mathbb{R}$-differentiable—the long annotated names are there to distinguish the two notions. Function being $\mathbb{R}$-differentiable but not $\mathbb{C}$-differentiable are however rear creature in our story, so we shall pretty soon drop the annotations in the complex case, just keeping them the in the real case.

(1.4) To tell when a complex differentiable[1] function is differentiable at a point $a \in \mathbb{C}$ and to define its derivative there, we mimic the good old definition of the derivative of a real-valued function. One forms the complex differential quotient associated to two nearby points, and tries to take the limit as the two points coalesce:

Defenition 1.1 *Let a be a point in Ω. We say that f is differentiable at a if the following limit exists:*

$$\lim_{h \to 0}(f(a+h) - f(a))/h. \tag{1.1}$$

If so is the case, the limit is denoted by $f'(a)$ and is called the derivative of f at a. If f is differentiable at all points in Ω one says that f is holomorphic in Ω. A function holomorphic in the entire complex plane (i.e., if $\Omega = \mathbb{C}$) is said to be entire.

An equivalent way of formulating this definition is to say that there exists a complex number $f'(a)$ such that for z in a vicinity of a one has

$$f(z) = f(a) + f'(a)(z-a) + \epsilon(z), \tag{1.2}$$

where the function $\epsilon(z)$ is such that $|\epsilon(z)/(z-a)| \to 0$ as $z \to a$.

(1.5) The usual elementary rules for computing derivatives that one learned once upon a time during calculus courses, are still valid in this context, and the proofs are *mutatitis mutandis* the same.

Taking derivatives is a complex linear operation: For complex constants α and β the linear combination $\alpha f + \beta g$ is differentiable at a when both f and g are, and it holds true that $(\alpha f + \beta g)'(a) = \alpha f'(a) + \beta g'(a)$.

Leibnitz' rule for a product still holds: If f and g are differentiable at a, the product fg is as well, and one has $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$. Similarly for a fraction: Assume f and g differentiable at a and that $g(a) \neq 0$, then the fraction f/g is differentiable and $(f/g)'(a) = (g(a)f'(a) - g'(a)f(a))/g(a)^2$.

The third important principle is the chain rule. If f is differentiable at a and g at $f(a)$, then the composition $g \circ f$ is differentiable at a with derivative given as $(g \circ f)'(a) = g'(f(a))f'(a)$.

[1] The annotation in the complex case did not survive particularly long!

(1.6) An obvious consequence of the elementary rules is that a polynomial $P(z)$ is holomorphic in the entire complex plane. Almost the same applies to rational functions. They are quotients P/Q between two polynomials P and Q and are holomorphic where they are defined; that is at at least[2] in the points where the denominator Q does not vanish.

The Cauchy-Riemann equations

Any function from Ω to $\mathbb{C}$ is also a function of two real variables taking values in $\mathbb{R}^2$ with component functions being the real part u and the complex part v of f. For such functions the derivative at the point $z = \alpha + i\beta$ is an $\mathbb{R}$-linear map $D_a f \colon \mathbb{R}^2 \to \mathbb{R}^2$, that is a map $D_a f \colon \mathbb{C} \to \mathbb{C}$ being linear over the reals.

The derivative, if it exists, satisfies a condition very much like condition (**??**) in the complex case, namely for z close to a one has

$$f(z) = f(a) + D_a f(z - a) + \epsilon(z), \tag{1.3}$$

where $\epsilon(z)$ is a function with $|\epsilon(z)/(z-a)|$ tending to zero when z tends to a. The difference from the condition (**??**) lies in the second term to the right: For f to be $\mathbb{C}$-differentiable, the map real linear $D_a f \colon \mathbb{C} \to \mathbb{C}$ must be multiplication by a complex number!

(1.7) Casting a glance on the two definitions (**??**) and (**??**) it seems clear that a $\mathbb{C}$-differentiable function is $\mathbb{R}$-differentiable as well. The Cauchy-Riemann equations are a pair of differential equations that guarantee that a $\mathbb{R}$-differentiable function is $\mathbb{C}$-differentiable, and they are in essence contained in the last sentence of the previous paragraph—that $D_a f$ be multiplication by a complex number. To give the equations a concrete form however, we must exhibit the matrices of the derivative-maps in the two cases, in both cases relative to the semi-canonical basis for $\mathbb{C}$ as a real vector space—*i.e.,* the basis the numbers 1 and i constitute[3].

Multiplication by at complex number $c = \alpha + i\beta$ send 1 to $\alpha + i\beta$ and i to $-\beta + i\alpha$, hence its matrix is

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}. \tag{1.4}$$

In the calculus courses (surely, calculus of several variables) we learned that the matrix of the derivative-map $D_a f$ in the semi-canonical basis is just the *Jacobian matrix*:

$$\begin{pmatrix} u_x(a) & v_x(a) \\ u_y(a) & v_y(a) \end{pmatrix}. \tag{1.5}$$

[2] Why "at least"!
[3] Why "semi-canonical"?

Comparing the two matrices, one sees that a function f, being differentiable in the real sense, is $\mathbb{C}$-differentiable if and only if the derivatives of its two component functions satisfy the relations

$$u_x(a) = v_y(a) \qquad u_y(a) = -v_x(a).$$

These are the famous Cauchy-Riemann equations. Remembering that $\partial_x f = \partial_x u + i\partial_x v$ and $\partial_y f = \partial_y u + i\partial_y v$, one observes they being equivalent to the single equation

$$\partial_x f(a) = -i\partial_y f(a), \tag{1.6}$$

and, of course, this common values equals $f'(a)$.

(1.8) So far we have considered differentiability in a point, but being $\mathbb{C}$-differentiable *e.g.,* in solely one isolated point, has no serious implications. If, for example, both partials of f vanishes there, the Cauchy-Riemann equations are trivially satisfied, and the only implication is that both the real and the imaginary part of f has a stationary point. The full weightiness of being differentiable[4] comes into play only when the function is differentiable[5] everywhere in a domain, that is, it is holomorphic. So, when summing up, we formulate the Cauchy-Riemann equations in that context:

Proposition 1.1 *Let Ω be a domain in $\mathbb{C}$ and let $f = u + iv$ be a complex valued function in Ω. Then f is differentiable throughout Ω if and only if it is differentiable in the real sense throughout Ω, and the real and imaginary parts satisfy the Cauchy-Riemann equations*

$$\partial_x u = \partial_y v \qquad \partial_y u = -\partial_x v \tag{1.7}$$

in Ω. If f is differentiable in Ω, one has

$$f' = \partial_x f = -i\partial_y f. \tag{1.8}$$

(1.9) Recall the differential operators ∂_z and $\partial_{\bar{z}}$ we defined by

$$\partial_z = (\partial_x - i\partial_y)/2 \qquad \partial_{\bar{z}} = (\partial_x + i\partial_y)/2.$$

In view of equation (**??**) the Cauchy-Riemann equations when formulated in terms of the operators ∂_z and $\partial_{\bar{z}}$, translate into the following proposition, the simplicity of the equation appearing is one virtue of the $\partial_{\bar{z}}$ and ∂_z notation:

[4] in the complex sense
[5] ditto

Proposition 1.2 *An $\mathbb{R}$-differentiable function f in the domain Ω is holomorphic in Ω if and only if it satisfies*

$$\partial_{\bar{z}} f = 0,$$

and in that case the derivative of f is given as $f' = \partial_z f$.

PROOF: This is indeed a simple observation. One has $\partial_{\bar{z}} f = (\partial_x f + i\partial_y f)/2$, which vanishes precisely when (**??**) is satisfied. One has $\partial_z f = (\partial_x f - i\partial_y f)/2$ which equals $\partial_x f$ (and $\partial_y f$ as well) whenever $\partial_{\bar{z}} f = 0$, *i.e.*, whenever $\partial_x f = -i\partial_y f$. ❑

Power series

Rational functions are, although they form very important class of functions, very special. A rather more general class of functions are those given by power series—and indeed, as we shall see later on, it comprise all functions holomorphic in a disk.

(1.10) Recall that a power series $f(z) = \sum_{n\geq 0} a_n (z-a)^n$ has a radius of convergence given as $R^{-1} = \limsup \sqrt[n]{|a_n|}$. That is, the series converges absolutely for $|z-a| < R$, and the convergence is uniform on compact sets included in $|z-a| < R$; *e.g.*, closed disks given by $|z-a| \leq \rho < R$. For short we say that the convergence is *normal*.

Indeed, if $|z-a| < \rho < R$, choose ϵ with $0 < \epsilon \leq (R-\rho)/R\rho$. By definition one has $\sqrt[n]{|a_n|} < 1/R + \epsilon$ for $n >> 0$, and this gives

$$\sqrt[n]{|a_n|}\,|z-a| < \rho/R + \rho\epsilon < 1.$$

Thus we may appeal to Weierstrass M-test comparing with the series $\sum_{n\geq 0} M^n$ where $M = \rho/R + \rho\epsilon$.

(1.11) It is a theorem of Abel's that f is holomorphic in the disk of convergence and that the derivative may be found by termwise differentiation:

Theorem 1.1 *Assume that the power series $f(z) = \sum_{n\geq 0} a_n (z-a)^n$ has radius of convergence equal to R. Then f is holomorphic in the disk D centered at a and with radius R, and the derivative is given as*

$$f'(z) = \sum_{n\geq 1} n a_n (z-a)^{n-1}. \tag{1.9}$$

that is, the power series can differentiated term by term.

PROOF: We may assume that $a = 0$. Since $\lim_{n\to\infty} \sqrt[n]{n} = 1$, the derived series has the same radius of convergence as the one defining f. Let R be the radius of convergence and denote by D the disk where the convergence takes place; that is, the disk given by $|z-a| < R$ and fix a point $z \in D$.

By the binomial theorem one has $(z+h)^n - z^n = nz^{n-1}h + h^2 R_n(z,h)$. It follows that the series $\sum_{n\geq 1} a_n R_n(z,h)$ converges normally for those h with $z+h \in D$, since both the series for f and the derived series converge normally in D.

Hence the sum $\sum_{n\geq 1} a_n R_n(z,h)$ is continuous and therefore bounded on a closed disk centered at z sufficiently small to be contained in D. We deduce that for h close to zero it holds true that

$$f(z+h) - f(z) = h \sum_{n\geq 1} a_n z^{n-1} + h^2 \sum a_n R_n(z,h),$$

where the term $\sum a_n R_n(z,h)$ is bounded, and the claim follows. ❑

(1.12) Successive applications of Abel's theorem shows that a function $f(z)$ i given by a power series has derivatives of all orders, and by an easy induction argument one finds the series

$$f^{(k)}(z) = \sum_{n\geq k} n(n-1)\dots(n-k+1)a_n(z-a)^{n-k}$$

for the k-derivative of f. The constant term of this series equals $k!a_k$, so substituting a for z gives $k!a_k = f^{(k)}(a)$. Hence we have the following result, which may informally be stated as if f has a power series expansion, the expansion is the Taylor series of f.

Proposition 1.3 *A function f given as a power series*

$$f(z) = \sum_{n\geq 0} a_n(z-a)^n$$

converging normally a disk D centered at a, has derivatives of all orders, and it hold true that

$$a_n = \frac{f^{(n)}(a)}{n!}.$$

Problem 1.4. Prove the Cauchy-Riemann equations by letting h approach zero through respectively real and purely imaginary values in (??) . ✷

Problem 1.5. Assume that $f = u + iv$ is holomorphic in the domain Ω. Use the Cauchy-Riemann equations to show that the gradient of u is orthogonal to the gradient of v and conclude that the level sets of the real part of f are orthogonal to the level sets of the imaginary part. ✷

Problem 1.6. Assume that V is a complex vector space and that $A\colon V \to V$ is an $\mathbb{R}$-linear map. One says that A is *$\mathbb{C}$-anti-linear* if $A(zv) = \overline{z}A(v)$ for all $z \in \mathbb{C}$ and all $v \in V$. Show that A is $\mathbb{C}$-anti-linear if and only if $A(iv) = -iA(v)$ for all vectors $v \in V$. Show that any A may be decomposed in a unique way as a sum $A = A_+ + A_-$, where A_+ is $\mathbb{C}$-linear and A_- is $\mathbb{C}$-anti-linear. Hint: Let $A_+(v) = (A(v) - A(iv))/2$ and $A_-(v) = (A(v) + A(iv))/2$. ✷

Problem 1.7. Assume that V is a one-dimensional complex vector space and that $A\colon V \to V$ is an $\mathbb{R}$-linear map. Show that A is multiplication by a complex number if and only if its $\mathbb{C}$-anti-linear part vanishes; *i.e.,* $A_- = 0$. ✷

Problem 1.8. Show that complex conjugation $\overline{z}$ is not $\mathbb{C}$-differentiable at any point. ✷

Problem 1.9. Show that for any complex $\mathbb{R}$-differentiable function it holds that $\overline{\partial_{\overline{z}} f} = \partial_z \overline{f}$. ✷

Problem 1.10. Show that $\partial_{\overline{z}}\overline{z} = 1$ and that $\partial_z \overline{z} = 0$. ✷

Problem 1.11. A function f $\mathbb{R}$-differentiable in the domain Ω is called *anti-holomorphic* if $\partial_z f = 0$ throughout Ω. Show that $f(z)$ is anti-holomorphic if and only if $\overline{f(z)}$ is holomorphic. ✷

Integration and Cauchy's formula

Recall that a *line integral* is an integral on the form $\int_\gamma pdx + qdy$ where γ is a path in the complex plane and p and q are two functions, real or complex, defined and continuous along the path γ. The path γ is a parametrization of a curve in $\mathbb{C}$, *i.e.,* a function $\gamma\colon [\alpha, \beta] \to \mathbb{C}$ that in our context always will be piecewise C^1; that is, in addition to γ being continuous, there should be a partition of the parameter-interval $[\alpha, \beta]$ such that γ is continuously differentiable on each of the closed subintervals.

Now and then, as a shortcut, we shall specify a curve C instead of a path in the integral; in that case it should be clearly understood from the context which way the curve should be parametrized. A frequently occurring example, is that of a circle C. The implied parametrization will be $\gamma(t) = a + re^{it}$ with the parameter t running from 0 to 2π and a being the center and r the radius of C—the circle is traversed once counterclockwise. Circles appear frequently in the disguise as boundaries of disks D; that is, as ∂D.

Differential forms

The integrand in a line integral, that is the expression $\omega = pdx + qdy$ is called a *differential form*, more precisely one should say a *differential one form*, since, as the name indicates, there are also *two-forms* and even n-forms for any natural number n. We shall make use two-forms, but no n-form with n larger than two will appear.

(1.13) You will find no mystery in the definition of a line integral if the path γ is C^1 and given as $\gamma(t) = x(t) + y(t)i$ with $t \in [\alpha, \beta]$. One simply proceeds in the direction the nose points, replacing x and y in the functions p and q with $x(t)$ and $y(t)$, and replacing dx and dy with $x'(t)dt$ and $y'(t)dt$. This gives a conventional integral over the interval $[\alpha, \beta]$:

$$\int_\gamma \omega = \int_\gamma pdx + qdy = \int_\alpha^\beta p(\gamma(t))x'(t)dt + q(\gamma(t))y'(t)dt.$$

In case γ is just piecewise C^1, one follows this procedure for each of the subintervals where γ is C^1, and at the end sums the appearing integrals.

(1.14) Given a real valued function u in the domain Ω. The differential du of u is the one-form

$$du = \partial_x dx + \partial_y u dy,$$

and forms of tis type are said to be *exact forms*. It is particularly easy to integrate exact forms, they behave just like derivatives (in some sense, they are derivatives). One has

$$\int_\gamma du = u(\gamma(\beta)) - u(\gamma(\alpha)), \tag{1.10}$$

The integral is just the difference between the values of u at the two ends of the path and does not depend on which path one follows, as long as it starts and ends where at the same places as γ. In particular if a path γ is closed, the integral of du round γ vanishes.

The formula **??** follows from the fundamental theorem of analysis and the chain rule. The chain rule immediately gives

$$\frac{d}{dt}u(\gamma(t)) = u_x(\gamma(t))x'(t) + u_y(\gamma(t))y'(t),$$

and one finishes off with fundamental theorem.

(1.15) Speaking about two-forms, in our case they are just expressions $pdx \wedge dy$ where p is a function of the appropriate regularity (*e.g.,* continuously differentiable) in the domain Ω where the form lives. The "wedge product" is anti-commutative, *i.e.,* $dx \wedge dy = -dy \wedge dx$, a feature that becomes natural when one defines the integral of w. To do this, let $r(s,t) = u(r,s) + iv(r,s)$ be a parametrization of Ω; *i.e.,* a continuously differentiable homeomorphism from some open set $U \subseteq \mathbb{R}^2$ (of course life could be as simple as U being equal to Ω and r being the identity). With the parametrization in place, one has the Jacobian determinant

$$\frac{\partial(u,v)}{\partial(s,t)} = \det \begin{pmatrix} u_s & u_t \\ v_s & v_t \end{pmatrix},$$

and one defines the integral $\int_\Omega \omega$ as

$$\int_\Omega \omega = \iint_U p(r(s,t))\frac{\partial(u,v)}{\partial(s,t)} dudv \tag{1.11}$$

Exchanging u and v changes the sign of the Jacobian determinant and by consequence the sign the double integral to the right in (**??**) . So the definition is consistent with $du \wedge dv = -dv \wedge du$, *i.e.,* the wedge product being anti-commutative.

