MAT4800 – Complex Analysis

Some real analysis

o and O-notation

Suppose f is defined in a neighborhood V of $0 \in \mathbf{R}^n$, $f : V \to \mathbf{R}^m$.

$f = o(|x|^k)$ iff $\lim_{x\to 0} \frac{|f(x)|}{|x|^k} = 0$. (The case $k = 0$ is called $o(1)$.)

$f = O(|x|^k)$ iff $\exists C > 0$ such that $|f(x)| \le C|x|^k$ for x small.

Definition

f is *differentiable* at a if there is a linear map $L : \mathbf{R}^n \to \mathbf{R}^m$ such that

$$\lim_{x\to 0} \frac{|f(a+x) - f(a) - L(x)|}{|x|} = 0.$$

Equivalently, $f(a+x) = f(a) + L(x) + o(|x|)$.

L is called the derivative of f at a, and is denoted by $\mathrm{d}f_a$.

If f is differentiable at a, then the partial derivatives $\frac{\partial f_j}{\partial x_i}(a)$ exist and satisfy

$$\mathrm{d}f_a(v) = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(a) v_i \right) e_j$$

$$\mathrm{d}f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

The last matrix is called the *Jacobian matrix*.

If the partial derivatives $\frac{\partial f_j}{\partial x_i}$ exist in a neighborhood of a and are continuous at a, then f is differentiable at a.

$C(\Omega) = \{f : \Omega \to \mathbf{C}\,; f \text{ is continuous}\}$

$C^1(\Omega) = \left\{f : \Omega \to \mathbf{C}\,; \frac{\partial f}{\partial x_i} \in C(\Omega), i = 1, \dots, n\right\}$

$C^k(\Omega) = \{f : \Omega \to \mathbf{C}\,; \text{all partial derivatives of order} \le k \text{ are continuous}\}$

Order does not matter

$\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}^n$ multiindex

$|\alpha| = \alpha_1 + \cdots + \alpha_n$, order of the multiindex.

$\mathrm{D}^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}$

$C^\infty(\Omega) = \cap_k C^k(\Omega)$

Complex function of a complex variable, $\Omega \subset \mathbf{C}$.

$f : \Omega \to \mathbf{C}$, $z = x + \mathrm{i}y$, $f = u + \mathrm{i}v$.

$$f(z) = f(x, y) = u(x, y) + \mathrm{i}v(x, y)$$

As a real function $f : \Omega \to \mathbf{R}^2$, where $\Omega \subset \mathbf{R}^2$, $f = (u, v)$.

Let $\lambda = \alpha + \mathrm{i}\beta \in \mathbf{C} \cong \mathbf{R}^2$. What is $\mathrm{d}f(\lambda)$?

$$\mathrm{d}f(\lambda) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x}\alpha + \frac{\partial u}{\partial y}\beta \\ \frac{\partial v}{\partial x}\alpha + \frac{\partial v}{\partial y}\beta \end{pmatrix} = \left(\frac{\partial u}{\partial x}\alpha + \frac{\partial u}{\partial y}\beta\right) + \mathrm{i}\left(\frac{\partial v}{\partial x}\alpha + \frac{\partial v}{\partial y}\beta\right) = \alpha\frac{\partial f}{\partial x} + \beta\frac{\partial f}{\partial y}$$

We want to express this in terms of λ.

$\alpha = \mathrm{Re}(\lambda) = \frac{1}{2}(\lambda + \bar{\lambda})$, $\beta = \mathrm{Im}(\lambda) = \frac{1}{2\mathrm{i}}(\lambda - \bar{\lambda})$

$$\mathrm{d}f(\lambda) = \frac{1}{2}(\lambda + \bar{\lambda})\frac{\partial f}{\partial x} + \frac{1}{2\mathrm{i}}(\lambda - \bar{\lambda})\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \mathrm{i}\frac{\partial f}{\partial y}\right)\lambda + \frac{1}{2}\left(\frac{\partial f}{\partial x} + \mathrm{i}\frac{\partial f}{\partial y}\right)\bar{\lambda} =: \frac{\partial f}{\partial z}\lambda + \frac{\partial f}{\partial \bar{z}}\bar{\lambda}$$

The first term is complex linear, $L(c\lambda) = cL(\lambda)$, the second term is complex antilinear, $L(c\lambda) = \bar{c}L(\lambda)$.

We have that $\mathrm{d}f$ is $\mathbf{C}$-linear iff $\frac{\partial f}{\partial \bar{z}} = 0$.

$\frac{\partial f}{\partial \bar{z}} = 0$ is called the Cauchy-Riemann equations, i.e., $\frac{\partial f}{\partial x} = -\mathrm{i}\frac{\partial f}{\partial y}$. On real form, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Exercise

(a) Show that $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ satisfy Leibniz rule!

(b) Suppose $L : \mathbf{C}^n \to \mathbf{C}^m$ is $\mathbf{R}$-linear. Show that L is $\mathbf{C}$-linear iff $L(\mathrm{i}v) = \mathrm{i}L(v)$ for all $v \in \mathbf{C}^n$, and that L is $\mathbf{C}$-antilinear iff $L(\mathrm{i}v) = -\mathrm{i}L(v)$ for all $v \in \mathbf{C}^n$.

(c) Show that every $\mathbf{R}$-linear $L : \mathbf{C}^n \to \mathbf{C}^m$ splits uniquely in a $\mathbf{C}$-linear and a $\mathbf{C}$-antilinear part

$$L = L_{\mathbf{C}} + L_{\bar{\mathbf{C}}}$$

where

$$L_{\mathbf{C}}(v) = \frac{1}{2}\big(L(v) - \mathrm{i}L(\mathrm{i}v)\big), \qquad L_{\bar{\mathbf{C}}} = \frac{1}{2}\big(L(v) + \mathrm{i}L(\mathrm{i}v)\big)$$

Definition

$f : \Omega \to \mathbf{C}$ is called $\mathbf{C}$-differentiable at a if

$$\lim_{\lambda \to 0} \frac{f(a+\lambda) - f(a)}{\lambda}$$

exists. This is denoted by $f'(a)$.

f is $\mathbf{C}$-differentiable at a iff $f(a+\lambda) = f(a) + f'(a)\lambda + o(|\lambda|)$ iff f is differentiable at a and $\mathrm{d}f_a$ is $\mathbf{C}$-linear.

Definition

Let Ω be an open subset of $\mathbf{C}$. We say that a complex function $f(z)$ defined in Ω is holomorphic if $f \in C^1(\Omega)$ and f is complex differentiable at all points in Ω, i.e., f satisfies the Cauchy-Riemann equations.

The set of holomorphic functions is denoted by $\mathcal{O}(\Omega)$.

It is not necessary to assume $f \in C^1(\Omega)$ (this follows automatically when f is $\mathbf{C}$-differentiable), but it makes things easier, because we can use Green's theorem in the plane.

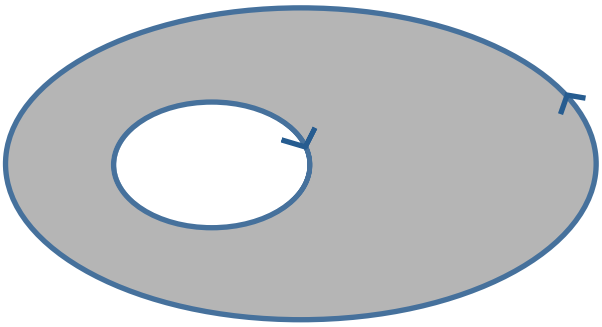
Green's theorem in the plane

If $\Omega \subset\subset \mathbf{R}^2$ is an open set with piecewise smooth boundary $\partial\Omega$ and M, N are two C^1 functions in $\overline{\Omega} = \Omega \cup \partial\Omega$, then

$$\int_{\partial\Omega} M\mathrm{d}x + N\mathrm{d}y = \iint_{\Omega} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \mathrm{d}x\mathrm{d}y$$

Remarks

1. $\partial\Omega$ is oriented such that Ω lies to the left of $\partial\Omega$.



2. It does not matter if M and N are real or complex valued.
3. $\int_{\partial\Omega} M\mathrm{d}x + N\mathrm{d}y$ is computed by parametrizing $\partial\Omega$ by $\big(x(t), y(t)\big)$, $a \le t \le b$. Then

$$\int_{\partial\Omega} M\mathrm{d}x + N\mathrm{d}y = \int_a^b M\big(x(t), y(t)\big)x'(t) + N\big(x(t), y(t)\big)y'(t)\mathrm{d}t$$

i.e. $\mathrm{d}x = x'(t)\mathrm{d}t$ and $\mathrm{d}y = y'(t)\mathrm{d}t$.

If $\gamma \subset \mathbf{C}$ is a curve parametrized by $z(t) = x(t) + \mathrm{i}y(t)$, $a \le t \le b$, and f is a complex function on γ, then the complex line integral is defined by

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b f\big(x(t) + \mathrm{i}y(t)\big)z'(t)\mathrm{d}t = \int_a^b f\big(x(t) + \mathrm{i}y(t)\big)\big(x'(t) + \mathrm{i}y'(t)\big)\mathrm{d}t = \int_\gamma f\mathrm{d}x + \mathrm{i}f\mathrm{d}y.$$

If $\gamma = \partial\Omega$ is as in Green's theorem, we get

$$\int_{\partial\Omega} f\mathrm{d}z = \iint_\Omega \left(\mathrm{i}\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right)\mathrm{d}x\mathrm{d}y = 2\mathrm{i}\iint_\Omega \frac{\partial f}{\partial \bar{z}}\mathrm{d}x\mathrm{d}y$$

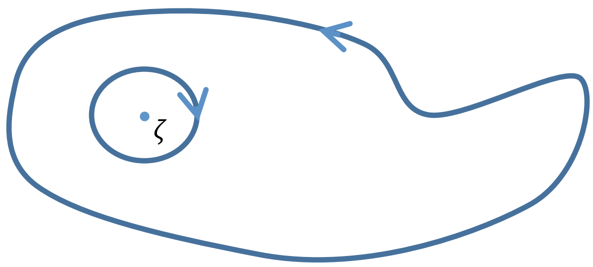
(Complex form of Green's theorem.)