(1.16) A one form $\omega = pdx + qdy$ in Ω with p and q C^1-functions, has a derivative d which is a two-form. It is given by the rules

$$d^2 = 0 \qquad d(u\omega) = du \wedge \omega + ud\omega \tag{1.12}$$

Hence with $\omega = pdx + qdy$ we find

$$\begin{aligned} d\omega &= dp \wedge dx + pd^2x + dq \wedge dy + qd^2y \\ &= (\partial_x pdx + \partial_y pdy) \wedge dx + (\partial_x qdx + \partial_y qdy) \wedge dy \\ &= (\partial_x q - \partial_y p)dx \wedge dy. \end{aligned}$$

Complex integration

(1.17) Now, let $f(z)$ be a complex function defined in the domain Ω whose real part is u and imaginary part is v, so that $f(z) = u(z) + iv(z)$. We want to make sense of integrals of the form

$$\int_\gamma f(z)dz,$$

where the complex differential dz is defined as $dz = dx + idy$. Introducing this into the expression $f(z)dz$, multiplying out and separating the real and imaginary parts, we find

$$\int_\gamma f(z)dz = \int_\gamma (udx - vdy) + i(vdx + udy), \tag{1.13}$$

which is just a combination of two ordinary real integrals.

(1.18) It is a fundamental principle (universally valid only interpreted with care[6]) principle "that integrating the derivative of a function gives us the function back", and in our context it remains in force—frankly speaking, any thing else would be unthinkable. A complex function f differentiable in the domain Ω whose derivative is continuous[7] satisfies the equality

$$\int_\gamma f'(z)dz = f(b) - f(a), \tag{1.14}$$

where γ is any path joining the point a to the point b. The chain rule and the Cauchy-Riemann equations give

$$\begin{aligned} du &= u_x dx + u_y dy = u_x dx - v_x dy \\ dv &= v_x dx + v_y dy = v_x dx + u_x dy \end{aligned}$$

[6] There are increasing real functions having a derivative that vanishes almost everywhere

[7] One of the marvels of complex function theory is, as we soon shall se, that this is always true

combining this with the definition of the integral (**??**) we obtain $f'(z)dz = du + idv$, and the formula follows by the corresponding formula for exact real forms.

For a *closed* path γ with parameter running from α to β one has $\gamma(\alpha) = \gamma(\beta)$, and consequently the integral around γ vanishes. We have

Proposition 1.4 *If f is differentiable in the domain Ω with a continuous derivative, and γ is a closed path in Ω, then*

$$\int_\gamma f' = 0.$$

Cauchy's integral theorem—the corner stone of complex function theory—states that under certain topological condition on the closed path γ and the domain Ω, a similar statement is valid for any holomorphic function—that is, its integral along γ vanishes. We are going to establish this, step by step in progressively more general variants. The start being the case when γ is the circumference of a triangle.

(1.19) As an illustration we cast a glance on the rational functions. Every polynomial $P(z)$ trivially has a primitive (as you should know, the derivative of $z^{n+1}/(n+1)$ equals z^n), and therefore $\int_\gamma P(z)dz = 0$ as long as the path γ is closed. The same is true for any rational function of the type $c(z-a)^{-n}$ where $n \geq 2$ (a primitive being $(z-a)^{1-n}/(1-n)$, as you should know). The only obstruction for a rational function having a primitive is therefore the occurrence of terms of type $c/(z-a)$ in its decomposition in partial fractions. When being free of such terms, the rational function $F(z)$ satisfies

$$\int_\gamma F(z)dz = 0$$

for closed paths γ avoiding the points where F is not defined.

(1.20) The converse of proposition **??** above also holds. One has

Proposition 1.5 *Let $f(z)$ be continuos in the domain Ω and assume that $\int_\gamma f(z)dz = 0$ whenever γ is a closed path in Ω. Then $f(z)$ has primitive in Ω, in other words, there is a function $F(z)$ defined in Ω with $F'(z) = f(z)$.*

Proof: We begin with choosing a point z_0 in Ω. Since the integral of f round any closed path vanishes, we may define a function $F(z)$ by declaring

$$F(z) = \int_\gamma f(z)dz,$$

where γ is any path from z_0 to z; Indeed, the integral has the same value whatever path of integration we chose, as long as it connects z_0 to z: If γ_1 and γ_2 are two of the kind, the path $\gamma_1\gamma_2^{-1}$ is closed, and thus we have

$$0 = \int_{\gamma_1\gamma_2^{-1}} f(z)dz = \int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz.$$

We have to verify that F is differentiable and that the equality $F'(z) = f(z)$ holds. The difference $F(z+h) - F(z)$ can be computed by integrating $f(z)$ along any path leading from z til $z+h$. As h is small in modulus, we may assume that $z+h$ lies in a disk centered at z. Then the line segment parametrized as $\gamma(t) = z + th$ with $0 \leq t \leq 1$ is contained in Ω. Now, $dz = ht$ along γ, and we find the following expression for the differential quotient of F:

$$h^{-1}\big(F(z+h) - F(z)\big) = h^{-1}\int_\gamma f(z)dz = \int_0^1 f(z+th)dt$$

It is a well known matter, and trivial to prove, that $\lim_{h\to 0}\int_0^1 f(z+th)d = f(z)$ when f is continuous at the point z, and with that, we are through. ❑

(1.21) Cauchy's approach to the his theorem was via what is now called Green's theorem, which by the way never is mentioned in any of Green's writings. The first time the statement occurs is in a paper by Cauchy from 1846. However Cauchy does not prove it, he promised a proof that never appeared, and the first proof was given by Riemann. For an extensive history of these matters one may consult [?]. The theorem is today stated in calculus courses as

$$\iint_\Omega (\partial_x q - \partial_y p)dxdy = \int_{\partial\Omega} pdx + qdy$$

where $\partial\Omega$ is the border of the domain Ω, and this form is very close to the way Cauchy stated it. In terms differential forms, it it takes the following appealing look:

$$\int_\Omega d\omega = \int_{\partial\Omega} \omega,$$

a formula that obtained by substituting the equality $d\omega = (\partial_x q - \partial_y p)dx \wedge dy$ from paragraph (??) in formula in Green's theorem.

There are two fundamental assumptions in Green's theorem. One about the functions involved, they must continuously differentiable (in the real sense) and one on the geometry. The border $\partial\Omega$ must be a curve that has a piecewise parametrized by continuously differentiable functions in a way that Ω lies to the left of $\partial\Omega$. This the current "calculus way" to state Green's theorem, but there are stronger versions around.

The general geometrical assumptions are notoriously fuzzy, and the proof in the general case is involved, but of course in simple concrete situations proof is simple. Just a combination of Fubini's theorem about iterated integration and the fundamental theorem of analysis. We shall not dive into general considerations about Green's theorem, but will only use it in clear cut situations.

(1.22) It is interesting to give Green's theorem a formulation adapted to the specific context of complex function theory; *i.e.*, a formulation in terms of the differential operators ∂_z and $\partial_{\bar z}$: As $d^2z = 0$ and $dz \wedge dz = 0$, one has

$$d(fdz) = (\partial_z f dz + \partial_{\bar z} f d\bar z) \wedge dz = \partial_{\bar z} f d\bar z \wedge dz$$

which gives

$$-\int_{\Omega} \partial_{\overline{z}} f dz \wedge d\overline{z} = \int_{\partial\Omega} f.$$

In view of the equality $d\overline{z} \wedge d_z = 2idx \wedge dy$, one obtains

$$\int_{\partial\Omega} f(z)dz = 2i \iint_{\Omega} \partial_{\overline{z}} f(z) dxdy$$

In view of the $\partial_{\overline{z}}$-formulation of the Cauchy-Riemann equations as in theorem **??** on page **??**; that is $\partial_{\overline{z}} = 0$ for holomorphic f's the form of Greens theorem in the form above, one obtains a version of the Cauchy's theorem:

Theorem 1.2 *Let f be a function that is holomorphic with continuous derivative in a domain Ω for which Green's theorem is valid; i.e., the border $\partial\Omega$ has a piecewise C^1-parametrization. Then it holds true that*

$$\int_{\partial\Omega} f(z)dz = 0.$$

This is of course a very nice result, but it is not entirely satisfying. In the days of Cauchy a holomorphic function had a continuous derivative by assumption, but nowadays that condition is dropped—as in our definition. The reason one can do this, is that Cauchy's theorem remains valid when the continuity of the derivative is not assumed; a result due to Edouard Jean-Baptiste Goursat, and which is the topic of the next section.

Moore's proof of Goursat's lemma

As announced, this paragraph is about Goursat's lemma the vanishing of integrals of holomorphic functions round triangles, of course without assumptions about continuity of the derivative. Goursat published this in 1884, and the simple and beautiful proof we give—really one of the gems in mathematics—is now standard and was found by Eliakim Hastings Moore in [**?**] from 1900 , and it is not due to Goursat as claimed in many texts. There is a point of exception occurring in the statement, which makes it easy to deduce Cauchy's formula from the lemma (which by the way we have promoted to a theorem).

Theorem 1.3 *Let Ω be a domain containing the triangle Δ and let $p \in \Omega$ be a point. Let f be a function continuous in Ω and differentiable through out $\Omega \setminus \{p\}$. Then*

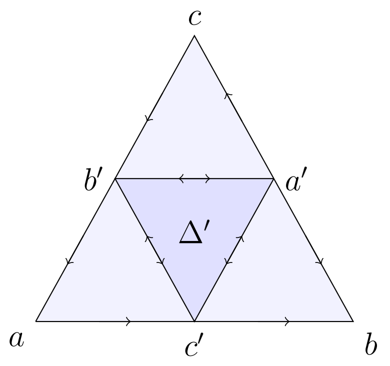
$$\int_{\partial\Delta} f(z)dz = 0.$$

Proof: In the first, and essential part, of the proof the special point p is assumed to lie outside the triangle Δ.

We shall describe a process that when fed with a triangle Δ, returns a new triangle Δ' contained in Δ and having the the following two properties:

1. $\left|\int_{\partial\Delta} f(z)dz\right| \leq 4\left|\int_{\partial\Delta'} f(z)dz\right|$
2. Both the diameter and the perimeter of Δ' is half of those of Δ.

Let the corners of Δ be a, b and c; and denote by c' the midpoint of the edge of Δ from a to b, by b' the midpoint of the edge from a to c and by a' the midpoint of the edge from b to c. These six points serve as corners of four new triangles that subdivide Δ; say Δ^i with $1 \leq i \leq 4$. As the new corners are the midpoints of the old edges, the perimeter of each of the triangles Δ^i is half that of Δ, and similarly for the diameters, they all equal half the diameter of Δ. So any of the four Δ^i-s satisfies the second requirement above.



Figur 1.1: *A triangles* $\Delta = abc$ *and the* $\Delta' = a'b'c'$

So to the first requirement. In the sum to the right in (**??**) below, the integrals of f along edges sheared by two of the four triangle cancel, and hence the equality in (**??**) is valid:

$$\int_{\partial\Delta} f(z)dz = \sum_i \int_{\partial\Delta^i} f(z)dz, \tag{1.15}$$

$$\left|\int_{\partial\Delta} f(z)dz\right| \leq \sum_i \left|\int_{\partial\Delta^i} f(z)dz\right|.$$

Among the four triangles Δ^i-s we pick the one for which $\left|\int_{\partial\Delta^i} f(z)dz\right|$ is maximal as the new triangle Δ', the output of the process. One obviously has $\left|\int_{\partial\Delta} f(z)dz\right| \leq 4\left|\int_{\partial\Delta'} f(z)dz\right|$, and the second requirement above is fulfilled as well.

Iterating this process one constructs a sequence of triangles Δ_n all contained in Ω having the three properties below (where as usual $\lambda(A)$ stands for the perimeter of a figure A and $d(A)$ for the diameter)

- $\Delta_{n+1} \subseteq \Delta_n$;
- $\left|\int_{\partial\Delta} f(z)dz\right| \leq 4^n \left|\int_{\partial\Delta_n} f(z)dz\right|$;
- $\lambda(\Delta_n) < 2^{-n}\lambda(\Delta)$;

□ $d(\Delta_n) < 2^{-n} d(\Delta)$.

The triangles form a descending sequence of compact sets with diameters shrinking to zero; their intersection is therefore one point, say a. By assumption f is differentiable at a, and we may write

$$f(z) = f(a) + f'(a)(z-a) + \epsilon(z-a)$$

where $|\epsilon(z-a)/(z-a)|$ tends towards zero when z approaches a; so if $\eta > 0$ is a given number, $|\epsilon(z-a)| < \eta\,|z-a|$ for z sufficiently close to a; that is for $z \in \Delta_n$ for $n >> 0$. As the integrals of both the constant $f'(a)$ and of $f'(a)(z-a)$ around any closed path vanish, one finds

$$\int_{\partial\Delta_n} f(z)dz = \int_{\partial\Delta_n} \epsilon(z-a)dz,$$

and hence

$$4^{-n}\left|\int_{\partial\Delta} f(z)dz\right| \leq \left|\int_{\partial\Delta_n} f(z)dz\right| = \left|\int_{\partial\Delta_n} \epsilon(z-a)dz\right| \leq$$
$$\leq \int_{\partial\Delta_n} \eta\,|z-a|\,|dz| \leq \eta \cdot 2^{-n} d(\Delta) \cdot 2^{-n}\lambda(\Delta),$$

Things are now so beautifully constructed that factor 4^{-n} cancels, and the inequality becomes

$$\left|\int_{\partial\Delta} f(z)dz\right| < \eta d(\Delta)\lambda(\Delta)$$

The positive number η being arbitrary, we conclude that $\int_{\partial\Delta} f(z)dz = 0$.

Finally, if the point p is among the corners of Δ, we may subdivide Δ in two triangles Δ' and Δ'', one of them, say Δ', containing the special point p and having perimeter as small we want. As the point p lies outside Δ'', the integral of f round $\partial\Delta''$ vanishes by what we have already done; hence $\int_{\partial\Delta} f(z)dz = \int_{\partial\Delta'} f(z)dz$. This integral can be maid arbitrarily small since f is bounded in Δ and the perimeter of Δ' can maid arbitrarily small.

At the very end, we get away with the case of p lying inside Δ, but not being a corner, by decomposing Δ into three (or two if p lies on an edge of Δ) new triangles, each having p as one corner and two of the corners of Δ as the other two. ❑

Problem 1.12. Let Ω be a domain and f a continuous function in Ω. Assume that for a finite set P of points in Ω, the function f is differentiable in $\Omega \setminus P$. Prove that $\int_{\partial\Delta} f(z)dz = 0$ for all triangles Δ in Ω. Hint: Induction and decomposition. ✷

Problem 1.13. Let Ω be a domain and f continuous and holomorphic in $\Omega \setminus P$ as in exercise ??. Show that the conclusion of ?? holds even if one only assumes that P is a closed subset without accumulation points in Ω. Hint: Triangles are compact. ✷

Cauchy's theorem in star-shaped domains

To continue the development of the Cauchy's theorem and expand its validity, we now pass to arbitrary closed paths in a *star-shaped domains* domain. Recall that the domain Ω is star-shaped if there is one point c, called the apex, such that for any z i Ω the line segment joining c to z is entirely contained in Ω. The point c is not necessarily unique, many domains have several apices.

Of course all convex domains are star-shaped, and this includes circular disks, the by far most frequently occurring domains in the course. The idea is to show that differentiable functions have primitives just by integrating them along line segments emanating from a fixed point. This is very close to the fact that continuous functions whose integral round any closed path vanishes, has a primitive (proposition **??** on **??**), in star-shaped domains the vanishing of integrals round triangles suffices.

(1.23) So assume that f is continuous throughout a star-shaped domain Ω with apex c and assume that f is differentiable everywhere in Ω except possibly at one point p.

For any two points a and b belonging Ω, we denote by $L(a,b)$ the line segment joining a to b, and we assume tacitly that it is parametrized in the standard way; that is as $(1-t)a+tb$ with the parameter t running from 0 to 1. The domain Ω being star-shaped with apex c by assumption the segment $L(c,a)$ is entirely contained in Ω. Now, we define a function F in Ω by integrating f along $L(c,z)$, that is we set

$$F(z)=\int_{L(c,z)} f(z)dz. \tag{1.16}$$

The claim is that F is continuos throughout Ω and differentiable except at p with derivative equal to f; in other words, the function F is what one usually calls a *primitive* for f:

Proposition 1.6 *Let Ω be a star-shaped domain and let p be a point in Ω. A continuous function f in Ω which is differentiable away from p, has a primitive in $\Omega\setminus\{p\}$.*

Proof: The task is to prove that $F(z)$ as defined by equation (**??**) is differentiable and that the derivative equals f. The proof is very close to the proof of proposition **??** (in fact, it is *mutatis mutandis* the same).

The obvious line of attack is to study the difference $F(a+h)-F(a)$ where a is an arbitrary point in Ω different from p and h is complex number with a small modulus. We fix disk centered at a contained in Ω. If $a+h$ lies in D, the line segment $L(a,a+h)$ lies in Ω as well.

We find

$$F(a+h)-F(a)=\int_{L(c,a+h)} f(z)dz-\int_{L(c,a)} f(z)dz=\int_{L(a,a+h)} f(z)dz, \tag{1.17}$$

the last and crucial equality holds true since the integral of f around the triangle with corners c, a and $a+h$ vanishes by Goursat's lemma (theorem **??** on page **??**).

The path $L(a, a+h)$ is parametrized as $a+th$ with the parameter t running from 0 to 1. Hence $dz = hdt$ along $L(a, a+h)$, and we find

$$\int_{L(a,a+h)} f(z)dz = h\int_0^1 f(a+th)dt.$$

The function f being continuous at a implies that given $\epsilon > 0$ there is $\delta > 0$ such that

$$|f(a+h) - f(a)| < \epsilon$$

whenever $|h| < \delta$. Hence

$$F(a+h) = F(a) + hf(a) + h\int_0^1 (f(a+th) - f(a))dt$$

where

$$\left|\int_0^1 (f(a+th) - f(a))dt\right| < \int_0^1 |f(a+th) - f(a)|\, dt < \epsilon,$$

once $|h| < \delta$. ❑

(1.24) Combining the theorem with the fact that the integral of a derivative round a loop vanishes, one obtains as an immediate corollary Cauchy's formula for star-shaped domains, namely:

Corollary 1.1 *If f is a function continuous in the star-shaped domain Ω and holomorphic in $\Omega \setminus \{p\}$, then $\int_\gamma f(z)dz = 0$ for all closed paths γ in Ω.*

Cauchy's formula in a star-shaped domain

By far the most impressive tool in the toolbox of complex function theory is Cauchy's formula, expressing the value of f at a point as the integral round a loop circling the point. Taking a step in the direction towards the general case, we proceed to establish this formula for star-shaped domains. This includes Cauchy's formula for disks. Albeit a modest version, it has rather strong implications for the local behavior of holomorphic functions. A crucial feature in the proof is the exceptional point p allowed in corollary **??** above—and this is the sole reason for including the exceptional point in the hypothesis of **??**.

(1.25) The setting is as follows: The domain Ω is star-shaped and a is any point Ω. Furthermore γ a closed path in Ω not passing through a and f is function holomorphic throughout Ω.

The auxiliary function

$$g(z) = \begin{cases} \dfrac{f(z) - f(a)}{z-a} & \text{when } z \neq a \\ f'(a) & \text{when } z = a \end{cases} \tag{1.18}$$

is continuous at a since f is differentiable there, and in $\Omega \setminus \{a\}$ it is obviously holomorphic. Hence g fulfills the hypothesis in Cauchy's theorem (corollary **??** on page **??**) and the integral of f round closed paths vanish. As a is not lying on the path γ it holds true that

$$\int_\gamma \frac{f(z) - f(a)}{z - a} dz = 0,$$

from which one easily deduces

$$\int_\gamma f(z)(z-a)^{-1} dz = f(a) \int_\gamma (z-a)^{-1} dz. \tag{1.19}$$

The integral $\int_\gamma (z-a)^{-1} dz$ is, as we shall see later on, an integral multiple of $2\pi i$, and we defines the integer $\mathrm{n}(\gamma, a)$ by setting

$$\mathrm{n}(\gamma, a) = \frac{1}{2\pi i} \int_\gamma (z-a)^{-1} dz.$$

It is called the *winding number* of g round a. We have thus establish the following version of Cauchy's formula for star-shaped domains:

Theorem 1.4 *Let f be holomorphic in the star-shaped domain Ω and a a point in Ω. For any closed path γ, one has*

$$\frac{1}{2\pi i} \int_\gamma f(z)(z-a) dz = \mathrm{n}(\gamma, a) f(a).$$

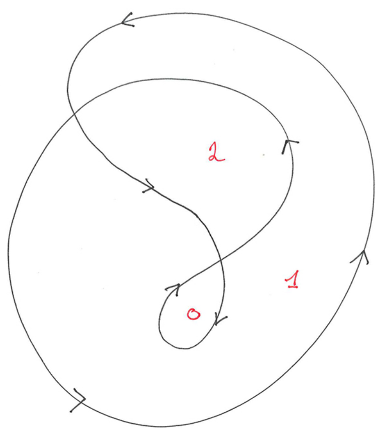
Of course this formula comes to its full force only when the winding number $\mathrm{n}(\gamma, a)$ is known. Hence it is worth while using some time and energy in studying the winding number and establish some of its general properties. We do that in the next paragraph.

(1.26) The following lemma is just a rephrasing in the lingo of function theory of a small lifting lemma from topology saying that any continuous map from an interval to the circle $\mathbb{S}^1$ lifts to universal cover $\mathbb{R}$ of $\mathbb{S}^1$. It is simple but crucial in our context, so we offer a proof.

Lemma 1.1 *Any path $\gamma(t)$ satisfying $|\gamma(t)| = 1$ for all values t of the parameter, may be brought on form $\gamma(t) = e^{i\phi(t)}$.*

If you wonder what kind of path γ is, it si just a movement on the unit circle. The function ϕ is a logarithm of $\gamma(t)$. So along portions of the path where the complex logarithm is defined, it is trivial that $\phi(t)$ exists. The function ϕ is also only unique up to additive constants of the form $2n\pi i$ with $n \in \mathbb{Z}$.

Proof: For simplicity, we assume that parameter interval of γ is the unit interval $[0, 1]$. Let τ be the supremum of the numbers s such that $\phi(t)$ exists for over $[0, s]$. In a neighbourhood U of $\gamma(\tau)$ the complex logarithm $\log w$ is well defined. We choose one of the branches and let $\psi(t) = \log\gamma(t)$ for $t \in \gamma^{-1}(U)$. One of the connected components of the inverse image $\gamma^{-1}(U)$ is an open interval J containing τ, and over $[0, \tau) \cap J$ the two functions ϕ and ψ differ only by an additive constant. Hence by adjusting ψ we can make them agree, and ϕ can be extended beyond τ, contradicting the definition of τ. ❑



The lemma allows paths to be parametrized with polar coordinates centered at points not on the path. The radius vector is just $r(t) = |\gamma(t) - a|$, and the angular coordinate is given as in the lemma; it is one of the functions $\phi(t)$ with $e^{i\phi(t)} = (\gamma(t)-a)\,|\gamma(t) - a|^{-1}$. Thus one has

$$\gamma(t) = a + r(t)e^{i\phi(t)}.$$

With this parametrization one finds $\gamma'(t) = r'(t)e^{i\phi(t)} + ir(t)e^{i\phi(t)}\phi'(t)$, and upon integration we arrive at

$$\begin{aligned}\int_\gamma (z-a)^{-1}dz &= \int_\alpha^\beta (r'(t)r(t)^{-1} + i\phi'(t))dt = \\ &= \log r(\beta) - \log r(\alpha) + (\phi(\beta) - \phi(\alpha))i.\end{aligned}$$

where log designates the good old and well behaved real logarithm. As the path γ is closed, $r(\beta) = r(\alpha)$ and $e^{i\phi(\beta)} = e^{i\phi(\alpha)}$, the latter equality implying that $\phi(\beta) - \phi(\alpha)$ is an integral multiple of 2π. We have establish

Lemma 1.2 *The winding number* $\mathrm{n}(\gamma, a) = \frac{1}{2\pi i}\int_\gamma (z-a)^{-1}dz$ *is an integer.*

Finally, we examine to which extent $\mathrm{n}(\gamma, a)$ varies with the point a, and we shall prove

Proposition 1.7 *If a and b belong to the same connected component of $\mathbb{C} \setminus \gamma$, then* $\mathrm{n}(\gamma, a) = \mathrm{n}(\gamma, b)$, *and the winding number* $\mathrm{n}(\gamma, a)$ *vanishes for a in the unbounded component.*

Assume that a and b are two different points and let z be any point in the plane. An elementary geometric observation is that the point z lies on the line through a and b if and only if the two vectors $z - a$ and $z - b$ are parallel or anti-parallel; phrased in other words, one is a real multiple of the other. They point in opposite directions if z belongs to the line segment $L(a, b)$ joining a to b, and in the same direction if not. The fractional linear transformation

$$A(z) = \frac{z - a}{z - b}$$

therefore maps the line segment between a and b onto the negative real axis.

Now, the principal branch $\log w$ of the logarithm is well defined and holomorphic in the split plane $\mathbb{C}^-$; that is in the complement of the negative real axis. Since the line segment $L(a, b)$ corresponds to the negative real axis under the map A, we conclude that $\log A(z) = \log(z - a)(z - b)^{-1}$ is well defined and holomorphic in the complement $\mathbb{C} \setminus L(a, b)$.

Lemma 1.3 *Let a and b bee different point in the complex plane and let γ be any closed path in $\mathbb{C}$. If γ does not intersect the line segment from a to b, the winding numbers of γ around a and b are the same, that is* $\mathrm{n}(\gamma, a) = \mathrm{n}(\gamma, b)$.

PROOF: The function $g(z) = \log(z - a)(z - b)^{-1}$ is defines and holomorphic along γ, and its derivative is given as

$$g'(z) = (z - a)^{-1} - (z - b)^{-1}.$$

As the integral of a derivative round a loop vanishes, we obtain

$$0 = \int_\gamma g'(z)dz = \int_\gamma (z - a)^{-1}dz - \int_\gamma (z - b)^{-1}dz$$

and we are happy! ❑

The proof of proposition **??** will be complete once we prove that any to points a and b in same component U of $\mathbb{C} \setminus \gamma$ can be connected by a piecewise *linear* path.

Connect a and b by a continuous path δ, and cover d by finitely many disks V_j all lying in U. By Lebesgue's lemma there is a partition $\{t_i\}$ of the parameter interval , such the portions of the path with parameter in the subintervals $[t_{i-1}, t_i]$ is contained in one of the V_j-s. But V_j being convex, the line segments between $\delta(t_{i-1})$ and $\delta(t_i)$ lie in V_j and a fortiori in U. Thus any two points in U can be connected by a piecewise path, and we are done.

As an illustration, be offer the nice curve drawn in figure xxx. It divides the plane into four connected components and the corresponding winding numbers are indicated in red ink. In two of the components the winding number is zero, and in two others they are 1 and 2 respectively.

Lemma 1.4 *If D is a disk and a is any point in D, then* $\mathrm{n}(\partial D, a) = 1$.

Proof: Winding numbers being constant throughout components (proposition **??** on page **??**) it suffices to check that winding number of ∂D round the center of the disk equals one, so we take a to be the center of D and parametrize ∂D as $z(t) = a + re^{it}$ with t running from 0 to 2π . One has $dz = ire^{it}dt$ and as $z - a = re^{ir}$ the integral becomes

$$\int_\gamma (z-a)^{-1}dz = i\int_0^{2\pi} dt = 2\pi i,$$

and $\mathrm{n}(\partial D, a) = 1$. ❑

Problem 1.14. Let C be the circle centered at a having radius r. Assume that γ is the path $a + re^{int}$ with n and integer and the parameter running from 0 to π—that is, it traverses the circle C n times in the direction indicated by the sign of n—then the winding number is $\mathrm{n}(\gamma, a) = n$. ✷

(1.27) A special case of theorem **??** is the Cauchy's formula for a circle:

Theorem 1.5 *Let D be a disk centered at a and f a function holomorphic in a domain containing the closure $\overline{D}$. The one has*

$$\frac{1}{2\pi i}\int_{\partial D} f(z)(z-a)^{-1}dz = f(a),$$

where the circumference ∂D is traversed once counterclockwise.

(1.28) In polar coordinates; *i.e.,* $z(t) = a + re^{it}$ this reads

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{it})dt$$

So the value of f at a equals the mean value of f along any circle centered at a on which f is holomorphic.

Consequences of the local version Cauchy's formula

The Cauchy formula has an impressive series of very strong consequences for holomorphic functions; the most important is that they will be infinitely many times differentiable; *i.e.,* have derivatives of all orders. Other important results are the maximum principle (which also has a global aspects) and the open mapping theorem, and finally Liouvilles theorem. This definitively a global statement saying that a bounded entire function is constant.

Derivatives of all orders and Taylor series

The setting in this section is slightly more general than in the previous section. Basically we introduce a way of getting hand on a lot of holomorphic functions by integration along curves, and we show that these functions are analytic, *i.e.*, they have well behaved Taylor expansions round every point where they are defined, and finally, by Cauchy's formula any f holomorphic in a disk, is obtained in this way.

(1.29) We start out with a path γ and a function ϕ defined on γ. The only hypothesis on ϕ is that it be integrable; that is the function $\phi(\gamma(t))$ must be a measurable function on the parameter interval $[\alpha, \beta]$, and the integral $\int_\gamma |\phi(z)|\,|dz| = \int_\alpha^\beta |f(\gamma(t))\gamma'(t)|\,dt$ must be a finite number. We reserve the letter M for that number. Integrating $\phi(z)(z-w)$ along γ gives us a function $\Phi(z)$ defined at every point z not lying on γ; that is, we have

$$\Phi(z) = \int_\gamma \phi(w)(w-z)^{-1}dw$$

for z not on γ. We shall see that Φ has derivatives of all orders, and we are going to give formula for the Taylor polynomials of Φ round any point a (not on γ) with a very good and practical estimate for the residual term. From this, we extract formulas for the derivatives of Φ analogous to Cauchy's formula and show that Taylor series converges to Φ.

Proposition 1.8 *The function $\Phi(z)$ is holomorphic and has derivatives of all orders off the path γ. Its n-th derivative is given as the integral*

$$\Phi^{(n)}(z) = n! \int_\gamma \phi(w)(w-z)^{-n-1}dw.$$

The Taylor series of Φ at any point not on γ converges normally to Φ in the largest disk centered at z not hitting γ.

Proof: We shall exhibit the Taylor series of Φ round any point a not lying on the curve γ. The tactics are simple and clear: Expand $(w-z)^{-1}$ in finite sum of powers of $(z-a)$ (with a residual term), multiply by $\phi(w)$, integrate along γ and hope that we control the residual term sufficiently well.

We begin carrying out this plan by recalling a formula from the old days in high school when one learned about geometric series, that is

$$\frac{1}{1-u} = 1 + u + \cdots + u^n + \frac{u^{n+1}}{1-u}, \tag{1.20}$$

where u is any complex number. We want to develop $(w-z)^{-1}$ in powers of $(z-a)$, and to that end we observe that

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{(w-a)}\frac{1}{1-\dfrac{z-a}{w-a}},$$

and in view of (??) above, we find by putting $u = (z-a)(w-a)^{-1}$

$$\frac{1}{w-z} = \sum_{k=0}^{n} \frac{(z-a)^k}{(w-a)^{k+1}} + \frac{(z-a)^{n+1}}{(w-z)(w-a)^{n+1}}.$$

Multiplying through by $\phi(w)$ and integrating along the path γ yields

$$\Phi(z) = \sum_{k=0}^{n} (z-a)^k \int_\gamma \phi(w)(w-a)^{-k-1} dw + R_n(z)(z-a)^{n+1}.$$

The factor $R_n(z)$ in the residual term equals

$$R_n(z) = \int_\gamma \phi(w)(w-z)^{-1}(w-a)^{-n-1} dw,$$

an expression that has a for our purpose a good upper bound. Indeed, let $d = \inf_{w\in\gamma} |w-a|$ be the distance from a to the curve γ. It is strictly positive since γ is compact and a does not lie on γ. Pick a positive number η with $\eta < 1$. For any z with $|z-a| < \eta d$ one has $|w-z| \geq |w-a| - |z-a| \geq (1-\eta)d$, and it is easily seen that these estimates give

$$|R_n(z)| < (1-\eta)^{-1} M/d^{-n-2}.$$

Hence

$$\left|R_n(z)(z-a)^{n+1}\right| < (1-\eta)^{-1} d^{-1} M \big(\frac{z-a}{d}\big)^{n+1} < (1-\eta)^{-1} d^{-1} M \eta^{n+1}.$$

The residual term tends uniformly to zero as n tends to infinity because $\eta < 1$, and we have established that $\Phi(z)$ has a power series expansion in any disk centered at a whose closure does not hit γ, and furthermore the n-th coefficient of the power series equals

$$\int_\gamma f(w)(w-z)^{-n-1} dw.$$

F The theorem now follows now from the principle that "every power series is a Taylor series" (proposition ?? on page ??). ❑

(1.30) In the theorem we assumed that γ parametrizes a compact curve, but the proof goes through more generally at least for points having a positive distance to γ; of course the main hypothesis is that ϕ be integrable along γ. For example, if γ is the real axis (strictly speaking, the parametrization of the real axis with the identity) and ϕ is any integrable function, the corresponding Φ is holomorphic off the axis.