Remarks

1. If f is holomorphic, we get Cauchy's theorem,
$$\int_{\partial\Omega} f\mathrm{d}z = 0$$
2. If γ is the circle $z = \zeta + r\mathrm{e}^{\mathrm{i}\theta}$, then $\mathrm{d}z = \mathrm{i}r\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta$ and
$$\int_\gamma \frac{f(z)}{z-\zeta}\mathrm{d}z = \int_0^{2\pi} \frac{f\big(\zeta + r\mathrm{e}^{\mathrm{i}\theta}\big)}{r\mathrm{e}^{\mathrm{i}\theta}} \cdot \mathrm{i}r\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta = \int_0^{2\pi} \mathrm{i}f\big(\zeta + r\mathrm{e}^{\mathrm{i}\theta}\big)\mathrm{d}\theta$$
$$= 2\pi\mathrm{i} \cdot (\text{average value of } f \text{ on } \gamma) \cong 2\pi\mathrm{i}f(\zeta)$$
3. Integral of a gradient; If γ is a curve from a to b and f is C^1 on γ, then
$$f(b) - f(a) = \int_\gamma \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y = \int_\gamma \frac{\partial f}{\partial z}\mathrm{d}z + \frac{\partial f}{\partial \bar{z}}\mathrm{d}\bar{z}$$
If f is holomorphic, then $f(b) - f(a) = \int_\gamma f'(z)\mathrm{d}z$.
If $|f'(z)| \le M$ on γ, then $|f(b) - f(a)| \le M\ell(\gamma)$.

Cauchy-Stokes' formula

Assume that f is C^1 in $\overline{\Omega}$, as in Green's theorem, and let $\zeta \in \Omega$. For small r, let $\Omega_r = \Omega \setminus \overline{D}(a,r)$. Then $\partial\Omega_r = \partial\Omega \cup \partial D(a,r)$, where $\partial D(a,r)$ is oriented clockwise.



Applying the complex form of Green's theorem to $\frac{f(z)}{z-\zeta}$ in Ω_r, we get

$$\int_{\partial\Omega} \frac{f(z)}{z-\zeta}\mathrm{d}z - \mathrm{i}\int_0^{2\pi} f\big(\zeta + r\mathrm{e}^{\mathrm{i}\theta}\big)\mathrm{d}\theta = 2\mathrm{i}\iint_{\Omega_r} \frac{\partial f/\partial\bar{z}}{z-\zeta}\mathrm{d}x\mathrm{d}y$$

The second integral will $\to 2\pi\mathrm{i}f(\zeta)$ as $r \to 0$, and the RHS will $\to 2\mathrm{i}\iint_\Omega \frac{\partial f/\partial\bar{z}}{z-\zeta}\mathrm{d}x\mathrm{d}y$ as $r \to 0$. (In the limit to the right, we have used the fact that $\frac{1}{z-\zeta}$ has a finite integral over Ω, i.e., is integrable, see Lemma 2 on page 99 of Narasimhan). This proves the following

Theorem

If f is C^1 in $\overline{\Omega}$ and $\zeta \in \Omega$ then

$$f(\zeta) = \frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega} \frac{f(z)}{z-\zeta} \mathrm{d}z - \frac{1}{\pi} \iint_{\Omega} \frac{\partial f/\partial \bar{z}}{z-\zeta} \, \mathrm{d}x\mathrm{d}y$$

In particular, if f is holomorphic, we get Cauchy's formula

$$f(\zeta) = \frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega} \frac{f(z)}{z-\zeta} \mathrm{d}z$$

Another particular case is if $f \in C^1(\mathbf{C})$ has compact support, then

$$f(\zeta) = -\frac{1}{\pi} \iint_{\mathbf{C}} \frac{\partial f/\partial \bar{z}}{z-\zeta} \mathrm{d}x\mathrm{d}y$$

for all $\zeta \in \mathbf{C}$.

Some consequences of the integral formulas

The first integral in the previous theorem is defined for all $f \in C(\partial\Omega)$. It is called the *Cauchy integral of* f. It is actually holomorphic for any curve. The following result follow immediately by differentiating under the sign of integration.

Proposition

Let $\gamma \subset \mathbf{C}$ be a piecewise smooth (C^1) curve and let $f \in C(\gamma)$. Then the function

$$\tilde{f}(\zeta) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma} \frac{f(z)}{z-\zeta} \, \mathrm{d}z$$

is holomorphic in $\mathbf{C} \setminus \gamma$. Moreover, $\tilde{f}$ is C^∞-smooth, $\tilde{f}'$ is holomorphic in $\mathbf{C} \setminus \gamma$, and

$$\tilde{f}^{(k)}(z) = \frac{k!}{2\pi \mathrm{i}} \int_{\gamma} \frac{f(z)}{(z-\zeta)^{k+1}} \, \mathrm{d}z$$

Definition

We say that a sequence of functions $\{f_n\}_{n=1}^{\infty}$ on Ω converges uniformly on compacts in Ω if there is a function f such that for any compact set $K \subset \Omega$ and $\epsilon > 0$ there is an integer N ($= N(K, \epsilon)$) such that

$$|f_n(z) - f(z)| < \epsilon \text{ for all } n \geq N \text{ and } z \in K.$$

Proposition

Let $f_n \in \mathcal{O}(\Omega)$ and assume that $f_n \to f$ uniformly on compacts in Ω. Then $f \in \mathcal{O}(\Omega)$ and $f_n^{(k)} \to f^{(k)}$ uniformly on compacts in Ω for any $k \in \mathbf{N}$.

Proof

Enough to prove on closed discs $\overline{D}(a, r) \subset \Omega$. This follows since f is given by an integral formula in $D(a, r)$ as in the previous proposition.

Definition

We say that a function f on Ω is analytic if f is given by a power series in all discs in Ω, i.e. if $D(a,r) \subset \Omega$ then

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \text{ for all } z \in D(a,r)$$

Proposition

If f is analytic in Ω then $f \in \mathcal{O}(\Omega)$.

Proof

Enough to prove that f is holomorphic in some disc $D(a,t)$ for all $a \in \Omega$. For simplicity of notation, assume $a = 0$ and that $D_r = \{|z| < r\} \subset \Omega$. If $0 < t < s < r$, then there exists $M > 0$ such that $|c_j s^j| < M$ for all $j \in \mathbf{N}$. Then for all $z \in \overline{D}_t$ we have

$$\left| \sum_{j=0}^{\infty} c_j z^j \right| \leq \sum_{j=0}^{\infty} |c_j s^j| \left(\frac{t}{s}\right)^j \leq M \sum_{j=0}^{\infty} \left(\frac{t}{s}\right)^j$$

The geometric series on the right converges. This shows that f is the limit of a sequence of polynomials on $\overline{D}_t$, hence f is holomorphic in D_t by proposition 3.2.

Proposition (Cauchy estimates)

If $f \in \mathcal{O}(D_r) \cap C(\overline{D}_r)$ then

$$\left|f^{(k)}(0)\right| \leq \frac{k!\, \|f\|_{\partial D_r}}{r^k}$$

Proof

By (3.2) we have that

$$\left|f^{(k)}(0)\right| \leq \frac{k!}{2\pi} \left| \int_{\partial D_r} \frac{f(z)}{z^{k+1}} \mathrm{d}z \right| = \frac{k!}{2\pi} \left| \int_0^{2\pi} \frac{f(re^{\mathrm{i}t})}{(re^{\mathrm{i}t})^{k+1}} \mathrm{i}re^{\mathrm{i}t}\, \mathrm{d}t \right| \leq \frac{k!\, \|f\|_{\partial D_r}}{r^k}$$

Corollary (Simple Maximum principle for a disc)

Let $f \in \mathcal{O}(D_r) \cap C(\overline{D}_r)$. Then $|f(0)| \leq \|f\|_{\partial D_r}$.

Theorem (Montel)

Let $\Omega \subset \mathbf{C}$ be an open set, and $\mathcal{F}$ be a family of holomorphic functions on Ω with the property that for each compact set $K \subset \Omega$ there exists a constant $C_K > 0$ such that $\|f\|_K \leq C_K$ for all $f \in \mathcal{F}$. Then for any sequence $\{f_j\}_{j \in \mathbf{N}} \subset \mathcal{F}$ there exists a subsequence $\{f_{n(j)}\}$ such that $f_{n(j)} \to f \in \mathcal{O}(\Omega)$ uniformly on compact subsets of Ω.

Proof

Let $A \subset \Omega$ be a dense sequence of points, and let $\{f_j\} \subset \mathcal{F}$ be a sequence such that $|f_j(a)|$ is convergent for all $a \in A$. We claim that the sequence $\{f_j\}$ converges to a holomorphic function f uniformly on compact subsets of Ω. Choose an exhaustion of Ω by compact sets $K_j \subset K_{j+1}^{\circ}$. For any j

we have that $\|f_i\|_{K_j} \leq M_j$ for all i. By the Cauchy estimates there is a constant N_j such that $\|f_i'\|_{K_j} < N_j$ for all i.

Now we fix K_j and show that $\{f_i\}\,|_{K_j}$ is a Cauchy sequence. Note that by the Mean Value Theorem we have for $z, z' \in K_{j+1}$ that $|f_i(z) - f_i(z')| \leq N_{j+1}|z - z'|$. Given any $\epsilon > 0$ we may choose a finite subset $\tilde{A} \subset K_{j+1}$ of A such that for any $z \in K_j$, there exists an $a \in \tilde{A}$ with $|z - a| < \frac{\epsilon}{4N_{j+1}}$. Furthermore, since $\{f_i\}\,|_{\tilde{A}}$ is Cauchy, we may find $N \in \mathbf{N}$ such that $|f_\ell(a) - f_m(a)| < \frac{\epsilon}{2}$ for all $\ell, m \geq N$. So given any $z \in K_j$ we may pick $a \in \tilde{A}$ to see that

$$|f_\ell(z) - f_m(z)| \leq |f_\ell(z) - f_\ell(a)| + |f_\ell(a) - f_m(a)| + |f_m(a) - f_m(z)| \leq 2N_{j+2}|z - a| + \frac{\epsilon}{2} < \epsilon$$

for all $\ell, m \geq N$, hence $\{f_i\}\,|_{K_j}$ is a Cauchy sequence.