Problem 1.15. Let $X \subseteq \mathbb{C}$ be a measurable subset and let ϕ be an integrable function on X. Define

$$\Phi(z) = \iint_X \phi(w)(w-z)^{-1} dxdy$$

where $dxdy$ is the two-dimensional Lebesgue measure. Show that Φ is holomorphic off X. ✷

PROBLEM 1.16. Assume that Γ is an "infinite contour", that is a path parametrized over the open interval $I = (0, \infty)$. Let $\phi(t)$ be an integrable function on Γ; that is, ϕ is measurable and the integral $\int_0^\infty |\phi(\Gamma(t))\Gamma'(t)|\,dt$ is finite. Define

$$F(z) = \int_\Gamma \phi(w)(w-z)^{-1}dw,$$

for z not on Γ. Show that this is meaningful; *i.e.*, both the real and the imaginary part of the integral are convergent. Show that $F(z)$ is a holomorphic function off Γ. ✱

(1.31) Our main interest in proposition **??** above are the implications it has for holomorphic functions. So let f be a function holomorphic in the domain Ω. For any point $z \in \Omega$ and any disk D contained Ω with center at z, Cauchy's local formula (theorem **??** on page **??**) tells us that

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} f(w)(w-z)^{-1}dw.$$

As usual, the boundary ∂D is traversed once counterclockwise. Hence we are in a good position to apply proposition **??** with the path γ being ∂D and the function ϕ being the restriction of f to ∂D—indeed, from Cauchy's formula we deduce that the function Φ then equals f, and **??** translates into the fundamental and marvelous

Theorem 1.6 *Assume that f is holomorphic in the domain Ω. Then f has derivatives of all orders, and for the n-th derivative the following formula holds true*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\partial D} f(w)(w-z)^{-n-1}dw,$$

where D is any disk centered at z and contained in Ω, and where, as usual, ∂D is traversed once counterclockwise. The Taylor series of f about any point z, converges normally to $f(z)$ in D.

Cauchy's estimates and Liouville's theorem

This paragraph is about *entire* functions; that is, functions being holomorphic in the entire complex plane. For such functions one may apply Cauchy's formula for the higher derivatives from the previous paragraph over any disk in $\mathbb{C}$. Using the disk centered at a point a with radius R one obtains upper bounds for the modulus of the higher derivative. These are famous the Cauchy estimates:

$$\left|f^{(n)}(a)\right| = \frac{n!}{2\pi}\int_{\partial D}\left|f(w)(w-a)^{-n-1}dw\right| < n! \sup_{w\in\partial D}|f(w)|/R^n, \qquad (1.21)$$

the perimeter of D being $2\pi R$ and $|z-a|$ being equal R on the circumference ∂D. One of the consequences of these estimates is that entire functions that are not constant must sustain a certain growth as z tends to infinity; they must satisfy growth conditions. The simples case is known as Liouville's theorem, and copes with bounded entire functions

Theorem 1.7 *Assume that f is a bounded entire function. Then f is constant.*

Proof: Assume that $|f|$ is bounded above by M. For any complex number a, one has the Cauchy estimate for the derivative of f, that is inequality (**??**) with $n = 1$,

$$|f'(a)| \leq M/R,$$

valid for all positive numbers R, as large as one wants. Hence $f'(a) = 0$, and consequently f is constant. ❏

(1.32) The next application of the Cauchy estimates, which we include as an illustration, is a slight generalization of Liouville's theorem. Functions having a *sublinear growth* musts be constant

Proposition 1.9 *Assume that $|f(z)| < A\,|z|^{\alpha}$ for some number $\alpha < 1$. Then f is constant.*

Proof: The proof is *mutatis mutandis* the same as for Liouville's theorem. The Cauchy estimate on a disk with radius R and center a gives

$$|f'(a)| < A \sup_{z \in \partial D} |z|^{\alpha} / R < A(R + |a|)^{\alpha}/R.$$

The term to the right tends to zero as R tends to infinity since $\alpha < 1$ (by l'Hôpital's rule, for example) , and hence $f'(a) = 0$. Since a was arbitrary, we conclude that f is constant. ❏

(1.33) The third application of Liouville's theorem along this line, it a result saying that entire functions with polynomial growth are polynomials; polynomial growth meaning that $|f|$ is bounded above by $A\,|z|^n$ for positive constant A and a natural number n. One can even say more, f must be a polynomial whose degree is less than n:

Proposition 1.10 *Let f be an entire function and assume that for a natural number n and a positive constant A one has $|f(z)| \leq A\,|z|^n$ for all z. Then f is a polynomial of degree at most n.*

Proof: The proof goes by induction on n, the case $n = 0$ being Liouville's theorem. The difference $f - f(0)$ is obviously a polynomial of degree at most n if and only if f is, so replacing f by $f - f(0)$, we may assume that f vanishes at the origin. Then $g(z) = f(z)/z$ is entire and satisfies the inequality $|g| \leq A\,|z|^{n-1}$. By induction g is a polynomial of degree at most $n - 1$, and we are through. ❏

(1.34) For any domain Ω it is important, but often challenging, to determine the group $\mathrm{Aut}(\Omega)$ of holomorphic automorphisms of Ω. It consists of maps $\phi\colon \Omega \to \Omega$ that are *biholomorphic*, that is, they are bijective with the inverse being holomorphic as well. It is a group under composition.

An illustrative example, but also an important result in it self, we shall show that all the automorphisms of the complex plane are the affine functions; *i.e.,* functions of the type $az + b$:

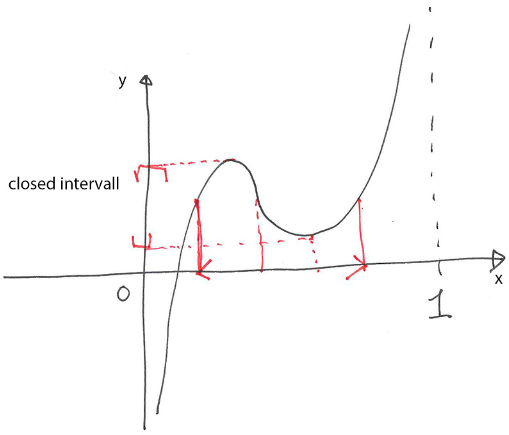
Proposition 1.11 *If $\phi\colon \mathbb{C} \to \mathbb{C}$ is biholomorphic, then there are complex constants such that $\phi(z) = az + b$.*

Proof: After having replaced ϕ by $\phi - \phi(0)$ we can assume that $\phi(0) = 0$, and have to prove that $\phi(z) = az$. The function $\phi(z)/z$ is holomorphic in the entire plane, and will turn out to be bounded. By Liouvilles theorem, it is therefore constant, say equal to a. Hence $f(z) = az$, and we are done.

It remains to see that $\psi(z)$ is bounded. Let $A_R = \{\, z \mid |z| > R \,\}$. Then $\phi(A_R) \cap \phi(\mathbb{C} \setminus A_R) = \emptyset$ and $\phi(\mathbb{C} \setminus A_R)$ is an open neighbourhood of 0. Hence ϕ does not have an essential singularity at infinity, but must have a pole. It must be order one, if not ϕ would not be injective, hence $\phi(z)/z$ is holomorphic at infinity and therefor bounded. ❑

The maximum modulus principle and the open mapping theorem

We start out in a laid back manner and consider a real function f in one variable defined on an open interval I. In general, there is no reason that $f(I)$ should be open, even if f is real analytic—any global maximum or minimum of f kills the openness of $f(I)$. A necessary criterion for f to be an open map (that is $f(U)$ is open for any open U) is that f have no local extrema, and in fact, this is also sufficient. Thus "having local maxima and minima" or "being an open mapping" are close-knit properties of f.



For holomorphic functions f the situation is in one aspect very different. The modulus of an holomorphic function never has local maxima, this is the renowned *maximum* *modulus principle*. The holomorphic functions are similar to real functions in the aspect that the maximum modulus principle is tightly knit to the functions being open mapping; and since the maximum modulus principle holds, they are indeed open maps.

(1.35) The maximum modulus principle can be approached in several ways, we shall present two. The first, presented in this paragraph, hinges on the Cauchy formula in a disk, and is a clean cut and the reason why the maximums principle holds is quite clear. The other one, which is in a sense simpler just using the second derivative test for maxima, comes at the end of this section.

Theorem 1.8 (The maximum modulus principle) *Let f be a function holomorphic in the domain Ω. Then $|f(z)|$ has no local maximum unless f is constant.*

PROOF: The crankshaft in this proof is the Cauchy's formula expressed in polar coordinates. If D_r is a disk contained in Ω, centered at a and with r, one has

$$f(a) = \frac{1}{2\pi i}\int_0^{2\pi} f(a + re^{it})dt. \tag{1.22}$$

This follows quickly by substituting $z = a + re^{it}$ in Cauchy's formula for a disk (theorem **??** on page **??**), and the identity may be phrased as the "mean value of f on the circumference equals the value at the center".

Aiming for an absurdity, assume that a is a local maximum for the modulus $|f|$, and chose r so small that $|f(a)| \geq |f(z)|$ for all z in D_r. Now, if $|f(a)| = |f(z)|$ for all $z \in D_r$, it follows that f is constant. Hence for at least one r there are points on the circle ∂D_r where $|f|$ assumes values less than $|f(a)|$, and by a well known and elementary property of integrals of continuous functions, we get the following contradictory inequality:

$$|f(a)| \leq \frac{1}{2\pi i}\int_0^{2\pi} \left|f(a + re^{it})\right| dt < \int_0^{2\pi} |f(a)| = |f(a)|$$

❑

The following two offsprings of the maximum modulus theorem are immediate corollaries:

Corollary 1.2 *Let f a function holomorphic in the domain Ω. Then for any point a in Ω it holds true that $|f(a)| < \sup_{z\in\Omega}|f(z)|$ unless f is constant.*

Corollary 1.3 *Let $K \subseteq \Omega$ be compact and f a function holomorphic in Ω. Then f achieves it maximum modulus at the boundary ∂K, and unless f is constant, the maximum is strict.*

(1.36) Knowing there is a maximum principle one is tempted to believe in a minimal principle as well. And indeed, at least for non-vanishing functions, there is one. The proof is obvious: As long as f does not vanish in Ω, the inverse function $1/f$ is holomorphic there, and the maximum modulus principle for $1/f$ yields a minimum modulus principle for f.

Theorem 1.9 (The minimum modulus principle) *Assume that the function f is a non-vanishing and holomorphic in the domain Ω. Then f has no local minimum in Ω unless f is constant.*

(1.37) We have now come to the open mapping theorem, which we deduce from the the minimum modulus principle:

Theorem 1.10 (Open mapping) *Let Ω be a domain and let f be holomorphic in Ω. Then $f(\Omega)$ is an open subset of $\mathbb{C}$ unless f is constant.*

Of course if $U \subseteq \Omega$ is open, it follows that $f(U)$ is open; just apply the theorem with $\Omega = U$. So the theorem is equivalent to f being an open mapping.

Proof: Let $a \in \Omega$ be a point. We shall show that $f(a)$ is an inner point of $f(\Omega)$.

After replacing f by $f - f(a)$ we may assume that $f(a) = 0$, and since the zeros of f are isolated, there are disks D about a where f has no other zeros than a, and whose boundary is contained in Ω. Our function f does not vanish on boundary ∂D and has a therefore a positive minimum ϵ there.

Now, let w be a point not in $f(\Omega)$ with $|w| < \epsilon/2$. The difference $f(z) - w$ does not vanish in Ω, and on ∂D we have

$$|f(z) - w| \geq |f(z)| - |w| \geq \epsilon - \epsilon/2 = \epsilon.$$

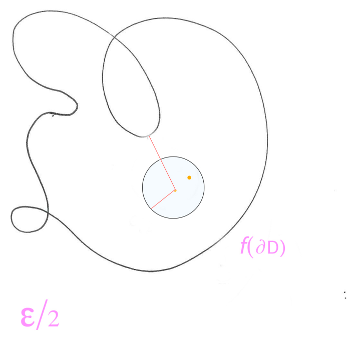
By the minimum modulus principle, $|f(z) - w| \geq \epsilon/2$ throughout D, in particular for $z = a$, which gives the absurd inequality $\epsilon/2 \leq |w| < \epsilon$. ❑

Problem 1.17. Prove that the open mapping theorem implies the maximum modulus principle. Hint: Every disk about $f(a)$ contains points whose modulus are larger than $|f(a)|$. ✷

(1.38) There is a simpler approach to the maximum modulus principle then the one we followed above that does not depend on relatively deep results like Cauchy's formula. The principle can be proven just by the good old second derivative test for extrema combined with the Cauchy-Riemann equations. We follow closely the presentation in [?] pages 24–26.

You probably remember from high school, that for a real function ϕ of one variable that is twice continuously differentiable the second derivative is non-positive at a local maximum; *i.e.*, if $a \in I$ is a local maximum for ϕ, then $\phi''(a) \leq 0$.

Now, if u is a twice continuously differentiable function of two variables defined in a domain $\Omega \subseteq \mathbb{C}$ and having a local maximum at $a = (\alpha, \beta)$, the Laplacian $\Delta u = u_{xx} + u_{yy}$



Figur 1.1:

is non-positive at a. Indeed, approaching a along lines parallel to the axes—that is applying the second derivative test to the two functions $u(\alpha, x)$ and $u(x, \beta)$—one sees that the second derivatives satisfy $u_{xx}(a) \leq 0$ and $u_{yy}(a) \leq 0$. With a small trick, this leads to:

Lemma 1.5 *Let the function u be defined and twice continuously differentiable in $\Omega \subseteq \mathbb{C}$ and assume that $\Delta u(z) \geq 0$ throughout Ω. Then for any disk D whose closure is contained in Ω, one has $u(a) \leq \sup_{z \in \partial D} u(z)$ for any $a \in D$. By consequence u has no local maximum in Ω.*

Proof: To begin with, assume that $\Delta u(z) > 0$ for all $z \in \Omega$, and let $u_0 = \sup_{z \in \overline{D}} u(z)$. If $u(a) > \sup_{z \in \partial D} u(z)$, the maximum point z_0 does not belong to the boundary ∂D and thus lies in D. But this is impossible as u does not have any local maximum after xxx above. If not, let $\epsilon > 0$ and look at the function $v(z) = u(z) + \epsilon |z|^2$. Then $\Delta v = \Delta u + 4\epsilon > 0$, and we have

$$u(a) < \sup_{z \in \partial D} (u(z) + \epsilon |z|^2) \leq \sup_{z \in \partial D} u(z) + \epsilon \sup_{z \in \partial D} |z|^2 ,$$

and letting ϵ tend to zero, we are done. ❑

Finally, to arrive at the maximum principle, we observe that if $u(z) = |f(z)|^2$, the Laplacian Δu is given as

$$\Delta u = \partial_z \partial_{\overline{z}} f \overline{f} = \partial_z (f \partial_{\overline{z}} \overline{f}) = \partial_z f \partial_{\overline{z}} \overline{f} = |f'(z)|^2 \geq 0.$$

Problem 1.18. Show that the Laplacian of the real and of the imaginary part of a holomorphic function vanish identically. ✷

Problem 1.19. Assume that f does not vanish in a Ω. Show that $u(z) = \log |f(z)|$ is well defined and with its Laplacian vanishing throughout Ω. ✷

Problem 1.20. Recall that the Hesse-determinant of a function u of two variable is $u_{xx}u_{yy} - u_{xy}^2$. Use the Cauchy-Riemann equations to show that the Hesse-determinant both of the real and of the imaginary part of a holomorphic function is non-positive. ✷

The order of holomorphic functions

A polynomial $P(z)$ has an order of vanishing at any point: The order is zero if P does not vanish at a and equals the multiplicity of the root a in case $P(a) = 0$. The order is characterized by being the largest number n with $(z-a)^n$ dividing P. Holomorphic functions resemble polynomials in this respect, they possess an order at every point where they are defined.

(1.39) Assume that f is a holomorphic function not vanishing identically near a. The Taylor series of f at a converges towards $f(z)$ in a vicinity of a, *i.e.*, one has

$$f(z) = f(a) + f'(a)(z-a) + \cdots + f^{(k)}(a)/k!(z-a)^n + \dots \tag{1.23}$$

for z near a. Hence if f together with all its derivatives vanish at a, the function f itself vanishes in a neigbourhood of a. So, if this is not the case, there is smallest non-negative integer n for which the n-th derivative $f^{(n)}(a)$ is non-zero. This integer is called the *order* of f at a and is written $\mathrm{ord}_a f$. The n first terms in the Taylor development will all be zero, and the remaining terms will all have $(z-a)^n$ as a factor; hence the Taylor series has the form

$$f(z) = (z-a)^n\big(f^{(n)}(a)/n! + f^{(n+1)}(a)/(n+1)!(z-a) + \dots\big),$$

where the series converges normally in a disk about a. We have proved

Proposition 1.12 *Assume that f is holomorphic near a and does not vanish identically in the vicinity of a. Let $n = \mathrm{ord}_a f$ denote the order of f at a. Then we may factor f as*

$$f(z) = (z-a)^n g(z),$$

where g is a holomorphic function near a not vanishing at a. The order of f is the largest non-negative integer for which such a factorization is possible.

Problem 1.21. Assume that f and g are two functions holomorphic near a.

a) Show that $\mathrm{ord}_a f = 0$ if and only if $f(a) \neq 0$.

b) Show that $\mathrm{ord}_a fg = \mathrm{ord}_a f + \mathrm{ord}_a g$.

c) Show that $\mathrm{ord}_a f + g \geq \min\{\mathrm{ord}_a f, \mathrm{ord}_a g\}$, with equality when the orders of f and g are different. Give examples with strict inequality but with $\mathrm{ord}_a f = \mathrm{ord}_a g$.

✷

(1.40) That holomorphic functions have factorizations like in ?? has some strong implications. The first is that the zeros of f must be isolated in Ω, another way expressing this is to say that the zero set $Z = \{ a \in \Omega \mid f(a) = 0 \}$ can not have any accumulation points in Ω. It might very well be infinite, even if Ω is a bounded domain, but its limit points all are situated outside Ω. This is a fundamental property of holomorphic functions, frequently use in sequel. It is called *identity principle*. An example is treated in exercise ?? below which is about the function $\sin \pi(z-1)(z+1)^{-1}$ which is holomorphic in the unit disk and has zeros at $(1-n)/(1+n)$ for $n \in N$. There are infinitely many and they accumulate at -1.

Theorem 1.11 *Let f be holomorphic in Ω. If the zero set Z of f has an accumulation point in Ω, then f vanishes identically.*

PROOF: Assume that f does not vanish identically, and let $a\imath\Omega$ be any point. Our function f has an order n at a and can be factored as $f(z) = (z-a)^n g(z)$, where g is holomorphic and does not vanish at a. The function g being continuous does not vanish in a vicinity of a, and of course $z - a$ only vanishes at a. We deduce that there is a neigbourhood of a where a is the only zero of $f(z)$, and consequently a is not a accumulation point of the zero set Z. ❑

The may be most frequently used form of the identity principle is the following, which by some authors is called the principle of "solidarity of values".

Corollary 1.4 *Assume that f and g are two functions holomorphic in Ω, if they coincide on a set with an accumulation point in Ω, they are equal.*

PROOF: Apply the identity principle ?? to the difference $f - g$. ❑

PROBLEM 1.22. Let f be holomorphic in Ω and assume that all but finitely many derivatives of f vanish at a point in Ω. Show that f is a polynomial. ✷

PROBLEM 1.23. Show that $\operatorname{Re}(1-z)(1+z)^{-1} = (1-|z|^2)\,|1+z|^{-2}$ and conclude that the map given by $z \to (1-z)(1+z)^{-1}$ sends the unit disk $\mathbb{D}$ into the right half plane. Let $f(z) = \sin \pi(1-z)(1+z)^{-1}$. Show that f has infinitely many zeros in $\mathbb{D}$ with -1 as an accumulation point. HINT: the zeros are those points in $\mathbb{D}$ such that $(1-z)(1+z)^{-1}$ is an integer. ✷

PROBLEM 1.24. Assume that f is holomorphic in the domain Ω. Show that the fibre $f^{-1}(a)$ is a discrete subset of Ω. Conclude that the fibre is at most countable. ✷

PROBLEM 1.25. Show that if f is holomorphic in D contained in Ω, and either $\operatorname{Re} f$ or the imaginary part $\operatorname{Im} f$ is constant in a disk $D \subseteq \Omega$, then f is constant. HINT: Use Cauchy Riemann equations and the identity principle. ✷

PROBLEM 1.26. Show that if $|f|$ is constant in a disk $D \subseteq \Omega$, then f is constant. HINT: Examine $\log f$. ✷

Isolated singularities

For a moment let $R(z) = P(z)/Q(z)$ be a rational function expressed as the quotient of two polynomials. It is of course defined in points where the denominator does not vanish, however, if a is a common zero of the denominator and the enumerator, one may cancel factors of the form $z - a$, and in case the multiplicity of a in numerator happens to be the higher, the rational function $R(z)$ has a well determined value even at a—it has a removable singularity there. Of course this definite value equals the limit $\lim_{z\to a} R(z)$. this not to happen, it is sufficient and necessary that $|R(z)|$ tends to infinity when z tends to a. Similar phenomenon, which we are about to describe, can occur for holomorphic functions.

Let Ω be a domain and $a \in \Omega$ a point. Suppose f is a function that is holomorphic in $\Omega \setminus a$. One sais that f has an *isolated singularity*. The isolated singularities come in three flavours; Firstly f can have a removable singularity (and at the end a is not a singularity at all). This is, as we shall see, equivalent to f being bounded near f. Secondly, the reciprocal $1/f$ can have a removable singularity while f has not; then one sais that f has a *pole* at a, and this occurs if and only if $\lim_{z\to a} |f(z)| = \infty$. In third case, that is if neither of the two first occurs, one says that f has an *essential singularity*.

(1.41) If f is holomorphic in a punctured disk D^* centered at a, one says that f has *a removable singularity* at a if it can be extended to a holomorphic function in D; that is, there is a holomorphic function g defined in D whose restriction to D^* equals f. Clearly this implies that $\lim_{z\to a}(z - a)f(z) = 0$ since f has a finite limit at a, and Riemann proved that also this is sufficient for f to be extendable. Nowadays this is called the *Riemann's extension theorem*:

Theorem 1.12 *Assume that f is holomorphic in the punctured disk D^* centered at a. Then f can be extended to a holomorphic function in D if and only if $\lim_{z\to a}(z - a)f(z) = 0$.*

Proof: If f can be extended, f has a limit at a and hence $\lim_{z\to a}(z - a)f(z) = 0$.

To prove the other implication, one introduces the auxiliary function

$$h(z) = \begin{cases} (z-a)^2 f(z) & \text{when } z \neq a, \\ 0 & \text{when } z = a. \end{cases}$$

Then h is holomorphic in the whole disk D and satisfies $h'(a) = 0$: For $z \neq a$ this is clear, and for $z = a$ one has

$$\big(h(z) - h(a)\big)/(z - a) = (z - a)^2 f(z)/(z - a) = (z - a)f(z),$$

which by assumption tend to zero when z approaches a. It follows that the order of h at a is at least two, and hence $h(z) = (z - a)^2 g(z)$ with g holomorphic near a. Clearly g extends f. ❑

If the function f is bounded near a, one certainly has $\lim_{z\to a}(z-a)f(z)=0$, and the Riemann's extension theorem shows that f can be extended. Riemann's criterion therefore has the following equivalent formulation:

Theorem 1.13 *Assume that f is holomorphic in the punctured disk D^* centered at a. Then f can be extended to a holomorphic function in D if and only if f is bounded in D^*.*

A familiar example of a function having removable singularity at the origin, is the function $\sin z/z$, and a little more elaborated one is $(2\cos z - 2 - z^2)/z^4$.

(1.42) A function f holomorphic in the punctured disk D^* is said to be *meromorphic* at a if $1/f(z)$ has a removable singularity there; or phrased equivalently: There is a neigbourhood U of a such that in the punctured neigbourhood $U^* = U \setminus \{a\}$ one may write $f(z) = 1/g(z)$ where $g(z)$ is holomorphic in U.

Two different cases can occur. If $g(a) \neq 0$, then $f(z)$ is holomorphic at a and nothing is new. On the other hand, if g vanishes at a, one says that f has a *pole* there, and the order of vanishing of g is called the *order of the pole* or the *pole-order*. One may factor g as

$$g(z) = (z-a)^n h(z),$$

where $n = \text{ord}_a g$ and h is holomorphic near a and $h(a) \neq 0$. Hence

$$f(z) = (z-a)^{-n} h_1(z),$$

where $h_1(z) = 1/h(z)$ is holomorphic and non-vanishing. The *order* of f at a is defined to be $-\text{ord}_a g$, so that at poles the order is negative[8]. For any function meromorphic at a this allows one to write

$$f(z) = (z-a)^{\text{ord}_a f} g(z),$$

where g is holomorphic and non-vanishing at a.

(1.43) In this paragraph we study more closely the third case when the singularity of f at a is an *essential* singularity, that is, it is neither removable nor a pole.

By the Riemann extension theorem $1/f$ has a removable singularity if and only if f is bounded near a, which translates into f being bounded away from zero in a neigbourhood of a. This is not the case if f has an essential singularity at a, meaning that for any $\epsilon > 0$ and any $\delta > 0$ there will always be points with $|z-a| < \delta$ with $|f(z)| < \epsilon$. Phrased in a different manner: The function f comes as close to zero as one wants as near a as one wants.

But even more is true. If α is *any* complex number, the difference $f - \alpha$ is meromorphic at a if and only if f is. This is trivial if f is holomorphic, and as the sequence

$$|f| - |\alpha| \leq |f - \alpha| \leq |f| + |\alpha|$$

[8]It is slightly contradictory that the order of f is the negative of the pole order, but it is common usage.

of inequalities shows, the difference $f - \alpha$ has a pole if and only if f has. So the end of the story is that f has an essential singularity if and only if $f - \alpha$ has. In the light of what we just did above, we have proven the following theorem, the *Casorati-Weierstrass theorem*

Theorem 1.14 *Assume that f has an essential singularity at a and let α be any complex number. Given $\epsilon > 0$ and $\delta > 0$, then there exists points z with $|z - a| < \delta$ and $|f(z) - \alpha| < \epsilon$.*

EXAMPLE 1.5. The archetype of an essential singularity is the singularity of $e^{1/z}$ at the origin. To get an idea of the behavior of $e^{1/t}$ we take a look at the function along the line where $\operatorname{Im} z = \operatorname{Re} z = t/2$, *i.e.*, where $z = (t + it)/2$. As $1/(1 + i) = (1 - i)/2$, and we find

$$e^{2/t(1+i)} = e^{1/t}(\cos 1/t - i \sin 1/t).$$

The ever accelerating oscillation of the trigonometric functions $\sin 1/t$ and $\cos 1/t$ as t approaches zero is a familiar phenomenon, and combined with the growth of $e^{1/t}$ illustrates eminently the Casorati-Weierstrass theorem. ❄

PROBLEM 1.27. Show that $f(z) = \sin \pi(1 - z)/(1 + z)$ has an essential singularity at $z = -1$. Show that for any real a with $|a| < 1$ there is a sequence $\{x_n\}$ of real numbers converging to -1 such that $f(x_n) = a$. HINT: Study the linear fractional transform $(1 - z)/(1 + z)$. ✱

PROBLEM 1.28. Let $g(z) = \exp -(1 + z)/(1 - z)$. Show that g has an essential singularity at $z = 1$. Show that $|g(z)|$ is constant when z approaches 1 through circles that are tangent to the unit circle at 1, and that any real constant can appear in this way. Show that g tends to zero when z approaches 1 along a line making an obtuse angle with A the real axis. HINT: Study the fractional linear transformation $(1+z)/(1-z)$. ✱

An instructive example

The theme of this paragraph, organized through exercises, is an entire function $F(z)$ with peculiar properties constructed by Gösta Mittag-Leffler and presented by him at the International Congress for Mathematicians in Heidelberg in 1905. When z tends to infinity, but stays away from a sector of the type $S_\alpha = \{\, z \mid -\alpha < \operatorname{Im} y < \alpha, \operatorname{Re} z > 0 \,\}$, the function $F(z)$ tends to zero. In addition $\lim_{x\to\infty} F(x) = 0$ where it is understood that x is real. In particular the limit of $F(z)$ is zero when z goes to ∞ along any ray emanating from the origin.

The construction is based on an "infinite contour" $\Sigma(u)$ where u is a positive real number. The path is depicted below in figure **??**. It has three parts: $\Sigma_1(u)$ is the segment from $x + \pi i$ to infinity, $\Sigma_2(u)$ the segment from ∞ to $x - \pi i$ and $\Sigma_0(u)$ the segment from $x - \pi i$ to $x + \pi i$. All three are parametrized in the simplest way by linear functions.

As a matter of language we say that a point z lies *inside* $\Sigma(u)$ if $\operatorname{Re} z > u$ and $-\pi < \operatorname{Im} z < \pi$; and of course, it lies *outside* $\Sigma(u)$ if it lies neither inside nor on $\Sigma(u)$.

Figur 1.2: *The path* $\Sigma(u)$.

We begin working with an entire function $f(z)$ merely assuming it be integrable along Σ; that is the integrals $\int_{\Sigma_i(u)} |f(z)|\,|dz|$ are finite for $i = 1, 2$. In the end we specialize f, as Mittag-Leffler did, to be the function

$$f(z) = e^{e^z} e^{-e^{e^z}}.$$

PROBLEM 1.29. Show that the integral

$$\int_{\Sigma(u)} f(w)(w-z)^{-1} dw.$$

is independent of u as long as z lies outside $\Sigma(u)$. ✷

Given an arbitrary complex number z and define a function

$$F(z) = \frac{1}{2\pi i} \int_{\Sigma(u)} f(w)(w-z)^{-1} dw. \tag{1.24}$$

where u is any real number such that z lies outside the contour $\Sigma(u)$. After the previous exercise this is a meaningful definition.

PROBLEM 1.30. Show that $F(w)$ is an entire function of w. HINT: See exercise **??** on page **??**. ✷

PROBLEM 1.31. Let $z = x + iy$ be any point not on the contour $\Sigma(u)$.

$$|w - z| \geq \begin{cases} |y - \pi| & \text{if } y \neq 0, \\ |x - u| & \text{if } y = 0. \end{cases}$$

Fix the number u and let $z = re^{i\phi}$. Show that

$$\lim_{r\to\infty} \int_{\Sigma(u)} f(w)(w-z)^{-1} dw = 0.$$

Show that $F(z)$ tends to zero when z tends to infinity along any ray emanating from the origin but being different from the positive real axis. HINT: For $|z|$ sufficiently large z stays outside of $\Sigma(u)$ and formula (**??**) is valid. ✷

Now we study what happens on the positive real axis, so assume that $z = x$ is real and positive. Fix a real and positive constant u_0 less than x, and let u be greater than x, and introduce the rectangular path R as illustrated in figure **??**.

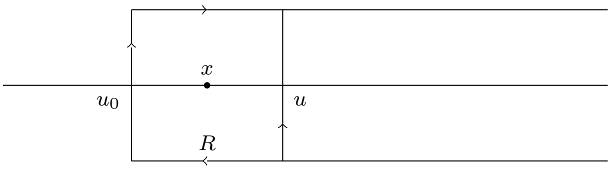
Problem 1.32. Show that as chains $\Sigma u_0 = R + \Sigma(u)$, and show that we have

$$F(x) - f(x) = \frac{1}{2\pi i}\int_{\Sigma(u_0)} f(w)(w-z)^{-1}dw.$$

Use this to show that

$$\lim_{x\to\infty} |F(x) - f(x)| = 0.$$

✷



Figur 1.3: *The rectangle R* .

In the last part of this exercise session, we specialize f to be the function $f(z) = e^{e^z}e^{-e^{e^z}}$.

Problem 1.33. Show that the integrals

$$\int_{\Sigma(u)} e^{e^w}e^{-e^{e^w}}dw$$

converge absolutely. Show that

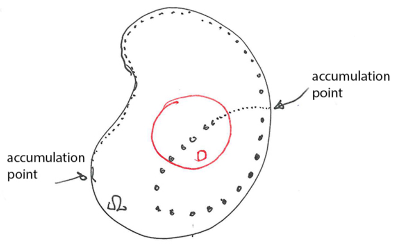
$$\lim_{x\to\infty} e^{e^x}e^{-e^{e^x}} = 0$$

and conclude that the associated function $F(z)$ tends to zero along any ray emanating from the origin. ✷

Problem 1.34. Show that F is not identically zero. ✷

The argument principle

It is classical that the multiplicities of the different roots of a polynomial add up to its degree. One can not hope for statements about holomorphic functions as strong as this. Already, there is no notion of degree for a holomorphic function in general. The order at a point is a sort of local degree; the degree of a polynomial is however a global invariant, and there is counterpart for holomorphic functions. And the number of zeros can very well be infinite, a simple example is $\sin \pi z$, which vanishes at all integers.



(1.44) However, there is a counting mechanism for the zeros, which goes under the name of the *argument principle*, which now and then is extremely useful.

So let f be holomorphic in Ω, and let D be any disk whose closure is contained in Ω. Then, as the zeros are isolated i Ω, there is at most finitely many of then in D.

Let $a_1, \ldots, a_r$ de the those of the zeros of f that are contained in the disk D, and denote by $n_1,, \ldots, n_r$ their multiplicities, *i.e.*, $n_i = \mathrm{ord}_{a_i} f$. By repeated application of proposition **??**, one may write

$$f(z) = \prod_i (z - a_i)^{n_i} g(z),$$

where the index i runs from 1 to r and where g is holomorphic an non-vanishing in D. Taking the logarithmic derivative gives

$$d \log f = \sum_{1 \leq i \leq r}^{r} n_i (z - a_i)^{-1} + d \log g.$$

(Recall that we write $d \log f$ for f'/f). The integral of $d \log f$ around the circumference ∂D, becomes

$$\frac{1}{2\pi i} \int_{\partial D} d \log f = \sum_{1 \leq i \leq r} n_i \, \mathrm{n}(\partial D, a_i) + \frac{1}{2\pi i} \int_{\partial D} d \log g.$$

Now, as g does not vanish in the disk D, it has a logarithm there, and hence $\int_\gamma d \log g = 0$. Consequently the integral $\int_\gamma d \log f$ satisfies

$$\frac{1}{2\pi} i \int_{\partial D} d \log f = \sum_i n_i \, \mathrm{n}(\partial D, a_i) = \sum_i n_i. \tag{1.25}$$

where the last equality holds since the winding nu,bers involved all equal one ∂D being traversed once counterclockwise and the a_i's all lying within ∂D. With the right interpretation the formula counts the total number of zeros of f contained in the disk D.

Problem 1.35. Denote by Z the set of zeros of f in Ω and for each $a \in Z$, let $n(a) = \mathrm{ord}_a f$. Show that

$$\int_{\partial D} d\log f = \sum_{a \in Z} n(a)\, \mathrm{n}(\partial D, a).$$

Hint: The sum is finite, even if it doesn't look like. ✷

(1.45) If a is any complex numbers, the zeros of the difference $f - a$ constitute the fibre $f^{-1}(a)$. Hence the technic in the last paragraph can as well be used to count points in fibres. Every point b in a fibre will contribute to the totality with a multiplicity equal to the multiplicity of the zero b of $f - a$. Denoting this multiplicity by $n(b)$ we have the formula

$$\frac{1}{2\pi i} \int_{\partial D} d\log(f - a) = \sum_{b \in f^{-1}(a) \cap D} n(b).$$

where of course $d\log(f - a) = f'(z)(f(z) - a)^{-1} dz$.