Theorem

Let $f \in \mathcal{O}(\Omega)$ and $\overline{D}(a, r) \subset \Omega$. Then

$$f(\zeta) = \sum_{j=0}^{\infty} c_j(\zeta - a)^j$$

in $D(a, r)$, where

$$c_j = \frac{1}{2\pi \mathrm{i}} \int_{\partial D_r} \frac{f(z)}{(z - a)^{j+1}} \mathrm{d}z$$

Proof

We may assume $a = 0$. Note that $\frac{1}{z-\zeta} = \frac{1}{z\left(1 - \frac{\zeta}{z}\right)} = \frac{1}{z}\sum_{j=0}^{\infty}\left(\frac{\zeta}{z}\right)^j$ as long as $|\zeta| < |z|$, and plug this into Cauchy's integral formula.

Proposition (Identity principle)

Let $f \in \mathcal{O}(\Omega)$, where $\Omega \subset \mathbf{C}$ is connected. If $\mathrm{Z}(f) = \{z \in \Omega : f(z) = 0\}$ has non-empty interior, then $f \equiv 0$ on Ω.

Proof

For each $a \in \Omega$ we have that $f(z) = \sum_{j=0}^{\infty} c_j(a)(z - a)^j$ on a small enough disk centered at a. By the formula above we see that $c_j(a)$ is continuous in a for all j. So the set of points $\{a \in \Omega : c_j(a) = 0 \text{ for all } j \in \mathbf{N}\}$ is non-empty, open, and closed in Ω.

Proposition

Let $f \in \mathcal{O}(\Omega)$. Then $\mathrm{Z}(f)$ is discrete unless f is constantly equal to zero.

Proof

We assume that f is not constant. Near a point $a \in \Omega$ with $f(0) = 0$ we have that $f(z) = \sum_{j=k}^{\infty} c_j(z - a)^j$, $k \geq 1$, $c_k \neq 0$, so we can write $f(z) = (z - a)^k\left(c_k + \sum_{j=1}^{\infty} c_{k+j}(z - a)^j\right)$.

Definition

Let $\mathcal{O}^*(\Omega) = \{f \in \mathcal{O}(\Omega) : f(z) \neq 0 \text{ for all } z \in \Omega\}$.

Theorem
Let $D = D(a,r)$ be a disc. If $f \in \mathcal{O}(D)$, then f has a holomorphic antiderivative, i.e., there is $F \in \mathcal{O}(D)$ such that $F' = f$. If $f \in \mathcal{O}^*(D)$ then f has a holomorphic logarithm and m-th root of any order.

Proof
We know that $f = \sum_{n=0}^{\infty} c_n (z-a)^n$ in D. Let $F = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(z-a)^{n+1}$.

If $f \in \mathcal{O}^*(D)$, then $\frac{f'}{f} \in \mathcal{O}(D)$ and there is $F \in \mathcal{O}(D)$ such that $F' = \frac{f'}{f}$. Then $g = f\mathrm{e}^{-F} \in \mathcal{O}^*(D)$ and $g' = f'\mathrm{e}^{-F} + f\mathrm{e}^{-F}\left(-\frac{f'}{f}\right) = 0$, hence $g = c \neq 0$, a constant. Pick $\alpha \in \mathbf{C}$ such that $\mathrm{e}^{\alpha} = c$. Then $f = \mathrm{e}^{F+\alpha}$, so $G = F + \alpha$ is a holomorphic logarithm and $\mathrm{e}^{\frac{1}{m}G}$ is a holomorphic m-th root for any $m \in \mathbf{N}$.

Remark
This result is true in any simply connected domain Ω.

Theorem
If Ω is a domain and $f \in \mathcal{O}(\Omega)$ is nonconstant, then $f(\Omega)$ is open.

Proof
Pick $a \in \Omega$. We have to show that $f(\Omega)$ contains a neighborhood of $f(a)$. We may assume that $a = 0 = f(a)$. Ω contains a disc $D = D(0,r)$, and f is not constant in D. If $f(D)$ does not contain a neighborhood of 0, there exist $a_j \to 0$ such that $f(z) \neq a_j$ in D, i.e. $g_j = \frac{1}{f-a_j} \in \mathcal{O}(D)$. If $r' < r$ is such that $f(z) \neq 0$ for all z with $|z| = r'$, then $|g_j|$ is uniformly bounded on this circle, but $|g_j(0)| = \frac{1}{|a_j|} \to \infty$ as $j \to \infty$. This contradicts the maximum principle on a disc.

Corollary (Maximum principle)
If Ω is a domain, $f \in \mathcal{O}(\Omega)$ and $a \in \Omega$ is such that $|f(z)| \leq |f(a)|$ for all $z \in \Omega$, then f is constant.

Proof
This follows from Open Mapping Theorem.

Proposition (Hurwitz' theorem)
If Ω is a domain, $f_j \in \mathcal{O}^*(\Omega)$, and $f_j \to f$ uniformly on compacts then either $f \in \mathcal{O}^*(\Omega)$ or $f \equiv 0$ in Ω.

Proof
If $f(a) = 0$ and $f \not\equiv 0$, pick $r > 0$ such that $f(z) \neq 0$ when $|z-a| = r$. Then $|f(z)| \geq \delta > 0$ when $|z-a| = r$, hence $|f_j(z)| \geq \frac{1}{2}\delta$ when $|z-a| = r$ for sufficiently large j. Therefore $g_j = \frac{1}{f_j} \in \mathcal{O}(\Omega)$ and $|g_j(z)| \leq \frac{2}{\delta}$ when $|z-a| = r$. But this is impossible, since $g_j(a) = \frac{1}{f_j(a)} \to \infty$ when $j \to \infty$.

Definitions
Punctured disc around a: $D^*(a,r) = \{z \in \mathbf{C} : 0 < |z-a| < r\}$.

If $a \in \Omega$ and $f \in \mathcal{O}(\Omega \setminus \{a\})$, we say that f has a pole of order $k \in \mathbf{N}$ at a if in some punctured disc around a we have

$$f(z) = \frac{g(z)}{(z-a)^k}$$

where $g(z) \neq 0$ in $D^*(a, r)$. We then have

$$f(z) = c_{-k}(z-a)^{-k} + c_{-k+1}(z-a)^{-k+1} + \cdots = \sum_{n=-k}^{\infty} c_n (z-a)^n$$

in $D^*(a, r)$.

The residue of f at a is defined by

$$\mathrm{res}_a f = c_{-1}$$

In $D^*(a, r)$ we then have

$$f(z) = \frac{c_{-1}}{z-a} + \frac{\mathrm{d}}{\mathrm{d}z}\left(\sum_{\substack{n=-k \\ n \neq -1}}^{\infty} \frac{c_n}{n+1} (z-a)^{n+1} \right)$$

Hence for $r' < r$ we have

$$\int_{|z-a|=r'} f(z)\mathrm{d}z = 2\pi \mathrm{i} c_{-1} = 2\pi \mathrm{i}\, \mathrm{res}_a(f)$$

Proposition

If $\Omega \subset\subset \mathbf{C}$ has piecewise smooth C^1 boundary, $f \in \mathcal{O}(\Omega) \cap C^1(\overline{\Omega})$, except for poles $a_1, \dots, a_N \in \Omega$, then

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega} f \,\mathrm{d}z = \sum_{i=1}^{N} \mathrm{res}_{a_i} f$$

(This is called the residue theorem).

Proof

Let $D_1, \dots, D_N$ be disjoint small discs around $a_1, \dots, a_N$ and put $\Omega' = \Omega \setminus \cup_{j=1}^{N} \overline{D}_j$. Then Cauchy's theorem gives

$$0 = \frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega'} f \,\mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega} f \,\mathrm{d}z - \sum_{j=1}^{N} \frac{1}{2\pi \mathrm{i}} \int_{\partial D_j} f \,\mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega} f \,\mathrm{d}z - \sum_{j=1}^{N} \mathrm{res}_{a_j} f$$

Definition

We say that $f \in \mathcal{O}(\Omega \setminus \{a\})$ has order k at a if $f(z) = (z-a)^k g(z)$, where $g \in \mathcal{O}(\Omega)$ and $g(a) \neq 0$.

If $k > 0$ then we call a a zero of order k. If $k < 0$ then a is a pole of order $-k$.

It follows that $\frac{f'}{f} = \frac{k}{z-a} + \frac{g'}{g}$ near a, and hence $\mathrm{res}_a \frac{f'}{f} = k = \mathrm{ord}_a f$.

Corollary

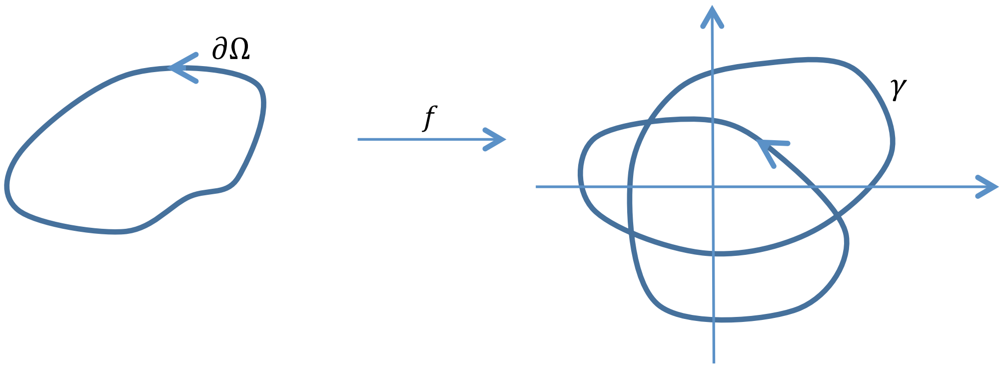
If $\Omega \subset\subset \mathbf{C}$ is as above, $f \in \mathcal{O}(\Omega) \cap C^1(\overline{\Omega})$ with $f(z) \neq 0$ on $\partial\Omega$, then

$$\int_{\partial\Omega} \frac{f'}{f} \mathrm{d}z = 2\pi\mathrm{i} \sum_{a \in \Omega} \mathrm{ord}_a f$$

If f only has simply zeroes and poles, this is

$$= \#\mathrm{zeroes} - \#\mathrm{poles}$$

This is also called the argument principle.



$$\int_{\partial\Omega} \frac{f'}{f} \mathrm{d}z = \int_{\gamma} \frac{1}{z} \mathrm{d}z = 2\pi\mathrm{i} \cdot (\text{winding number of } \gamma \text{ around zero})$$

This is still true if f has poles in Ω.