If γ is a parametrization of ∂D, the composition $f \circ \gamma$ is parametrizes a path Γ in $\mathbb{C}$; *i.e.*, we have $\Gamma = f \circ \gamma$. The winding number $\mathrm{n}(\Gamma, a)$ is given by an integral, and substituting $w = f(z)$ this integral changes in the following way:

$$\mathrm{n}(\Gamma, a) = \frac{1}{2\pi i} \int_{\Gamma} (w - a)^{-1} dw = \frac{1}{2\pi i} \int_{\gamma} f'(z)(f(z) - a)^{-1} dz,$$

hence

$$\mathrm{n}(\Gamma, a) = \sum_{b \in f^{-1}(a) \cap D} n(b).$$

We sum up in the following proposition:

Proposition 1.13 *Let f holomorphic in Ω and let D be a disk whose closure lies in Ω. Let a be any complex number. The number of points in the fibre $f^{-1}(a)$ lying within the disk D is finite, and counted with multiplicities, equals the winding number $\mathrm{n}(\Gamma, a)$ where Γ is the image of the boundary circle ∂D under f, traversed once counterclockwise.*

The winding number of a closed path is, as we saw, constant within each connected component of the complement of the path. Applying this to the image Γ of ∂D under f, we conclude that the number of points in $f^{-1}(a) \cap D$ and in $f^{-1}(b) \cap D$—counted appropriately—are the same as long as a and b belongs to the same connected component of $\mathbb{C} \setminus \Gamma$.

In particular, if A is a disk about a contained in the image of ϕ and not intersecting Γ, the two sets $f^{-1}(a) \cap D$ and $f^{-1} \cap D$ have equally many members. This leads to

Proposition 1.14 *Assume that f is a holomorphic map and that a is a solution of $f(z) = f(a)$ of multiplicity n. Then there is a disk D about a such that for b sufficiently close to $f(a)$, all solutions of $f(z) = b$ in D are simple and their number equals n.*

The theorem says that there are disks D and B about a and $f(a)$ respectively such that B lies in the image $f(D)$, and such that the fibers $f^{-1}(b) \cap D$ all are simple—that is every point occurs with multiplicity one—except the central fibre $f^{-1}(a) \cap D$ which reduces to just one point with multiplicity n.

Proof: As the derivative f' is holomorphic, its zeros are isolated and there is a disk D about a where it does not vanish in other points than a. Making D smaller, if necessary, it will also avoid the points in the fiber $f^{-1}(f(a))$ other than a.

The image $f(D)$ is open, and we chose a disk B containing $f(a)$ and lying in a connected component of the complement $\mathbb{C} \setminus \partial A$. As f' has no zeros in D, except at a, all fibers $f^{-1}(b) \cap D$ for $b \in B$, except $f^{-1}f(a) \cap D$, are simple, and by proposition **??** above they all have n points, as fibers over points from the same component of $\mathbb{C} \setminus f(\partial A)$. ❑

(1.46) The case $n = 1$ in **??** is a very important special case. Then the statement is that a functions f with $f'(a) \neq 0$ is injective in a disk containing a. This is also a consequence of the inverse function theorem, $f'(a)$ being the jacobian map at a of f; but there is a stronger statement that the inverse map f^{-1} is holomorphic. One has

Proposition 1.15 *Let f be holomorphic in Ω and let $a \in \Omega$ be a point with $f'(a) \neq 0$. There is a disk D about a on which f is biholomorphic. That is f is injective and the inverse map $f^{-1}\colon f(D) \to D$ is holomorphic, moreover the its derivative at $f(a)$ equals $1/f'(a)$.*

Proof: The inverse map f^{-1} is continuous since f is open, and the usual argument for the existence of the derivative of f^{-1} we know from calculus goes trough, letting $w = f(z)$ and $b = f(a)$ we have

$$(f^{-1}(w) - f^{-1}(b))/(w - b) = (z - a)/(f(z) - f(a)) \tag{1.26}$$

and as w tends to b continuity of f^{-1} implies that z tends to a, and the right side of (**??**) tends to $1/f'(z)$. ❑

Another way of proving this, is to appeal to the inverse function theorem. It says that f^{-1} is C^∞ near $f(a)$ and that its jacobian map at a point $f(z)$ equals the inverse of that of f. But of course, the inverse of multiplication by a complex number c is multiplication by c^{-1}, and we conclude by the Cauchy-Riemann equations.

(1.47) A biholomorphic map is frequently called *conformal*, a term coming from cartography and alluding to the fact that a holomorphic function with a non-vanishing derivative at a point a infinitesimally preserves the angel and orientation between vectors at a. This is due to the jacobian map being multiplication by $f'(a)$, so if $f'(a) = re^{i\phi}$ all angels are shifted by ϕ, so the difference between the two is conserved. The proposition **??** may be phrased as if f is holomorphic near a with non-vanishing derivative at a, then f is biholomorphic near a.

PROBLEM 1.36. Let γ and γ' be two paths that pass by a both with a non-vanishing tangents at a. Let ψ be the angle between the two tangents. Let f be holomorphic near a with $f'(a) \neq 0$. Show that the paths $f \circ \gamma$ and $f \circ \gamma'$ both have non-vanishing tangents at $f(a)$ and that the angle between them equals ψ. ✷

(1.48) Now, consider the case that $f(a) = 0$ and that $f'(a)$ vanishes, say with multiplicity n, Then f may be factored near a as

$$f(z) = z^n g(z)$$

where $g(z)$ is holomorphic and non-vanishing in a neigbourhood of a. It follows that g has an n-th root in a disk about a; say $g = h^n$. We may thus write

$$f(z) = (zh(z))^n = \rho(z)^n$$

where $\rho(z) = zh(z)$. Now $\rho' = h(z) + zh'(z)$ does not vanish at a since g does not, and hence ρ is biholomorphic near ρ. We therefore have

Proposition 1.16 *Assume that $f(a)$ has a zero of multiplicity n at a. Then there is a biholomorphic map ρ defined in a neigbourhood U of a such that*

$$f(z) = \rho(z)^n$$

for $z \in U$. In particular, f is locally injective at a if and only if $f'(a) \neq 0$.

PROOF: Only the last sentence is not proven. We have seen that if $n = 1$, then f is locally conformal and, in particular, it is locally injective. So assume that $n > 1$ and we must establish that f is *not* injective. The map ρ is open so the image $\rho(U)$ contains a disk A about the origin. If $\rho(z) \in A$ and η is an n-th root of unity, $\eta\rho(z)$ lies in A as well. Now, A being contained in $\rho(U)$ one has $\eta\rho(z) = \rho(z')$ for some $z' \in U$, and z' is different from z since ρ is injective. It follows that $f(z') = (\eta f(z))^n = f(z)$. ❑

PROBLEM 1.37. Show that

$$f^{-1}(b) = \int_{\partial D} zf'(z)(f(z) - b)^{-1} dz$$

✷

The general argument principle

At the end of this section, we give generalization of the formula (**??**) on page **??**. extending it to meromorphic functions. In this case one is forced to take both poles and zeros into account—it is their difference in number (with multiplicities) that is a categorical quantity, or said in clear text, a quantity that can be computed formally. This difference is just the sum

$$\sum_{a \in D} \mathrm{ord}_a f$$

where D is a disk whose closure lies within the domain Ω. We can only count the zeros and poles if they are finite in number, and $\overline{D}$ lying in Ω ensures this. Poles and zeros are isolated—that is the points a where $\operatorname{ord}_a f$ does not vanish—hence in the compact disk $\overline{D}$ there can only finitely many of them.

A second generalization is the introduction of a closed path γ in D. Loosely speaking, we count the difference of the number of poles and the number of zeros of f "lying with in γ". The precise meaning is the sum

$$\sum_{a\in D} \mathrm{n}(\gamma, a)\operatorname{ord}_a f.$$

where now D is any disk with encompassing γ and with $\overline{D} \subseteq \Omega$—and it essential that f has neither poles nor zeros lying on the path γ. We can safely factor f as a product

$$f(z) = \prod_{a\in D} (z-a)^{\operatorname{ord}_a f} g(z)$$

where $g(z)$ is holomorphic and without zeros in D and, of the course, the product is finite. Taking logarithmic derivatives we get the formula

$$d\log f = \sum_{a\in D} (z-a)^{-1}\operatorname{ord}_a f + d\log g \tag{1.27}$$

and integrating along the closed path γ:

$$\frac{1}{2\pi i}\int_\gamma d\log f = \sum n(\gamma, a)\operatorname{ord}_a f,$$

as $\int_\gamma d\log g = 0$, the function g having a logarithm in D.

(1.49) Let us now introduce a second holomorphic function $h(z)$ in Ω, and consider the integral $\int_\gamma g d\log f$. Multiplying (**??**) on page **??** by h gives

$$d\log f(z) = \sum_{a\in D} h(z)(z-a)^{-1}\operatorname{ord}_a f + h(z)d\log g(z).$$

Now, g is holomorphic and without zeros and $d\log g(z)$ is holomorphic as well. Hence $h(z)d\log g(z)$ is holomorphic and consequently its integrals round closed paths vanish by the Cauchy theorem. To integrate the terms in the sum, we appeal to Cauchy's formula which can be applied since h is holomorphic. This gives

$$\frac{1}{2\pi i}\int_\gamma h(z)d\log f(z) = \sum_{a\in D} h(a)\,\mathrm{n}(\gamma, a)\operatorname{ord}_a f, \tag{1.28}$$

which one may interpret as a counting formula for zeros and poles, but this time they are weighted by the function h. We sum up these computations in

Theorem 1.15 *Let f be a meromorphic function and h a holomorphic function in Ω. Then for any disk D with $\overline{D} \subseteq \Omega$ and any closed path γ in D, one has the equality*

$$\frac{1}{2\pi i}\int_\gamma h d\log f = \sum_{a\in\Omega} h(a)\,\mathrm{n}(\gamma, a)\mathrm{ord}_a f.$$

There is a still more general version of this theorem. Working with paths being null-homotopic, and this is the most natural hypothesis, one can get rid of the disk D, but for the moment we do not know that integrals of holomorphic functions only depends on the homotopy class of the integration path. Once that is established, the theorem **??** in its full force follows easily, but that is for the next section.

The Riemann sphere

The Riemann-sphere $\hat{\mathbb{C}}$ or the extended complex plane is just the one point compactification of the complex plane. We add one point at the infinity, naturally denoted by ∞, so as a set $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The topology is defined a for any one point compactification. The open sets containing ∞ are the sets $K^c \cup \{\infty\}$ where K is any compact subset of $\mathbb{C}$ (and K^c is its complement in $\mathbb{C}$), and the rest of topology, *i.e.*, those open sets not containing the point at infinity, are the open sets in the finite plane $\mathbb{C}$.

One has a coordinate function round ∞ defined by

$$w(P) = \begin{cases} 1/z & P = z \neq \infty \\ 0 & P = \infty. \end{cases}$$

A disk D_R centered at ∞ with radius R, that is $\{\, w \mid |w| < R \,\}$ corresponds to $\{\infty\} \cup \{\, z \mid |z| > R \,\}$, and it intersects the finite plane in $\{\, z \mid |z| > R^{-1} \,\}$.

By using the coordinate w we may extend all theory about the local behavior of a complex function at a finite points, to be valid at infinity as well.

(1.50) Assume that f is a function defined for $|z| > R^{-1}$. We say that f is *holomorphic at ∞* if $f(w^{-1})$ has a removable singularity at $w = 0$. By the Riemann extension theorem this is equivalent to $f(z)$ being bounded as $z \to \infty$, or if you want, to $f(z)$ having a limit when $z \to \infty$. And of course this limit is the value of f at ∞

In the same vain the function f has a pole at infinity if $f(w^{-1})$ has one at the origin. The pole has order n if $f(w^{-1}) = w^{-n}g(w)$ where the function $g(w)$ is holomorphic and non-vanishing at the origin. Substituting $w = z^{-1}$ we see that this becomes $f(z) = z^n g(z^{-1})$ where $g(z^{-1})$ is bonded, but with a non-zero limit when $z \to \infty$. An of course, f has a zero at infinity if $f(z) = z^{-n}g(z^{-1})$ where g has a non-zero limit as z tends to infinity.

Example 1.6. A polynomial of degree n has a pole of order n at infinity. Indeed, we have assuming that polynomial p is monic,

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0 = z^n(1 + a_{n-1}z^{-1} + \cdots + a_0 z^{-n}) = z^n g(z)$$

where $g(z)$ tends to 1 as $z \to \infty$ ❋

(1.51) In the last paragraph we discussed function defined at infinity, we take a closer look at functions taking the value infinity. Saying that f has a pole at a is the same a saying that $\lim_{z\to a} |f(z)| = \infty$. This is equivalent to saying that $f(z)$ tends to ∞ in the Riemann-sphere $\hat{\mathbb{C}}$, so setting $f(a) = \infty$ gives a continuous function into $\hat{\mathbb{C}}$.

Using the coordinate $w = z^{-1}$ at infinity, the behavior of f is described, by the behavior of $1/f(z)$, and it is easily seen that the order of vanishing of f at infinity equals the pole order at a.

Finally, a function f might have a pole at infinity, and its behavior is described by $1/f(z^{-1})$.

The general homotopy version of Cauchy's theorem

A type problems invariably arising in complex function theory are variants of the following "patching problem": Given a certain number of open subsets $\{U_i\}$ indexed by the set I and covering a domain Ω and for each U_i a function F_i holomorphic in U_i. Assume that any pair F_i and F_j differ by constant on each connected component of the intersection $U_i \cap U_j$—*e.g.*, a situation like this arises when F_i is a primitive for a given function f holomorphic in the union $\Omega = \bigcup_i U_i$.

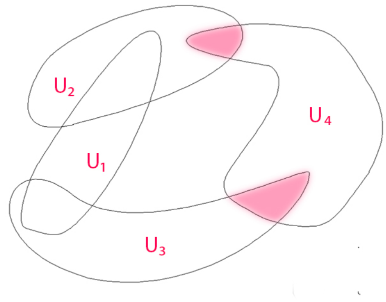
The big question is: When can one change each F_i by a constant such that any pair F_i and F_j agree on the whole $U_i \cap U_j$? Or phrased in precise manner: When can one find complex constants c_i such that for all pairs of indices the equality

$$F_i(z) + c_i = F_j(z) + c_j$$

holds true for all $z \in U_i \cap U_j$? The condition is clearly necessary and sufficient for the existence of a "patch" of the F_i's, meaning a function F defined in the whole of Ω restricting to F_i on each U_i. That is, F satisfy $F|_{U_i} = F_i$ for each i in I.

The question is a non-trivial one; illustrated by the simple situation with just two opens U_1 and U_2, but with the intersection $U_i \cap U_j$ being disconnected. In this situation the answer is positive if and only if F_1 and F_2 differ by the same constant on all the connected components of $U_i \cap U_j$.

In the figure below, for example, F_2 and F_3 on the open sets U_2 and U_3 are easily adjusted to coincide with F_1 on the intersections $U_1 \cap U_2$ and $U_3 \cap U_4$. Of course one can make F_4 mach F_2 on $U_2 \cap U_4$ but at the same time F_4 matches F_3 on $U_3 \cap U_4$, one has extremely lucky or very clever at the choices of F_2 and F_3.



Figur 1.4:

This section offers a variation of this theme. In the bigger picture on has the *cohomology groups* invented precisely for tackling challenges as described this flavour, but those will be for later.

We start out with a short recapitulation of a notion from topology, namely the homotopy of paths, and proceed with the main theme, a general Cauchy type theorem, stating that the integral of a holomorphic function only depends of the homotopy type of the path of integration.

Homotopy

For a moment we take on a topologist glasses and review —in a short and dirty manner — the notion of homotopy between two paths in a domain Ω of the complex plane. Homotopy theory has grown to big theory, nowadays it is a lion's share of algebraic topology, but it originated in complex function theory, and a lot of the results specific for elementary function theory of can be developed in an ad hoc manner without any reference to homotopy. However, let what belongs to the king belong to the king, and more important, pursuing the study of Riemann surfaces one will find that fundamental groups are omnipresent.

For a more thorough treatment one may consult Allan Hatchers book [?].

(1.52) For a topologist a path in Ω is a *continuous* path, that is a continuous map $\gamma : [0,1] \to \Omega$. It is convenient in this context to let all parameter intervals be the unit interval $I = [0,1]$. As $[0,1]$ is mapped homeomorphically onto any interval $[\alpha,\beta]$ by the affine function $(1-t)\alpha + t\beta$, this does not impose any serious principal restriction.

Observe that with this definition a constant map $\gamma(t) = a$ is path— a *constant path*. The *reverse path* of γ denoted γ^{-1}, is the path $\phi(1-t)$. If γ_1 and γ_2 are two paths such that the end-point of γ_1 coincides with the starting point of γ_2, one has *the composite path* $\gamma = \gamma_2\gamma_1$ given as

$$\gamma(t) = \begin{cases} \gamma_1(2t) & \text{when } 0 \leq t \leq 1/2 \\ \gamma_2(2t-1) & \text{when } 1/2 < t \leq 1, \end{cases}$$

one first traverses γ_1 and subsequently γ_2.

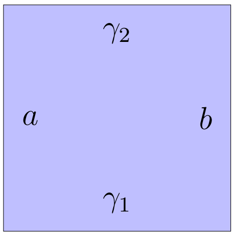
Closed paths, *i.e.,* loops ending where the started, are called loops in topology. And one usually specifies the common end- and start-point and speaks about loops at a point a. Two loops at a can always be composed.

(1.53) The intuitive meaning of two paths being homotopic in the domain Ω is that one can be deformed continuously into the other without leaving Ω. Let γ_0 and γ_1 be the two paths in the domain Ω. They are assumed to continuous and to have a common starting point, say a, and a common end-point b. That is, one has $\gamma_0(0) = \gamma_1(0) = a$ and $\gamma_0(1) = \gamma_1(1) = b$. It is a feature of the notion of homotopy that the starting points and the end-points stay fixed during the deformation.

The precise definition is as follows:

Defenition 1.2 *Let γ_0 and γ_1 be two continuous paths in the domain Ω both with starting point a and both with end-point b, are homotopic if there exists a continuous function $\phi\colon I \times I \to \Omega$ with $\phi(0,t) = \gamma_0(t)$ and $\phi(1,t) = \gamma_2(t)$ and $\phi(s,0) = a$ and $\phi(s,1) = b$.*

In figure below we have depicted $I \times I$ with the behavior of the homotopy ϕ on the boundary indicated.



Figur 1.5: A homotopy

(1.54) It is common to write $\gamma_1 \sim \gamma_2$ if γ_1 and γ_2 are homotopic, and it is not difficult to show that homotopy is an equivalence relation. The algebraic operation of forming the composite of two paths is compatible with homotopy. The composition is associative up to homotopy meaning that $(\gamma_1\gamma_2)\gamma_3 \sim \gamma_1(\gamma_2\gamma_3)$ where of course it is understood that the γ_i's are mutually composabel, and one may show that the homotopy classes of loops at a form a group under composition with the constant path as unit element and, of course, with the inverse path as inverse. It is called the *fundamental group* of Ω at a and it is written $\pi_1(\Omega, a)$.

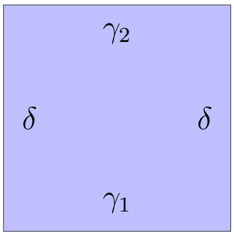
Example 1.7. If Ω is star-shape, say with a as the central point, then every loop at a is homotopic to the constant loop at a. Indeed, if γ is a loop, the convex combination $\phi(s,t) = (1-s)\gamma(t) + sa$ is a homotopy as required. ❋

Example 1.8. Assume that ϕ is a homotopy between γ_1 and γ_2, and assume that the final point of γ coincides with the common initial point of γ_1 and γ_2. Show that $\gamma_1\gamma \sim \gamma_2\gamma$, and with the appropriate hypothesis on γ, that $\gamma\gamma_1 \sim \gamma\gamma_2$. Conclude that if $\gamma_1' \sim \gamma_2'$, and γ_i''s satisfy the right composability condition, one has $\gamma_1\gamma_1' \sim \gamma_2\gamma_2'$.

Hint: Define a homotopy ψ by $\psi(s,t) = \gamma(2t)$ for $0 \leq t \leq 1/2$ and $\psi(s,t) = \phi(s, 2t-1)$ for $1/2 < t \leq 1$.

❄

(1.55) One can relax the condition on a homotopy and not require that the end-points be fixed. In that case one speaks about *freely homotopic paths*. Although, if the two paths are closed, one requires that the homotopy be a homotopy of closed paths; that is, the deformed paths are all closed. To be precise, one requires that $\phi(s,0) = \phi(s,1)$ for all s. This implies that the two paths $\delta_1(s) = \phi(s,0)$ and $\delta_2(s) = \phi(s,1)$ are the same.



(1.56) Let γ_1 and γ_2 be two piecewise C^1-curves that are composable—the end-point of the first being the start point of the other—and let γ the composite. Clearly γ is also piecewise C^1 and one has

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz.$$

In the same vain, if γ is piecewise C^1, the inverse path γ^{-1} is as well, and one has

$$\int_{\gamma^{-1}} f(z)dz = -\int_\gamma f(z)dz.$$

Integration behaves a little like a group homomorphism, so to speak. It takes composites to sums and inverse to negatives. And in the next section the main result is that integration of holomorphic functions also is compatible with homotopy—that is, the integral only depends on the homotopy class of the path of integration.

Homotopy invariance of the integral I

We come to main concern in this section, the general Cauchy theorem. In the usual setting, we are given a domain Ω and a function f holomorphic in Ω. The main result of the section basically says that the integral of f along a path γ (that must be piecewise C^1 to serve as a path of integration) only depends on the homotopy class of γ, and this means a homotopy that fixes the end points. There is also a version with the homotopy being a free homotopy, but it is only valid for close curves.

From the homotopy invariance we extract the general Cauchy's theorem and with the use of a few results about homotopy groups (that we do not prove) we obtain the general formulation of Cauchy's formula and the counting formula for zeros and poles.

Problem 1.38. Give an example of two freely homotopic paths and a holomorphic function whose integrals along the to paths differ. ✷

Problem 1.39. Give an example of two homotopic paths (fixed end-homotopic) and function that is not holomorphic whose integrals along the two paths differ. ✷

(1.57) It is slightly startling that although a homotopy between two piecewise C^1-curves is just required to be continuous (so no integration is allowed along the deformed paths), the integral of f along them remains the same.

If the homotopy is continuously differentiable, however, the independence of the integrals is not difficult to establish. Let $\phi\colon I \times I \to I$ denote the homotopy, and that assume it to be C^∞ in the interior of $I \times I$ and to restrict to piecewise-continuous paths on the boundary $\partial I \times I$.

We cover $\phi(I \times I)$ with finitely many disks ($I \times I$ is compact!). Furthermore we choose a partition $\{t_i\}_{0 \leq i \leq r}$ of the unit interval I such that if R_{ij} denotes the rectangle $[t_{i-1}, t_i] \times [t_{j-1}, t_j]$, it holds true that each R_{ij} is mapped into one of the covering disks. The restriction of ϕ to the boundary ∂R_{ij} is a closed path lying in the covering disk in which the image of R_{ij} lies, and we denote this path by $\phi(\partial R_{ij})$. The function f is holomorphic in the covering disk, so Cauchy's theorem for disks gives us

$$\int_{\phi(\partial R_{ij})} f(z)dz = 0. \tag{1.29}$$

By a simple and classical cancellation argument, which should be clear from the figure **??** below, it follows that

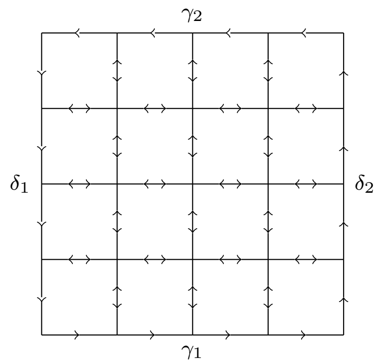
$$\int_{\gamma_1} f(z) - \int_{\gamma_2} f(z) - \int_{\delta_1} f(z)dz + \int_{\delta_2} f(z)dz = \sum_{i,j} \int_{\phi(\partial R_{ij})} f(z)dz = 0$$

the last equality stemming from (**??**) above. Hence we have

$$\int_{\gamma_1} f(z) - \int_{\gamma_2} f(z) = \int_{\delta_1} f(z)dz - \int_{\delta_2} f(z)dz. \tag{1.30}$$

Now assume that the γ_i's are closed paths. If we require that the the deformation of γ_1 into γ_2 should be through closed paths, we must have δ_1 and δ_2 to be the same paths. Then the right side in (**??**) above vanishes, and we can conclude that

$$\int_{\gamma_1} f(z) = \int_{\gamma_2} f(z).$$



Figur 1.6:

Homotopy invariance of the integral II

We closely follow the presentation of Reinholdt Remmert (page 169–174 in the book [?]), and proof is inspired by the proof of the so called van Kampen theorem in algebraic topology—a important theorem used to compute the fundamental group of unions—one would find in most textbooks in algebraic topology (*e.g.,* in [?]).

(1.58) The proof we present seems long and complicated, but the core is very simple. Most of it consists of rigging (which is the same rigging as we did in the case of a C^∞ homotopy)—one might be tempted to compare it to assembling a full orchestra to play a ten second jingle.

Theorem 1.16 *If γ_1 are γ_2 are two homotopic piecewise C^1-paths in the domain Ω and $f(z)$ is a holomorphic function in Ω, then one has the equality*

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

Proof: The basic rigging is as follows: Let $\{U_k\}$ be cover of Ω by open disks. Then f has a primitive function over each U_k; that is, there are functions F_k holomorphic in U_k with $F_k' = f$ in U_k, and these functions are unique up to an additive constant.

The inverse images $\phi^{-1}(U_k)$ form an open cover of $I \times I$ and by Lebesgue's lemma there is a partition $0 = t_0 < \cdots < t_r = 1$ of I such that each of the subrectangles $R_{ij} = [t_{i-1}, t_i] \times [t_{j-1}, t_j]$ are contained in $\phi^{-1}(U_k)$ for at least one k. We rename the R_{ij}'s and call them R_k indexed with k increasing t te left and upwards; that is, R_0 is the bottom left rectangle and R_n, say, the upper right one. The U_k's are renumbered accordingly. (Two U_k' for different k's can be equal).

The point of the proof is to construct a continuous function $\psi\colon I \times I \to \Omega$ with the property

$$\psi(s,t) = F_k(\phi(s,t)) \text{ for } (s,t) \in R_k, \tag{1.31}$$

where each F_k is a primitive function for f in U_k. which is as close to finding a primitive to $f(\phi(s,t))$ we can come. A crucial fact is that in the intersections $U_i \cap U_j$, which are connected, the functions F_i and F_j differ by a constant both being a primitive for f, and the salient point in the construction of ψ is to change the F_k's by appropriate constants (it might even happen that U_k and $U_{k'}$ are equal for $k \neq k'$ but the two functions F_k and $F_{k'}$ are different).

Lemma 1.6 *Once we have established the existence of a function ψ satisfying (??) the theorem follows.*

Proof: The map ϕ is just a continuous map, but on the boundary of $I \times I$ it restricts to the two original piecewise C^1-paths; so $\phi(0,t)$ is just the parametrization γ_1. Hence we get

$$\int_{\gamma_1} f(z)dz = \sum_{i=1}^{r} \int_{t_{i-1}}^{t_i} f(\phi(0,t))\phi'(0,t)dt = \sum_{i=1}^{r} \int_{t_{i-1}}^{t_i} F_i'(\phi(0,t))\phi'(0,t)dt =$$
$$= \sum_{i=1}^{r} \psi_i(t_{i-1}) - \psi_i(t_i) = \psi(0,0) - \psi(0,1).$$

In a similar way, one finds

$$\int_{\gamma_2} f(z)dz = \psi(1,0) - \psi(1,1).$$

Now, the homotopy ϕ fixes the end-points, which means that $\psi(s,0)$ and $\psi(s,1)$ are independent of s, in particular it follows that $\psi(0,1) - \psi(1,1) = \psi(0,0) - \psi(0,1)$, and in view of the computations above, that is exactly what we want. ❏

We carry on with the jingle, the construction of the mapping ψ so as to satisfy the condition (??) above. The tactics consist in using induction and successive extensions to exhibit, for each m, a function ψ_m on the union $\bigcup_{0\leq k\leq m} R_k$ extending ψ_{m-1} and satisfying (??) for $k \leq m$. And at the end of the process, we let ψ be equal to ψ_r — the last of the functions ψ_m.

So we assume that ψ_m is constructed on $\bigcup_{0\leq k\leq m} R_k$ subjected to (??) and try to extend it to

$$\bigcup_{0\leq k\leq m+1} R_k = R_{m+1} \cup \bigcup_{0\leq k\leq m} R_k.$$

The most difficult case is when R_{m+1} is located in a corner, as depicted in the figure below. We concentrate on that situation, leaving to the zealous student the easier case when R_m is situated in the bottom row or at the leftmost boundary of $I \times I$ and only intersects one of the previous rectangles in one edge.

The image of the edge $R_m \cap R_{m+1}$ under ϕ is contained in $U_m \cap U_{m+1}$. After possibly having changed F_{m+1} by a constant, we may assume that F_m and F_{m+1} coincide in

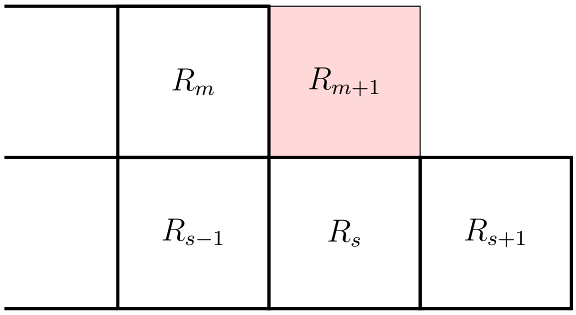
$U_m \cap U_{m+1}$ (which is connected), and hence $F_m(\phi(s,t))$ and $F_{m+1}(\phi(s,t))$ are equal along $R_m \cap R_{m+1}$.

By induction, $F_s(\phi(s,t))$ and $F_m(\phi(s,t))$ coincide in the corner $R_m \cap R_s \cap R_{s-1}$, both being equal to $\psi_m(s,t)$ there.

So along the edge $R_m \cap R_{m+1}$ the functions $F_m(\phi(s,t))$ and $F_{m+1}(\phi(s,t))$ agree, and along $R_m \cap R_s$ the functions $F_m(\phi(s,t))$ and $F_{m+1}(\phi(s,t))$ agree, hence $F_s(\phi(s,t))$ and $F_{m+1}(\phi(s,t))$ take the same value in the corner-point!

The salient point is to see that the functions F_s and F_{m+1} agree along the edge $R_{m+1} \cap R_s$, because then they patch up to a continuous function on $R_{m+1} \cup \bigcup_{0 \leq k \leq m} R_k$.

Luckily, they differ only by a constant in the intersection $U_{m+1} \cap U_s$, and the image of the corner lies there. As F_m and F_s agree in the corner, as do F_m and F_{m+1}, it follows that F_{m+1} and F_s are equal in the corner. Since their difference in $U_{m+1} \cap U_s$ is a constant, it follows that they are equal there, and in particular they coincide along the edge $R_{m+1} \cap R_s$. And that is what we were aiming for!



❏

(1.59) One can relax the condition on a homotopy and not require that the end-points be fixed in which case one speaks about *freely homotopic paths*. Although, if the two paths are closed, one requires that the homotopy be a homotopy of closed paths; that is, the deformed paths are all closed. To be precise, one requires that $\phi(s,0) = \phi(s,1)$ for all s.

In general integrals are obviously not invariant under free homotopy for non-closed paths, but for closed paths it holds true. One has

Theorem 1.17 *Let γ_1 and γ_2 be two closed piecewise C^1-paths in the domain Ω that are freely homotopic. Let f be holomorphic in Ω. Then*

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

Proof: The proof is the virtually same as for theorem **??**, with only one small exception: The maps $\psi(s,0)$ and $\psi(s,1)$ are no longer constant. However, we know that

$\phi(s,0) = \phi(s,1)$ for all s, which is sufficient to save the proof. As a matter of notation we let δ denote this path.

For each s it holds true that $\psi(s,0) = F_{i_s}(\phi(s,0))$ for some index i_s, however this index may change along the path δ. In an analogous manner, $\psi(s,1) = F_{j_s}(\phi(s,1))$ with the index j_s possibly varying with s. Now, $\phi(s,0) = \phi(s,1)$ and the F_k's differ only by constants. Therefore the difference $\psi(s,0) - \psi(s,1)$ is locally constant along δ and hence constant by continuity. It follows that

$$\psi(0,0) - \psi(0,1) = \psi(1,0) - \psi(1,1),$$

and by reference to the proof of lemma **??** we are done. ❑

(1.60) As an example, but important example, let us show that any closed path $\gamma(t)$ in the star-shaped domain Ω with apex a is freely homotopic to any circle contained in Ω and centered at a —traversed a certain number of times, in any direction. That γ is freely homotopic to a path of the form re^{it}, the parameter t running from 0 to $2n\pi$ and n being an integer and r sufficiently small so the circle lies in Ω. Express the path $\gamma(t)$ in polar coordinate as

$$\gamma(t) = a + r(t)e^{i\phi(t)},$$

with t running from 0 to $2n\pi$. Define a homotopy Φ by

$$\Phi(s,t) = (1-s)r(t)e^{i(1-s)\phi(t)} + sre^{ist},$$

where t runs from 0 to $2n\pi$—since the segment from a to $\gamma(t)$ is contained in Ω, clearly the segment from $a + re^{it}$ is as well. This shows that two closed curves are freely homotopic in Ω if and only if their winding numbers about a are equal.

(1.61) The previous example can be generalized using van Kampen's theorem. One may show that if Ω is any domain and Ω' is obtained from Ω by removing a point a (or a closed disk $\overline{D}$) contained in Ω, there is an exact sequence of fundamental groups

$$1 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \pi_1(\Omega') \longrightarrow \pi_1(\Omega) \longrightarrow 1 \tag{1.32}$$

and where the map α sends the generator 1 of $\mathbb{Z}$ to a circle around a contained in Ω and being traversed once counterclockwise, so a closed path γ lying in Ω' and being null-homotopic in Ω, has a homotopy class in Ω' that is a multiple of $\alpha(1)$; that is, the path is homotopic in Ω' to a small circle round a traversed a certain number of times, in one direction or the other.