If $f(z) \neq w$ on $\partial\Omega$, i.e., $w \notin \gamma$, we have that the number of solutions of the equation $f(z) = w$ in Ω, counted with multiplicity, is given by

$$\frac{1}{2\pi\mathrm{i}} \int_{\partial\Omega} \frac{f'}{f - w} \mathrm{d}z = \frac{1}{2\pi\mathrm{i}} \int_{\gamma} \frac{\mathrm{d}z}{z - w} = \text{winding number of } \gamma \text{ around } w$$

In the figure above, $f'(z) = w$ has two solutions in the component of zero of $\mathbf{C} \setminus \gamma$, none in the unbounded component, and one in each of the remaining components.

Theorem (Rouché's theorem)

Let $\Omega \subset\subset \mathbf{C}$ be as above, $f, g \in \mathcal{O}(\Omega) \cap C^1(\overline{D})$ such that $|f(z) - g(z)| < |f(z)|$ for all $z \in \partial\Omega$. Then f and g have the same number of zeroes in Ω, i.e.,

$$\sum_{z \in \Omega} \mathrm{ord}_z f = \sum_{z \in \Omega} \mathrm{ord}_z g$$

Proof

Clearly f has no zeroes on $\partial\Omega$ and $\left|1 - \frac{g(z)}{f(z)}\right| < 1$ on $\partial\Omega$, so $F = \frac{g}{f}$ takes values in the disc $D(1,1)$ on $\partial\Omega$ and therefore has a holomorphic logarithm near $\partial\Omega$. We have

$$(\log F)' = \frac{F'}{F} = \frac{\frac{g'f - f'g}{f^2}}{\frac{f}{g}} = \frac{g'}{g} - \frac{f'}{f}$$

Hence

$$0 = \int_{\partial\Omega} (\log F)' \mathrm{d}z = \int_{\partial\Omega} \frac{g'}{g} - \int_{\partial\Omega} \frac{f'}{f} = \sum_{z\in\Omega} \mathrm{ord}_z g - \sum_{z\in\Omega} \mathrm{ord}_z f$$

Proposition

If Ω is a domain, $f_j \in \mathcal{O}(\Omega)$ are injective for all j, and $f_j \to f$ uniformly on compacts, then either f is injective or f is constant.

Proof

Assume that $a, b \in \Omega$ and that $f(b) = f(a)$. Let $g_j(z) = f_j(z) - f_j(a)$. Then $g_j \in \mathcal{O}^*(\Omega \setminus \{a\})$ and $g_j \to f - f(a)$ uniformly on compacts. Then either $f - f(a)$ is constant, which must be zero, so $f \equiv f(a)$, or $f - f(a)$ is without zeroes, which contradicts the fact that $f(b) = f(a)$.

Proposition

If $f \in \mathcal{O}(\Omega)$ is injective, then $f'(z) \neq 0$ for all $z \in \Omega$ and f has a holomorphic inverse $f^{-1} \in \mathcal{O}(f(\Omega))$.

Proof

We may assume that $z = 0$ and that $f(z) = 0$. We shall show that f has a zero of order 1 at 0. We have that $f(z) = z^k g(z)$ with $g \in \mathcal{O}(\Omega)$, $g(0) \neq 0$, $k \in \mathbf{N}$. In a disc D_r, g has a holomorphic kth root, i.e., there is $h \in \mathcal{O}(D_r)$ with $g(z) = h(z)^k$ and $h(0) \neq 0$. We get $f(z) = \left(z\, h(z)\right)^k$. The function $zh(z)$ is nonconstant, hence open. But then f takes values in a small disc at least k times in D_r. Hence $k = 1$.

By the inverse mapping theorem f has a C^∞ smooth inverse $f^{-1} : f(\Omega) \to \Omega$. The derivative $\mathrm{d}f^{-1}$ is the inverse of $\mathrm{d}f$, hence it is complex linear and f^{-1} is holomorphic.

Define $A(r, s) = \{\zeta \in : r < |\zeta| < s\}$ for $0 \leq r < s \leq \infty$.

Proposition (Laurent expansion)

If $f \in \mathcal{O}\left(A(r, s)\right)$ then f has a unique Laurent series expansion in $A(r, s)$,

$$f(\zeta) = \sum_{j=-\infty}^{\infty} c_j \zeta^j$$

where $c_j = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{f(z)}{z^{j+1}} \mathrm{d}z$, any $\rho \in (r, s)$. The series $\sum_{j\geq 0} c_j \zeta^j$ converges for $|\zeta| < s$, and the series $\sum_{j<0} c_j \zeta^j$ converges for $|\zeta| > r$.

Proof

The Cauchy theorem gives that $\int_{|z|=\rho} \frac{f(z)}{z^{j+1}} \mathrm{d}z$ is independent of $\rho \in (r, s)$. Let $\zeta \in A(r, s)$ and pick r', s' such that

$$r < r' < |\zeta| < s' < s$$

By the Cauchy-Stokes formula, we have

$$\begin{aligned} f(\zeta) &= \frac{1}{2\pi \mathrm{i}} \int_{|z|=s'} \frac{f(z)}{z-\zeta} \mathrm{d}z - \frac{1}{2\pi \mathrm{i}} \int_{|z|=r'} \frac{f(z)}{z-\zeta} \mathrm{d}z \\ &= \frac{1}{2\pi \mathrm{i}} \int_{|z|=s'} \frac{f(z)}{z} \frac{1}{1-\frac{\zeta}{z}} \mathrm{d}z + \frac{1}{2\pi \mathrm{i}} \int_{|z|=r'} \frac{f(z)}{\zeta} \frac{1}{1-\frac{z}{\zeta}} \mathrm{d}z = I + II \end{aligned}$$

$$I = \frac{1}{2\pi \mathrm{i}} \int_{|z|=s'} \frac{f(z)}{z} \sum_{j=0}^{\infty} \left(\frac{\zeta}{z}\right)^j \mathrm{d}z = \sum_{j=0}^{\infty} \left(\frac{1}{2\pi \mathrm{i}} \int_{|z|=s'} \frac{f(z)}{z^{j+1}} \right) \zeta^j$$

$$\begin{aligned} II = \frac{1}{2\pi \mathrm{i}} \int_{|z|=r'} \frac{f(z)}{\zeta} \sum_{j=0}^{\infty} \left(\frac{z}{\zeta}\right)^j \mathrm{d}z &= \sum_{j=0}^{\infty} \left(\frac{1}{2\pi \mathrm{i}} \int_{|z|=r'} f(z) z^j \mathrm{d}z \right) \zeta^{-(j+1)} \\ &= \sum_{j'<0} \left(\frac{1}{2\pi \mathrm{i}} \int_{|z|=r'} f(z) z^{-(j'+1)} \mathrm{d}z \right) \zeta^{j'} \end{aligned}$$

where $j' = -(j+1)$.

Exercise

If $r = 0$, $A(r,s)$ is the punctured disc $D_s^* = \{\zeta : 0 < |\zeta| < s\}$. f has a singularity at 0. There are three types:

(1) Removable singularity: $a_n = 0$ for $n < 0$. This happens iff f is bounded in D_s^*.
(2) Pole of order k: $a_{-k} \neq 0$, $a_n = 0$ for $n < -k$. This happens iff $|f| \to \infty$ when $z \to 0$.
(3) Essential singularity: $a_n \neq 0$ for infinitely many $n < 0$. This happens iff $f(D_t^*)$ is dense in $\mathbf{C}$ for all $0 < t \leq s$.

Liouville's theorem

If $f \in \mathcal{O}(\mathbf{C})$ is bounded, then f is constant.

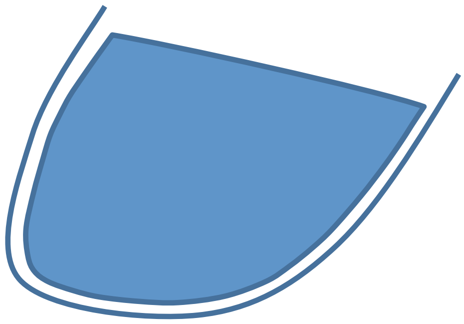
This follows easily from Cauchy estimate of f'.

Partitions of unity

If $U \subset \mathbf{R}^n$ is open, then there exists an exhaustion $\{K_j\}_{j=1}^{\infty}$ of U by compacts such that $K_j \subset K_{j+1}^{\circ}$, $\cup_j K_j = U$.

Proof

If $U = \mathbf{R}^n$ this is trivial. If not, let $K_j = \left\{z \in U : \mathrm{d}(z, \mathbf{R}^n \setminus U) \geq \frac{1}{j}\right\} \cap \bar{B}(j)$.



Definition

We say that a family $\mathcal{F}$ of subsets of $\mathbf{R}^n$ is locally finite if every $a \in \mathbf{R}^n$ has a neighborhood $B(a,r)$ such that $B(a,r) \cap E \neq \emptyset$ for only a finite number of sets $E \in \mathcal{F}$.

This is equivalent to $K \cap E \neq \emptyset$ for only a finite number of sets $E \in \mathcal{F}$ for any compact K.

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a collection of open sets. We say that $\mathcal{V} = \{V_j\}_{j \in J}$ is a refinement of $\mathcal{U}$ if for each V_j there is a U_i with $V_j \subset U_i$ and $\cup_{j \in J} V_j = \cup_{i \in I} U_i$.

Theorem

If $\mathcal{U} = \{U_i\}$ is an open covering of U (i.e., $U = \cup U_i$), then there is a locally finite refinement $\mathcal{V} = \{V_j\}$ of $\mathcal{U}$ and compacts $L_j \subset V_j$ such that $\cup_{j \in J} L_j = U$.

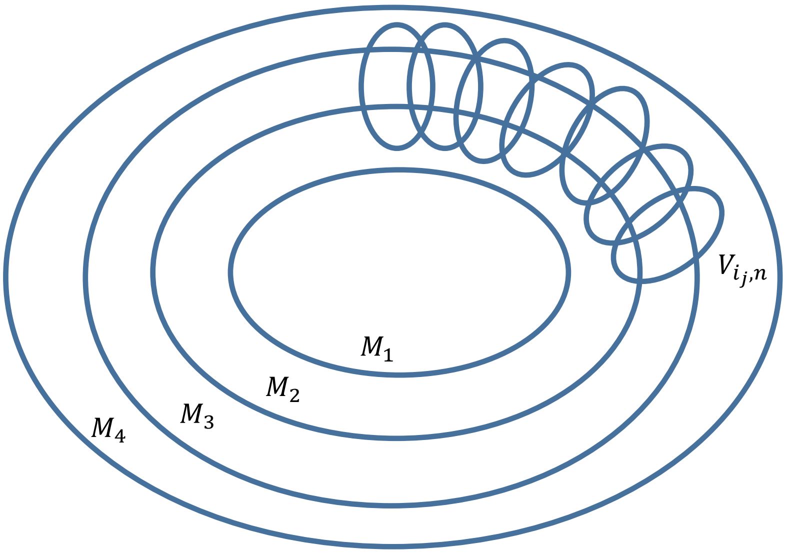
Proof

Let $\{K_n\}_{n=1}^{\infty}$be an exhaustion of U. We shall divide U into compact "rings" M_n like this:

$$M_1 = K_1, \qquad M_{n+1} = K_{n+1} \setminus K_n^{\circ}, \qquad \text{so } \cup_{n=1}^{\infty} M_n = U$$

We then define open sets W_n containing M_n which can only intersect the previous and next ring:

$$W_1 = K_2^{\circ}, \qquad W_2 = K_3^{\circ}, \qquad W_n = K_{n+1}^{\circ} \setminus K_{n-2} \text{ for } n \geq 3$$



Now $\mathcal{V}_n = \{V_{i,n} = U_i \cap W_n\}$ is an open cover of M_n and there exist $V_{i_j,n} \in \mathcal{V}_n$, $j = 1, \dots, p_n$ which cover M_n. Then there is some $\delta = \delta(n)$ such that for any $x \in M_n$ there is some i_j such that $B(x, \delta) \subset V_{i_j,n}$. This gives that the compacts

$$L_{i_j,n} = \left\{x \in M_n : \mathrm{d}\left(x, \mathbf{R}^n \setminus V_{i_j,n}\right) \geq \delta\right\} \subset V_{i_j,n}$$

cover M_n. Now, let

$$\mathcal{V} = \left\{V_{i_j,n} : n \in \mathbf{N}, j = 1, \dots, i_n\right\}$$

$\mathcal{V}$ is a refinement of $\mathcal{U}$ and since any compact K is contained in some K_n and therefore will not intersect any $V_{i_j,m}$ when $m > n + 1$, it is locally finite. The corresponding $L_{i_j,n}$ cover M_n and hence U.

If ϕ is a function defined on U, we define $\operatorname{supp} \phi = \overline{\{x : \phi(x) \neq 0\}}$, where we take the closure in U.

$C_0^\infty(U) = \{\phi \in C^\infty(U) : \phi \text{ is real and supp } \phi \text{ is a compact subset of } U\}$.

Definition. Partition of unity relative to $\mathcal{U}$.

If $\mathcal{U} = \{U_i\}_{i \in I}$ is an open cover of U, then a partition of unity relative to $\mathcal{U}$ is a family $\phi_i \in C^\infty(U)$ such that $\phi_i \geq 0$, $S_i = \operatorname{supp} \phi_i \subset U_i$, S_i of ϕ_i is locally finite, $\sum \phi_i \equiv 1$ in U.

Lemma

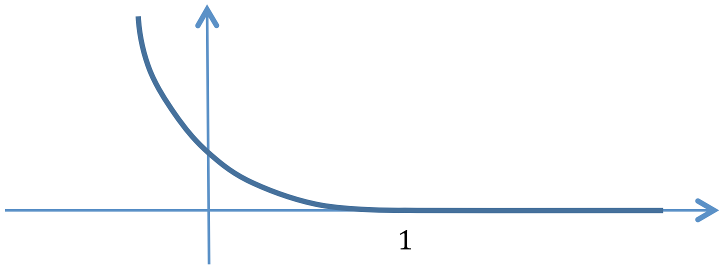
If U is open, $K \subset U$ is compact, then there is a positive function $\phi \in C_0^\infty(U)$ such that $\phi(x) > 0$ for $x \in K$.

Proof

The function

$$\psi(t) = \begin{cases} \mathrm{e}^{-1/(1-t)}, & t \leq 1 \\ 0, & t \geq 1 \end{cases}$$

is in $C^\infty(\mathbf{R})$.



There exists $\delta > 0$ such that $\operatorname{dist}(K, \mathbf{R}^n \setminus U) \geq 2\delta$. There are a finite number of points $a_1, \dots, a_N \in K$ such that $K = \cup_{i=1}^N B(a_i, \delta)$. Let

$$\phi(x) = \sum_{i=1}^n \psi\left(\frac{|x - a_i|^2}{\delta^2}\right)$$

Theorem
If $\mathcal{U} = \{U_i\}_{i \in I}$ is an open cover of U, then there is a partition of unity relative to U.

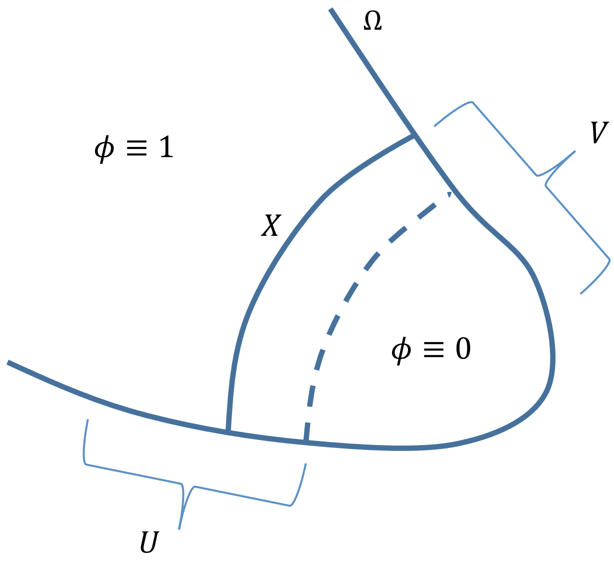
Proof
Let $\mathcal{V} = \{V_j\}_{j \in J}$ be a locally finite refinement of $\mathcal{U}$ and $L_j \subset V_j$ compacts which cover U. Then there are $\psi_j \in C_0^\infty(V_j) \subset C^\infty(U)$ such that $\psi_j > 0$ in K_j.

Let $\psi = \sum_j \psi_j$. The sum is locally finite, hence $\psi \in C^\infty(U)$ and $\psi > 0$ in U. If we let $\chi_j = \psi_j/\psi$, then χ_j is a partition of unity relative to V_j. For each $j \in J$ pick $\tau(j) \in I$ such that $V_j \subset U_{\tau(j)}$ and for each $i \in I$ define $\phi_i = \sum_{j \in \tau^{-1}(i)} \chi_j \in C^\infty(U)$. Clearly, $\{\operatorname{supp} \phi_i\}$ is locally finite.

If $x \in U \setminus U_i$ there is a neighborhood V of x such that $C \cap \operatorname{supp} \chi_j \neq \emptyset$ for only finitely many j. If $j \in \tau^{-1}(i)$ then $\operatorname{supp} \chi_j$ is a compact subset of U_i, hence $\phi_i \equiv 0$ in $V \setminus \cup_{j \in \tau^{-1}(i)} \operatorname{supp} \chi_j$ and $x \notin \operatorname{supp} \phi_i$. This proves that $\operatorname{supp} \phi_i \subset U_i$.

Theorem (Separation of closed sets)
If $\Omega \subset \mathbf{R}^n$ is open, $X \subset \Omega$ closed (relatively), $X \subset U$ open, then there exists $\phi \in C^\infty(\Omega)$ such that $0 \leq \phi \leq 1$, $\phi|_x = 1$, $\phi|_{\Omega \setminus U} = 0$.

Proof
Let ϕ_U, ϕ_V be a partition of unity relative to the covering $\{U, V\}$ with $V = \Omega \setminus X$. We must have $\phi_V|_X = 0$, so $\phi_U = 1$ on X. Also $\phi_U = 0$ in $\Omega \setminus U$.



Theorem (Patching C^∞ functions on disjoint closed sets)

If $\Omega \subset \mathbf{R}^n$ is open, $X_1, X_2 \subset \Omega$ two disjoint closed sets and $\phi_1, \phi_2 \in C^\infty(\Omega)$, then there exists $\phi \in C^\infty(\Omega)$ such that $\phi|_{X_1} = \phi_1$, $\phi|_{X_2} = \phi_2$.

Proof

Pick $\alpha \in C^\infty(\Omega)$, $0 \le \alpha \le 1$, $\alpha|_{X_1} = 1$, $\alpha|_{X_2} = 0$, and let $\phi = \alpha\phi_1 + (1-\alpha)\phi_2$.

The $\bar{\partial}$-equation, $\frac{\partial u}{\partial \bar{z}} = \phi$.

Recall Cauchy-Stokes formula in $\Omega \subset \mathbf{C}$. $(z = x + \mathrm{i}y, \zeta = \xi + \mathrm{i}\eta)$

If $f \in C^1(\bar{\Omega})$, $z \in \Omega$ then $f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta - \frac{1}{\pi} \iint_\Omega \frac{\partial f / \partial \bar{\zeta}}{\zeta - z} \mathrm{d}\xi \mathrm{d}\eta$.

If f is also holomorphic in Ω then $f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$.

If $f \in C_0^1(\mathbf{C})$, $z \in \mathbf{C}$ then $f(z) = -\frac{1}{\pi} \iint_{\mathbf{C}} \frac{\partial f / \partial \bar{\zeta}}{\zeta - z} \mathrm{d}\xi \mathrm{d}\eta$.

Given $\phi \in C_0^1(\mathbf{C})$, we want to find f such that $\frac{\partial f}{\partial \bar{z}} = \phi$. It is natural to try

$$f(z) = -\frac{1}{\pi} \iint_{\mathbf{C}} \frac{\phi(\zeta)}{\zeta - z} \mathrm{d}\xi \mathrm{d}\eta = -\frac{1}{\pi} \iint_{\mathbf{C}} \frac{\phi(\zeta + z)}{\zeta} \mathrm{d}\xi \mathrm{d}\eta$$

If we can differentiate under the sign of integration, then

$$\frac{\partial f}{\partial \bar{z}} = -\frac{1}{\pi} \iint_{\mathbf{C}} \frac{\frac{\partial \phi}{\partial \bar{z}}(\zeta + z)}{\zeta} \mathrm{d}\xi \mathrm{d}\eta = \phi(z)$$

Differentiation is allowed. Differentiate with respect to x, let $h \in \mathbf{R}$.

$$\frac{f(z+h) - f(z)}{h} = -\frac{1}{\pi} \iint_{\mathbf{C}} \frac{\frac{1}{h}\left(\phi(\zeta + z + h) - \phi(\zeta + z)\right)}{\zeta} \mathrm{d}\xi \mathrm{d}\eta \to -\frac{1}{\pi} \iint_{\mathbf{C}} \frac{\frac{\partial \phi}{\partial x}(\zeta + z)}{\zeta} \mathrm{d}\xi \mathrm{d}\eta$$

by the dominated convergence theorem, since $\frac{1}{\zeta} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^2)$. We can do the same in the y-direction, and hence we have proved

Theorem (Solving $\bar{\partial}$ with compact support)

If $\phi \in C_0^\infty(\mathbf{C})$ and

$$f(z) = -\frac{1}{\pi}\iint_{\mathbf{C}} \frac{\phi(\zeta)}{\zeta - z}\mathrm{d}\xi\mathrm{d}\eta$$

then $f \in C^\infty(\mathbf{C})$ and $\frac{\partial f}{\partial \bar{z}} = \phi$.