General Cauchy theorem

From the invariance of the integral, we immediately obtain the following fundamental theorem:

Theorem 1.18 *Let Ω be a domain and f a function holomorphic in Ω and let γ be a piecewise closed C^1-path in Ω. Assume that γ is null-homotopic. Then*

$$\int_\gamma f(z)dz = 0.$$

PROOF: Let α be "half" the path γ, that is $\alpha(t) = \gamma(t/2)$ for $t \in [0,1]$, and let β be the other half, that is the one given by $\beta(t) = \gamma(t/2 + 1/2)$. Then of course γ is the composite $\beta\alpha$. The composite being null-homotopic implies that $\alpha \sim \beta^{-1}$, and hence by theorem **??** one has

$$\int_\alpha f(z)dz = -\int_\beta f(z)dz,$$

but then

$$\int_\gamma f(z)dz = \int_\alpha f(z)dz + \int_\beta f(z)dz = 0.$$

❑

For simply connected domains we get the general Cauchy theorem as an immediate corollary

Corollary 1.5 *Let Ω be a simply connected domain and let f be holomorphic ion Ω. Then for any closed path γ it holds true that*

$$\int_\gamma f(z)dz = 0.$$

(1.62) An in view of the existence criterion for primitives (proposition **??** on page **??**) we see holomorphic functions in simply connected domains all have primitives:

Corollary 1.6 *If f is a holomorphic function in the simply connected domain Ω, then f has a primitive.*

(1.63) In particular, and of particular interest, this applies to the logarithm. Any holomorphic function f vanishing nowhere in the simply connected Ω has a logarithm; *i.e.,* there is a function, which we denote by $\log f$, and that satisfies the equation

$$\exp \circ \log f = f \tag{1.33}$$

throughout Ω. Indeed, as f is without zeros in Ω, the logarithmic derivative f'/f is holomorphic there and, Ω being simply connected, has a primitive there. We temporarily denote this primitive by L (as long as (**??**) is not verified, it does not deserve to be titled $\log f$). A small and trivial computation using standard rules for the derivative, shows that

$$\partial_z f^{-1}(z) \exp(L(z)) = 0.$$

Hence $\exp(Lz) = Af(z)$ for some constant A. Of course it might be that $A \neq 1$, but then we change the primitive L into $L - \log A$, which is another primitive for f'/f.

As usual $\log f$ is unique only up to whole multiples of $2\pi i$.

When the logarithm $\log f$ is defined, the function f also possesses roots of all types. More generally for any complex constant α, the power f^α is defined; it is given as $f^\alpha = \exp(\alpha \log f)$.

The Genral Cauchy formula

Using the remark in example **??**, we obtain the general form of the formula of Cauchy, valid for null-homotopic paths in any domain Ω:

Theorem 1.19 *Assume that f is a holomorphic function in the domain Ω, and let $a \in \Omega$ be a point. Then for any closed path γ being null-homotopic in Ω, it holds true that*

$$\mathrm{n}(\gamma, a) f(a) = \frac{1}{2\pi i} \int_\gamma f(z)(z-a)^{-1} dz.$$

PROOF: By the homotopy invariance of the integral (theorem **??** on page **??**) and the remark in paragraph **??**, the integral in the theorem equals

$$\frac{1}{2\pi i} \int_{n\partial D} f(z)(z-a)^{-1} dz$$

for a certain integer n. In this integral D denotes a disk whose closure is contained in Ω, and $n\partial D$ indicates the path that is the boundary circle of D traversed n times. ❏

(1.64) There is also a generalization of the argument principle—giving us the ultimate formulation. However, it needs some preparation, the first being a common technic, which as well will be useful later, called *exhausting by compacts*. Recall the notation A° for the set of interior points of a set A.

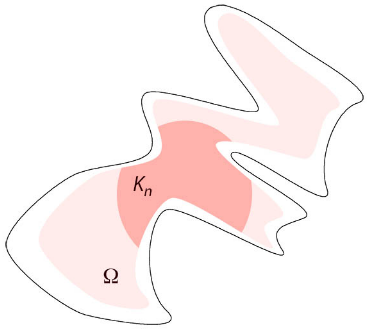
Lemma 1.7 *Assume that Ω is a domain in the complex plane. Then there exists a sequence of compact sets K_n all contained in Ω satisfying the two properties*

- ☐ *The sequence is increasing: $K_n \subseteq K_{n+1}$;*
- ☐ *Their interiors cover Ω, that is: $\bigcup_n K_n^\circ = \Omega$.*

PROOF: For each n we put

$$K_n = \{\, z \in \Omega \mid d(z, \partial D) \geq 1/n \,\} \cap \{\, z \mid |z| \leq n \,\}.$$

Then K_n is closed and bounded (the distance function being continuos) and the K_n-s form an increasing sequence. For every point z in Ω one has $d(z, \partial D) > 1/n$ and $|z| < n$ for n sufficiently large, hence the interiors of the K_n cover Ω. ❏



Figur 1.7: One of the compact exhausting sets.

(1.65) The reason we are interested in this process of exhausting by compacts at this stage, is that it guaranties there only being finitely poles and zeros of f having non-vanishing winding number with respect to a given closed path γ in Ω.

Indeed, γ is compact and hence must be contained in some K_n. Points outside K_n belong to the unbounded component of the complement $\mathbb{C}\setminus\gamma$ and the winding numbers of γ round them vanish. But zeros and poles of f are isolated, so in compact sets there is only finitely many. Hence

Lemma 1.8 *Assume that γ is a closed path in the domain Ω and that f is meromorphic in Ω. Then there is only a finite number of points $a \in \Omega$ such that $\mathrm{n}(\gamma, a) \neq 0$.*

(1.66) The second preparation is a formula from homotopy theory analogous to the exact sequence in example **??** on page **??**, but involving not only one point, and just as is the case with **??**, it hinges on the van Kampen theorem. We shall not prove it, so if you do not know the van Kampen theorem, you have no choice but trusting us.

Given a finite number $a_1, \ldots, a_r$ of points in the domain Ω and given r little disks D_i, centered at a_i respectively and so little that they are contained in the domain Ω. LetΩ' be Ω with the r given points deleted; *i.e.*, $\Omega' = \Omega \setminus \{a_1, \ldots, a_r\}$.

Denote by c_i the homotopy class in Ω' of the boundary circle ∂D_i traversed once counterclockwise. Then there is an exact sequence

$$Z \star \cdots \star \mathbb{Z} \longrightarrow \pi_1(\Omega') \longrightarrow \pi_1(\Omega) \longrightarrow 1.$$

Don't let the stars frighten you, they stand for something called a free product of groups. If you want to dig into these questions Alan Hatchers book [**?**] can be recommended. In clear text the sequence means that the fundamental group $\pi_1(\Omega)$ equals the quotient of $\pi_1(\Omega)$ by the normal subgroup generated by the r classes c_i.

The form of this statement, useful for us, is that if γ is a closed path null-homotopic in Ω and avoiding the points a_i, its homotopy class equals an integral combination $n_1 c_1 + \cdots + n_r c_r$ of the classes of the little circles round the a_i-s. By applying the homotopy invariance to the different integrals $\int_\gamma (z-a_i)^{-1}dz$, one sees that $n_i = \mathrm{n}(\gamma, a_i)$, so in the fundamental group of Ω', one has the equality

$$[\gamma] = \sum_i \mathrm{n}(\gamma, a_i)c_i$$

whenever γ is a closed and null-homotopic path in Ω (and $[\gamma]$ denotes its homotopy class); indeed, one has

$$\frac{1}{2\pi i}\int_{c_i}(z-a_j)^{-1}dz = \delta_{ij}.$$

(1.67) We have come to the scene of the ultimate formula in the context of counting poles and zeros: The setting is a domain Ω, a function f meromorphic in Ω and a function g holomorphic there. Finally, a closed path null-homotopic in Ω is an important player, and here comes the hero of the play, the ultimate formula:

$$\frac{1}{2\pi i}\int_\gamma g(z)d\log f(z) = \sum_{a\in\Omega} g(a)\,\mathrm{n}(\gamma, a)\mathrm{ord}_a f \tag{1.34}$$

This formula looks suspiciously like the formula (**??**) on page **??**, but the difference is of course the relaxed conditions on the domain and the path. The proof is simple once the preparations are in place.

We know that only for only finitely many points $a_1, \ldots, a_r$ in Ω the following product $\mathrm{n}(\gamma, a)\mathrm{ord}_a f$ is non-zero, hence the sum in the formula is finite. We know that γ is homotopic to an an integral combination $c = \sum_i n_i c_i$, with $c_i = \mathrm{n}(\gamma, a)_i$, and by the homotopy invariance of the integral we can replace $\int_\gamma g d\log f$ by $\sum_i n_i \int_{c_i} g d\log f$. Finally, in each of the terms in the latter sum the integral equals $g(a)\mathrm{ord}_{a_i} f$ by Cauchy's formula for a disk.

Laurent series

Recall that an annulus is a region in the complex plane bounded by two concentric circle. If the two radii are R_1 and R_2 with R_1 the smaller, and a is their common center, the annulus consists of the points z satisfying $R_1 < |z-a| < R_2$. In case $R_1 = 0$ or $R_2 = \infty$, the annulus is *degenerate* and equals to either the punctured disk $0 < |z-a| < R_2$, the complement of a closed disk $R_1 < |z-a|$ or the whole complex plane (in case $R_1 = 0$ and $R_2 = \infty$).

This section is about functions that are holomorphic in an annulus. They have a development into a double series analogous to the Taylor development of a function holomorphic in a disk.

(1.68) Let a_n be a sequence of complex numbers that is indexed by $\mathbb{Z}$; that is n can take both positive and negative integral values. Consider the *double series*

$$\sum_{n\in\mathbb{Z}} a_n(z-a)^n, \tag{1.35}$$

which for the moment is just a formal series. It can be decomposed in the sum of two series , one comprising the terms with non-negative indices, and the other the terms having negative indices. That is we one has

$$\sum_{n\in\mathbb{Z}} a_n(z-a)^n = \sum_{n<0} a_n(z-a)^n + \sum_{n\geq 0} a_n(z-a)^n. \tag{1.36}$$

One says that the series Σ is convergent for the values of z belonging to set S if and only if each of the two series in the decomposition above converges for z in the given S, and we say that the convergence is uniform on compacts if it is for each of the two decomposing series.

In case the series (**??**) converges for z in the set S, the "positive" and the "negative" series in (**??**) converges to functions f_+ and f_- respectively, and we say that double series converges to the function $f = f_+ + f_-$.

The "positive"series

$$\sum_{n\geq 0} a_n(z-a)^n$$

is an ordinary power series centered at the point a, and has, as every power series has, a radius of convergence. Call it R_2. The series thus converges in the disk D_{R_2} given by $|z-a| < R_2$, and diverges in the region $|z-a| > R_2$. It converges uniformly on compact sets contained in D_{R_2}, and as we know very well, defines a holomorphic function there.

On the other hand, the "negative" series

$$\sum_{n<0} a_n(z-a)^n$$

is a power series in $w = (z-a)^{-1}$; indeed, performing this substitution we obtain the expression

$$\sum_{n>0} a_{-n}w^n$$

for the "negative" series. This power series has a radius of convergence, that we for a reason soon to become clear call R_1^{-1}, so it converges for $|w| < R_1^{-1}$ and diverges if $|w| > R_1^{-1}$. Translating these conditions on w into conditions on z, we see that the "negative" series converges for $|z-a| > R_1$ and diverges for $|z-a| < R_1$. The convergence is uniform on compacts and therefore the sum of the series is a holomorphic function f_- in the region $|z-a| > R_1$.

The interesting constellation of the two radii of convergence is that $R_1 < R_2$, in which case the double series converges in the region sandwiched between the two circles centered at a and having radii R_1 and R_2 respectively, and there it represents the holomorphic function $f = f_+ + f_-$.

(1.69) Now, let $R_1 < R_2$ be two positive real numbers and let a be a complex number. We shall work with a function f that is holomorphic in the annulus $A(R_1, R_2)$, and we are going to establish that f has what is called a *Laurent series* in A, that , it can be represented as double series like the one in (**??**). We shall establish the following result:

Theorem 1.20 *Assume that f is holomorphic in the annulus $A = A(R_1, R_2)$. Then f is represented by a double series*

$$f(z) = \sum_{n\in\mathbb{Z}} a_n (z-a)^n$$

which converges uniformly on compacts in A. The coefficients a_n are given by

$$a_n = \frac{1}{2\pi i}\int_{c_r} f(w)(w-a)^{-n-1}dw$$

where c_r is any circle centered at a and having a radius r with $R_1 < r < R_2$.

Proof: To begin with, we let r_1 and r_2 be two real numbers with $R_1 < r_1 < r_2 < R_2$. The two circles c_1 and c_2 centered at a and with radii r_1 and r_2 respectively (and both traversed once counterclockwise) are clearly two freely homotopic paths in A, a homotopy being $\phi(s,t) = sc_{1(t)} + (1-s)c_2(t)$ (where c_i as well denotes the standard parametrization of c_i). Hence for any z lying between c_1 and c_2 the general Cauchy formula gives

$$f(z) = \frac{1}{2\pi i}\int_{c_2} f(w)(w-z)^{-1}dw - \frac{1}{2\pi i}\int_{c_1} f(w)(w-z)^{-1}dw \tag{1.37}$$

indeed, the winding number of the composite path $c_2 - c_1$ round z equals one.

Now, the point is that the two integrals appearing in (**??**) above, will be the two functions f_+ and f_-. To see this we shall apply the proposition **??** on page **??** twice.

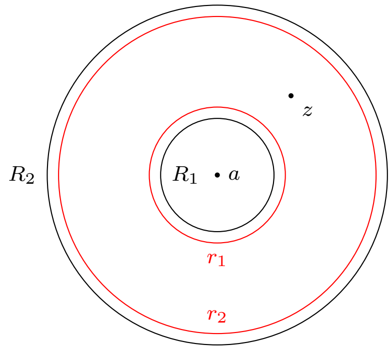
We start be examining the first integral, whose path of integration is c_2, and we take $\phi(w) = f(w)$ in proposition **??**. Hence

$$f_+(z) = \frac{1}{2\pi i}\int_{c_2} f(w)(w-z)^{-1}dw$$

is holomorphic in the disk $|z-a| < c_2$, and its Taylor series about a has the coefficients

$$a_n = \frac{1}{2\pi i}\int_{c_2} f(w)(w-a)^{-n-1}dw.$$

According to proposition **??**, the Taylor series converges in the largest disk not hitting the path of integration, that is the disk $|z-a| < c_2$.



Figur 1.8: *The annulus and the two auxiliary circles*

Next we the examine the second integral, and to do this, we perform the substitution $u = (w-z)^{-1}$. Then $dw = -u^{-2}du$, and the new path of integration is $|u| = r_1^{-1}$, a circle centered at the origin which designate by d. Upon the substitution, the integral becomes

$$f_-(z) = \frac{1}{2\pi i}\int_{c_1} f(w)(w-z)^{-1}dw = -\frac{1}{2\pi i}\int_d f(u^{-1}+z)u^{-1}du$$

Applying once more the proposition **??**, this time with $\phi(u) = -f(u^{-1}+z)$ and the path of integration equal to d (positively oriented), we conclude that the integral is a holomorphic function in the disk $|u| < r_1^{-1}$, or equivalently for $|z-a| > r_1$. Its Taylor series about the origin has, according to proposition **??**, coefficients b_n given by the integrals below, where we as well, reintroduce the variable w:

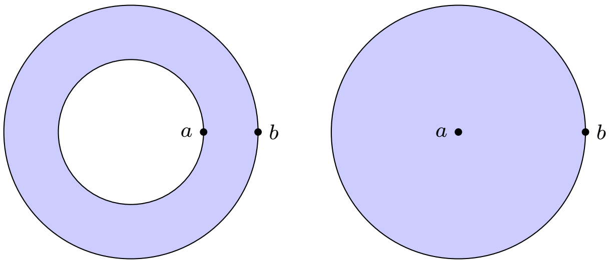
$$b_n = -\frac{1}{2\pi i}\int_d f(u^{-1}+a)u^{-n-1}du = \frac{1}{2\pi i}\int_{c_1} f(w)(w-a)^{n-1}dw,$$

And in fact, that will be all! ❑

Problem 1.40. Determine the Laurent series of the function $f(z) = (z-a)^{-1}(z-b)^{-1}$ in the annulus $A(|a|, |b|)$ centered at the origin. ✷

Problem 1.41. Determine the Laurent series of $f(z) = (z-a)^{-1}(z-b)^{-1}$ in the annulus $A(0, |b-a|)$ centered at a. ✷

Problem 1.42. Let f have an isolated singularity in a and be holomorphic for $0 < |z-a| < r$. Show that f has a pole at a if and only if the series for f_- in the Laurent development of f in annulus the $A(0,r)$ centered at a has a finite number of terms. ✷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Bibliografi

[Rem] Reinhold Remmert. *Complex functions.*

[Rot38] Alice Roth. Approximationseigenschaften und strahlengrenzwert meromorpher und ganser funktionen. *Comment. Math. Helv.*, 11:77–125, 1938.