Notice that in general f does not have compact support, since for large R

$$0 = \int_{|z|=R} f\mathrm{d}z = 2\mathrm{i}\iint_{|z|\leq R} \frac{\partial f}{\partial \bar{z}}\mathrm{d}x\mathrm{d}y = 2i\iint_{|z|\leq R} \phi\mathrm{d}x\mathrm{d}y$$

would imply that $\int_{\mathbf{C}} \phi\mathrm{d}x\mathrm{d}y = 0$.

Theorem (Smeared out Cauchy integral formula)

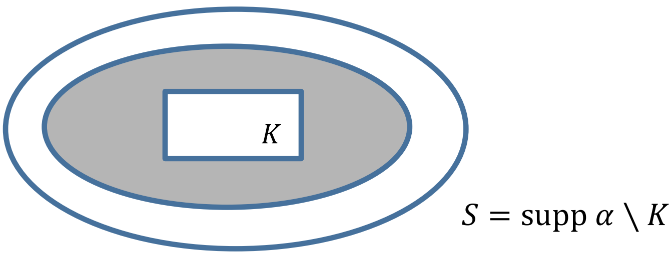
If $K \subset \Omega$ is compact, $f \in \mathcal{O}(\Omega)$ and $\alpha \in C_0^\infty(\Omega)$ is $\equiv 1$ on K, then for $z \in K$

$$f(z) = -\frac{1}{\pi}\iint_{\Omega} f(\zeta)\frac{\partial \alpha}{\partial \bar{\zeta}}\frac{1}{\zeta - z}\mathrm{d}\xi\mathrm{d}\eta$$

In particular, $\iint_{\Omega} f(\zeta)\frac{\partial \alpha}{\partial \bar{\zeta}}\mathrm{d}\xi\mathrm{d}\eta = 0$.

Proof

Apply Cauchy-Stokes to $\phi = \alpha f$.



Definition

Let $K \subset \mathbf{C}$ be compact. Then

$$\mathcal{O}(K) = \{f \in \mathcal{O}(U_f) : U_f \text{ open neighborhood of } K\}$$

Example

$$K = \left\{|z| = \frac{1}{2}\right\}$$

Then $f(z) = z$ and $g(z) = \frac{1}{z}$ are both in $\mathcal{O}(K)$.

The Runge problem

Let $K \subset \Omega$ be compact and $f \in \mathcal{O}(K)$. Is it possible to approximate f on K by $f_n \in \mathcal{O}(\Omega)$?

Example

Let K, f and g be as above.

(a) Let $\Omega = D = \{|z| < 1\}$. Then $f \in \mathcal{O}(\Omega)$, so there are no problems with f. We claim that g cannot be approximated: If $h \in \mathcal{O}(D)$ and $h \sim g$ on K (close to), then
$$1 = zg(z) \sim zh(z) \text{ on } K$$
If $k(z) = zh(z)$ is close to 1 on K, then it also is close on $D_{\frac{1}{2}} = \left\{|z| < \frac{1}{2}\right\}$ by the maximum modulus theorem. But this is not true, since $k(0) = 0$.

(b) Let $\Omega = D^* = D \setminus \{0\}$. Then both f and g are in $\mathcal{O}(\Omega)$, so there are no problems with approximation.

The problem in (a) is that $\Omega \setminus K$ has a component, $D_{\frac{1}{2}}$, which is relatively compact in Ω. In (b), the corresponding component is $D_{\frac{1}{2}} \setminus \{0\}$, which is not relatively compact since it goes all the way up to $0 \in \partial\Omega$.

Exercise

Let $\Omega \subset \mathbf{C}$ be open, let $K \subset \Omega$ be compact, and let U be a bounded connected component of $\Omega \setminus K$. Then the following are equivalent:

(1) $\exists \delta > 0$ such that $|z - w| \geq \delta$ for all $z \in U, w \notin \Omega$
(2) $U \subset\subset \Omega$
(3) $\partial U \subset K$
(4) U is also a connected component of $\mathbf{C} \setminus K$

If we negate this, the following are equivalent:

(1) For all $\delta > 0$ there exist $z \in U$ and $w \notin \Omega$ such that $|z - w| < \delta$
(2) U is not relatively compact in Ω
(3) $\partial U \cap (\mathbf{C} \setminus K) \neq \emptyset$
(4) The connected component U' of $\mathbf{C} \setminus K$ containing U is not contained in Ω, i.e., $U' \cap (\mathbf{C} \setminus \Omega) \neq \emptyset$

Theorem (Runge)

Let $\Omega \subset \mathbf{C}$ be open and $K \subset \Omega$ compact. The following are equivalent:

(1) $\mathcal{O}(\Omega)|_K$ is dense in $\mathcal{O}(K)$.
(2) No connected component of $\Omega \setminus K$ is relatively compact in Ω.
(3) $\forall a \in \mathbf{C} \setminus K$ there is $f \in \mathcal{O}(\Omega)$ such that $|f(a)| > |f|_K$.

Proof

(1) $\Rightarrow$ (2) If U is a connected component of $\Omega \setminus K$ which is relatively compact in Ω, then $\partial U \subset K$, because otherwise we could attach a disc to $z \in \partial U \setminus K$ to obtain a bigger connected set. If

$z_0 \in U$ and $f(z) = \frac{1}{z-z_0} \in \mathcal{O}(K)$, then f cannot be approximated by $f_n \in \mathcal{O}(\Omega)$, because if $\frac{1}{z-z_0} - f_n \to 0$ on K, then $g_n = 1 - (z-z_0)f_n \to 0$ on K, but $g_n(z_0) = 1$. This violates the maximum modulus theorem, since $\partial U \subset K$.

(2) $\Rightarrow$ (1) We must prove that every $f \in \mathcal{O}(K)$ can be approximated uniformly on K by $f_n \in \mathcal{O}(\Omega)$. Pick $f \in \mathcal{O}(W)$ for some open neighborhood W of K.

Step 1. Approximation of f by rational functions with poles outside K.

Pick $\alpha \in C_0^\infty(W)$ such that $\alpha = 1$ in a neighborhood W_0 of K. For $z \in K$ we have by Cachy-Stokes formula

$$f(z) = \frac{1}{\pi}\iint_{\mathbf{C}} f(\zeta)\frac{\partial\alpha}{\partial\bar{\zeta}}\frac{1}{z-\zeta}\mathrm{d}\zeta\mathrm{d}\eta = \frac{1}{\pi}\iint_{L=\operatorname{supp}\alpha\setminus W_0} f(\zeta)\frac{\partial\alpha}{\partial\bar{\zeta}}\frac{1}{z-\zeta}\mathrm{d}\xi\mathrm{d}\eta$$

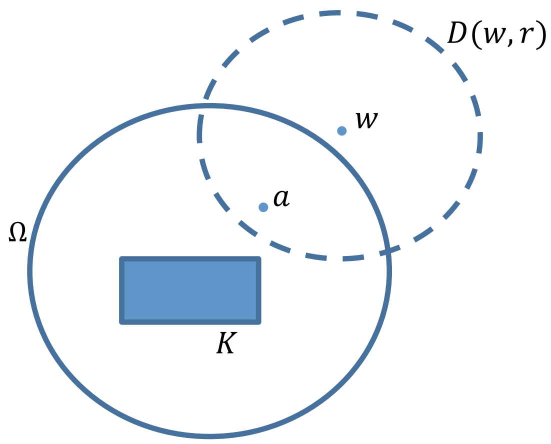
If we subdivide $\mathbf{C}$ by small squares and form the corresponding Riemann sums for the integral,

$$\frac{1}{\pi}\sum_{\nu} f(z_\nu)\frac{\partial\alpha}{\partial\bar{\zeta}}(z_\nu)\frac{1}{z-z_\nu}$$

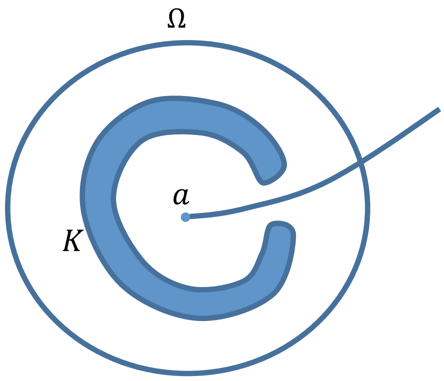
Then these Riemann sums will approximate the integrals, uniformly on K, since the integrand is compactly supported, hence uniformly continuous in $\mathbf{C}$. The z_ν's will be close to $L = \operatorname{supp}\alpha \setminus W_0$, hence in $\Omega \setminus K$. It follows that f can be approximated on K by a finite sum $\sum_\nu c_\nu \frac{1}{z-z_\nu}$ with $z_\nu \in \Omega \setminus K$.

Step 2. We now look at terms of the form $\frac{1}{z-a}$ with $a \in \Omega \setminus K$. We shall approximate these by functions which are holomorphic in Ω by "pushing the poles out of Ω".

Examples



$\frac{1}{z-a}$ is holomorphic outside $D(w,r)$ and is given there by a power series in $\frac{1}{z-w}$.



The pole a can be gradually pushed out of Ω.

Therefore, let $a \in \Omega \setminus K$ and let U be the connected component of $\mathbf{C} \setminus K$ containing a. Let

$$U_a = \left\{w \in U\,;\frac{1}{z-a}\text{ can be approximated on }K\text{ by polynomials in }\frac{1}{z-w}\right\}$$

We will show that $U_a = U$. We will do this by showing that U_a is both open and closed in $\mathbf{C} \setminus K$.

U_a is open: Suppose $w \in U_a$ and $D(w,r) \cap K = \emptyset$. If P_ϵ is a polynomial in $\frac{1}{z-w}$ which approximates f on K and $w' \in D\left(w, \frac{r}{2}\right)$, then $P_\epsilon\left(\frac{1}{z-w}\right)$ is holomorphic outside $\overline{D}\left(w', \frac{r}{2}\right)$ and can therefore be developed in a power series in $\frac{1}{z-w'}$ there. A finite sum of this power series will approximate P_ϵ on the compact $K \subset \mathbf{C} \setminus \overline{D}\left(w', \frac{r}{2}\right)$.

U_a is closed in $\mathbf{C} \setminus K$: Assume $w_n \in U_a$ and $w_n \to w \in \mathbf{C} \setminus K$. Then there is a disc $\overline{D}(w,r) \subset \mathbf{C} \setminus K$ and a $w_n \in \overline{D}(w,r)$. $\frac{1}{z-a}$ can be approximated on K by polynomials in $\frac{1}{z-w_n}$. These are holomorphic outside $\overline{D}(w,r)$ and the same argument as above gives that $w \in U_a$.

This proves the claim.

We now prove that $\frac{1}{z-a}$ can be approximated on K by a function which is holomorphic in Ω.

If U_a is bounded, then we claim that $U_a \cap (\mathbf{C} \setminus \Omega) \neq \emptyset$. Otherwise, $U_a \subset \Omega$ and $U_a$is a connected component of $\Omega \setminus K$. But $\partial U_a \subset K$, hence U_a would be relatively compact in Ω, which is impossible. Hence there is some $w \in U_a \setminus \Omega$ and by definition $\frac{1}{z-a}$ can be approximated by a polynomial in $\frac{1}{z-w}$, which is holomorphic in Ω.

If U_a is unbounded, then there is $w \in U_a$ with $|w| > \sup\{|z|, z \in K\}$. Let $r = |w|$. In this case a polynomial in $\frac{1}{z-w}$ is holomorphic in the disc $D(0,R)$, hence is given by a power series there, and can be approximated by a polynomial on K.

(3) $\Rightarrow$ (2) is analogous with (1) $\Rightarrow$ (2): If $U \subset\subset \Omega$ is a connected component of $\Omega \setminus K$, then $\partial U \subset K$ and for all $a \in U$ we have by the maximum modulus principle $|f(a)| \leq |f|_{\partial U} \leq |f|_K$ which contradicts (3).

(2) $\Rightarrow$ (3) If $a \in \Omega \setminus K$, then $L = K \cup \{a\}$ has the same property and by the implication (2) $\Rightarrow$ (1), $\mathcal{O}(\Omega)|_L$ is dense in $\mathcal{O}(L)$. If U and V are disjoint open sets, $K \subset U$, $a \in V$ and ϕ is defined by $\phi = 0$ in U, $\phi = 1$ in V, then $\phi \in \mathcal{O}(L)$, hence there exists $f \in \mathcal{O}(\Omega)$ such that $|f - \phi|_L < \frac{1}{2}$. But then $|f|_K < \frac{1}{2} < |f(a)|$.

This completes the proof of the theorem.

Remark

From the implication (2) $\Rightarrow$ (1) we get that if

- No connected component of $\Omega \setminus K$ is relatively compact in Ω
- $A \subset \mathbf{C}$ is a set which contains at least one point in every bounded component of $\mathbf{C} \setminus \Omega$
- $f \in \mathcal{O}(K)$

then f can be approximated uniformly on K by rational functions with poles in A.

The polynomials are dense in $\mathcal{O}(\mathbf{C})$. Hence if we let $\Omega = \mathbf{C}$ in Runge's theorem, we get:

Corollary

For a compact set $K \subset \mathbf{C}$ the following are equivalent:

(1) Every $f \in \mathcal{O}(K)$ can be approximated by polynomials.
(2) $\mathbf{C} \setminus K$ is connected (i.e., K has no holes).
(3) For any $z \notin K$ there is a polynomial P such that $|P(z)| > |P|_K$.

Such K are called polynomially convex.

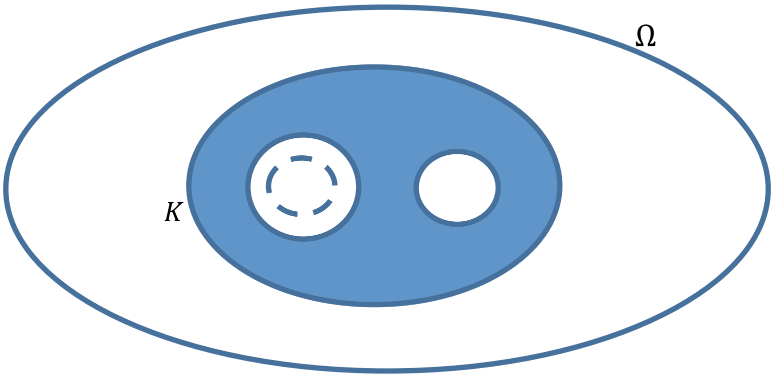
Definition

Let $K \subset \Omega$ be compact. The holomorphically convex hull of K in Ω is defined by

$$\widehat{K}_\Omega = \{z \in \Omega : |f(z)| \le |f|_K \text{ for all } f \in \mathcal{O}(\Omega)\}$$

Condition (3) in Runge's theorem states that $\widehat{K}_\Omega = K$, in which case we call K holomorphically convex in Ω. We have $\widehat{\widehat{K}}_\Omega = \widehat{K}_\Omega$. We shall see that $\widehat{K}_\Omega$ fills in the holes in K which do not contain holes in Ω.

Example



$\widehat{K}_\Omega$ fills in the hole to the right, not the left. (Ω does not contain the dashed little hole.)

Exercise

$\widehat{K}_\Omega$ does not get closer to $\partial\Omega$, i.e., $\mathrm{d}\big(\widehat{K}_\Omega, \partial\Omega\big) = \mathrm{d}(K, \partial\Omega)$.

$\widehat{K}_\Omega$ is compact.

Theorem

$\widehat{K}_\Omega$ is the union of K and all relatively compact components of $\Omega \setminus K$.

Proof

If U is such a component, then $\partial U \subset K$ and therefore $U \subset \widehat{K}_\Omega$ by the maximum modulus theorem. This shows that

$$K_1 \coloneqq K \cup \big(\cup_{U_\alpha \subset\subset \Omega} U_\alpha\big) \subset \widehat{K}_\Omega$$

Also, $\Omega \setminus K_1 = \cup_{U_\alpha \subset\subset \Omega} U_\alpha$ is open, hence K_1 is closed in Ω and therefore compact. Also, no components of $\Omega \setminus K_1$ are relatively compact. Runge's theorem gives that any $z \notin K_1$ can be separated from K_1 (and hence K) by a holomorphic function in Ω. This proves that $z \notin \widehat{K}_\Omega$, i.e., $\widehat{K}_\Omega \subset K_1$.

Lemma

If $\Omega \subset \mathbf{C}$ is open, then

$$K_n = \left\{z \in \Omega \,;\, \mathrm{d}(z, \mathbf{C} \setminus \Omega) \ge \frac{1}{n}, |z| \le n\right\}$$

is a holomorphically convex exhaustion of Ω.

Theorem (Classical Runge theorem)
If $\Omega \subset \mathbf{C}$ is open, $A \subset \mathbf{C}$ is a set which contains one point from each bounded component of $\boldsymbol{C} \setminus \Omega$, then every $f \in \mathcal{O}(\Omega)$ can be approximated uniformly on compacts by rational functions with poles in A.

Proof
Pick $f \in \mathcal{O}(\Omega)$ and a compact set $K \subset \Omega$. Replacing K by $\widehat{K}_\Omega$, we may assume that K is holomorphically convex in Ω. The result follows from the remark to Runge's theorem.

Mittag-Leffler's theorem

Definition
Let $\mathbf{C}_a^* = \mathbf{C} \setminus \{a\}$. The set $\mathbf{C}_0^*$ is denoted by $\mathbf{C}^*$.

If f is holomorphic in a punctured disc around a, we have

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

The negative powers $p_a = \sum_{n=-\infty}^{-1} c_n (z-a)^n$ is called the principal part of f at a. We have $p_a \in \mathcal{O}(\mathbf{C}_a^*)$.

Theorem 1 (Mittag-Leffler) Prescribing principal parts
If $E \subset \Omega$ is discrete and for every $a \in E$ there is given a principal part $p_a \in \mathcal{O}(\mathbf{C}_a^*)$, then there is $f \in \mathcal{O}(\Omega \setminus E)$ such that $f - p_a$ is holomorphic in a neighborhood of a for all $a \in E$.

Proof
Let $\{K_n\}$ be a holomorphically convex exhaustion of Ω and put $K_0 = \emptyset$. Let $E_n = E \cap \{K_n \setminus K_{n-1}\}$. Each E_n is finite. Put

$$g_n = \sum_{a \in E_n} p_a \in \mathcal{O}(\mathbf{C} \setminus E_n) \supset \mathcal{O}(K_{n-1})$$

Let $f_1 = g_1$. Then $f_1 - p_a$ is holomorphic in a for all $a \in E_1$ and is holomorphic outside K_1. We would like to add g_2, but the problem is convergence. However, since $g_2 \in \mathcal{O}(K_1)$ and K_1 is holomorphically convex, we can find $h_2 \in \mathcal{O}(\Omega)$ such that $|g_2 - h_2|_{K_1} < 2^{-2}$. If we let $f_2 = g_1 + (g_2 - h_2)$, then $f_2 - p_a$ is holomorphic at all $a \in E_1 \cup E_2$. We proceed inductively to find $h_n \in \mathcal{O}(\Omega)$ such that $|g_n - h_n|_{K_{n-1}} < 2^{-n}$. It follows that

$$f = \lim f_n = g_1 + \sum_{n=2}^{\infty} (g_n - h_n)$$

solves the problem.

If every $p_a \in \mathcal{M}(\mathbf{C})$, i.e., only has a pole at a, then $f \in \mathcal{M}(\Omega)$.

It is enough to assume that $p_a \in \mathcal{O}\big(D^*(a,r)\big)$ for some $r > 0$.

Equivalent formulation:

Theorem 1'

If $E \subset \Omega$ is discrete, $\Omega = \cup_{j \in J} U_j$ and $g_j \in \mathcal{O}(U_j \setminus E)$ are such that $g_j - g_k \in \mathcal{O}(U_j \cap U_k)$ for all j, k, then there is $g \in \mathcal{O}(\Omega \setminus E)$ such that $g - g_j \in \mathcal{O}(U_j)$ for all j.

(1') ⇒ (1): Put $E = \{z_j\}$, $U_j = (\Omega \setminus E) \cup \{z_j\}$ and $g_j = p_{z_j}$.

(1) ⇒ (1'): For $a \in E$, pick $j(a)$ such that $a \in U_{j(a)}$ and let p_a be the principal part of $g_{j(a)}$ at a. This is independent of the choice of $j(a)$. If $g \in \mathcal{O}(\Omega \setminus E)$ such that $g - p_a$ is holomorphic at a for all $a \in E$, then $g - g_j \in \mathcal{O}(U_j)$.

In theorem 1', suppose we can find the "holomorphic correction terms", $f_j = g - g_j \in \mathcal{O}(U_j)$ directly. How can we be sure that they patch together to a global g? We must have

$$f_i + g_i = f_j + g_j \text{ in } (U_i \cap U_j) \setminus E$$

$$f_i - f_j = g_j - g_i \text{ in } U_i \cap U_j$$

Let $f_{ij} = g_j - g_i \in \mathcal{O}(U_i \cap U_j)$. The existence of f_i follows from:

Theorem 2

If $\{U_j\}_{j=1}^{\infty}$ is an open covering of Ω and $f_{ij} \in \mathcal{O}(U_i \cap U_j)$ satisfy the cocycle condition

$$f_{ij} + f_{jk} + f_{ki} = 0 \text{ in } U_i \cap U_j \cap U_k$$

for all indices i, j, k. Then there exist $f_j \in \mathcal{O}(U_j)$ such that $f_{ij} = f_i - f_j$ in $U_i \cap U_j$ for all i, j.

Notice that the cocycle condition implies that $f_{ii} = 0$ and $f_{ji} = -f_{ij}$ for all i, j.

The argument above shows that Theorem 2 ⇒ Theorem 1'.

We shall now prove Theorem 2.

Step 1

We first prove that there are smooth solutions to the problem, i.e., there are $\phi_i \in C^{\infty}(U_i)$ such that $f_{ij} = \phi_i - \phi_j$ in $U_i \cap U_j$. For this, it is sufficient that $f_{ij} \in C^{\infty}(U_i \cap U_j)$.

Proof

Let α_i be a partition of unity relative to $\mathcal{U} = \{U_i\}$ and define in U_i

$$\phi_i = \sum_k \alpha_k f_{ik}$$

This is in $C^{\infty}(U_i)$, since supp $\alpha_k \subset U_k$ and the sum is locally finite. In $U_i \cap U_j$ we have

$$\phi_i - \phi_j = \sum_k \alpha_k (f_{ik} - f_{jk}) = \sum_k \alpha_k f_{ij} = f_{ij}$$

Step 2

We now correct the ϕ_i to make a holomorphic solution. Notice that since $\phi_i - \phi_j$ differ by a holomorphic function on $U_i \cap U_j$, the function

$$\psi(z) = \frac{\partial \phi_i}{\partial \bar{z}} \text{ for } z \in U_i$$

is globally defined in Ω. If we can find $u \in C^\infty(\Omega)$ such that

$$\frac{\partial u}{\partial \bar{z}} = \psi$$

then $f_i = \phi_i - u \in \mathcal{O}(U_i)$ and solves the problem. Hence Theorem 2 follows from the following result:

Theorem (Solution of $\bar{\partial}$-equation)

If $\psi \in C^\infty(\Omega)$ then there exist $u \in C^\infty(\Omega)$ such that $\frac{\partial u}{\partial \bar{z}} = \psi$.

Proof

Notice that we can solve the equation in a neighborhood of any compact set $K \subset \Omega$. Just chop off ψ with a smooth function. The solution is in $C^\infty(\mathbf{C})$.

We shall now build the solution inductively as in Mittag-Leffler's theorem. Let $\{K_n\}_{n=1}^\infty$ be a holomorphically convex exhaustion of Ω. First, solve $\frac{\partial u_j}{\partial \bar{z}} = \psi$ in an open neighborhood V_1 of K_1, and get $u_1 \in C^\infty(\mathbf{C})$. We now want to correct u_1 so the equation is satisfied in an open neighborhood V_2 of K_2. Let $\phi = \psi - \frac{\partial u_1}{\partial \bar{z}}$. Then $\phi \in C^\infty(\Omega)$ and $\phi = 0$ in V_1. Now solve $\frac{\partial v_2}{\partial \bar{z}} = \phi$ in V_2, $v_2 \in C^\infty(\mathbf{C}) \cap \mathcal{O}(V_1)$. Then $u_1 + v_2$ solves the problem in V_2, but we want the process to converge, so we pick $f_2 \in \mathcal{O}(\Omega)$ such that $|v_2 - f_2|_{K_1} < 2^{-2}$ and let $u_2 = v_2 - f_2$.

Now, proceed to find $u_3, \dots, u_n \in C^\infty(\mathbf{C})$ and open neighborhoods V_j of K_j, $j = 3, \dots, n$, such that

$$u_j \in \mathcal{O}(V_{j-1}), \qquad |u_j|_{K_{j-1}} < 2^{-j}$$

$$\frac{\partial u_1}{\partial \bar{z}} + \cdots + \frac{\partial u_n}{\partial \bar{z}} = \psi \text{ in } V_n$$

Then $u = \sum_{n=1}^\infty u_n$ is the required solution.

The winding number

Let γ be a closed piecewise C^1 curve in $\mathbf{C}$. Then for $z \in \mathbf{C} \setminus \gamma$,

$$\operatorname{Ind}(\gamma, z) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{\mathrm{d}\zeta}{\zeta - z}$$

is called the winding number of γ around z. Clearly, $\operatorname{Ind}(\gamma, z) \in \mathcal{O}(\mathbf{C} \setminus \gamma)$.

Lemma

$\operatorname{Ind}(\gamma, z) \in \mathbf{Z}$

Proof

Assume γ is parametrized over $[0,1]$, so $\gamma(0) = \gamma(1)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\exp \int_a^t \frac{\zeta'(u)}{\zeta(u) - z} \mathrm{d}u}{\zeta(t) - z} = \frac{\exp(\int \quad) \cdot \frac{\zeta'(t)}{\zeta(t) - z} \cdot (\zeta(t) - z) - \exp(\int \quad) \cdot \zeta'(t)}{(\zeta(t) - z)^2} = 0$$

Hence it is constant, which must be $\frac{1}{\zeta(0) - z}$. Then

$$\exp \int_0^1 \frac{\zeta'(s)}{\zeta(s) - z} \mathrm{d}s = \frac{\zeta(1) - z}{\zeta(0) - z} = 1$$

And hence $\int_0^1 \frac{\zeta'(s)}{\zeta(s) - z} \mathrm{d}s = 2\pi \mathrm{i} \cdot n$ for some $n \in \mathbf{Z}$.

$\operatorname{Ind}(\gamma, z)$ is constant in each connected component of $\mathbf{C} \setminus \gamma$ and it is 0 in the unbounded component.

Definition

Ω is simply connected if any closed curve is homotopic to a constant curve.

Exercise

The following are equivalent:

(1) Ω is simply connected
(2) Any two curves between two points a and b are homotopic.
(3) For any closed curve $\gamma \subset \Omega$ and $z \notin \Omega$, $\operatorname{Ind}(\gamma, z) = 0$.
(4) $\mathbf{C} \setminus \Omega$ has no compact components
(5) $\mathbf{P}^1 \setminus \Omega$ is connected

Lemma

Suppose $g \in \mathcal{O}^*(\Omega)$. Then the following are equivalent:

(1) g has a holomorphic logarithm in Ω ($\mathrm{e}^f = g$)
(2) $\frac{g'}{g}$ has a holomorphic primitive in Ω
(3) $\int_\gamma \frac{g'}{g} \mathrm{d}z = 0$ for all closed curves in Ω

Proof

(1) ⇒ (2) If $\mathrm{e}^f = g$ then $\frac{g'}{g} = f'$

(2) ⇒ (1) If $\frac{g'}{g} = f'$, let $h = \mathrm{e}^{-f} g$. Then $h' = \mathrm{e}^{-f}(g' - f'g) = 0$, hence $h \equiv c$, so $g = c\mathrm{e}^{f} = \mathrm{e}^{f+\alpha}$

The equivalence of (2) and (3) is well known from calculus.

If Ω is simply connected then g has a holomorphic logarithm because (3) holds.

Lemma

If z_0 and z_1 are in the same component of $\mathbf{C} \setminus K$, then $g(z) = \frac{z-z_0}{z-z_1}$ has a holomorphic logarithm in a neighborhood of K. If z_0 is in the unbounded component of $\mathbf{C} \setminus K$ then $g(z) = z - z_0$ has a holomorphic logarithm.

Proof

Pick a neighborhood Ω of K such that z_0, z_1 are in the same component of $\mathbf{C} \setminus \Omega$. Then

$$\frac{g'(z)}{g(z)} = \frac{1}{z - z_0} - \frac{1}{z - z_1}$$

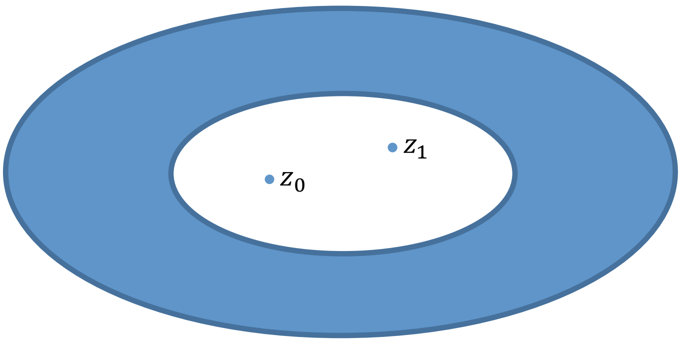
Hence if $\gamma \subset \Omega$ is a closed curve, then

$$\int_\gamma \frac{g'(z)}{g(z)} \mathrm{d}z = \int_\gamma \frac{\mathrm{d}z}{z - z_0} - \frac{\mathrm{d}z}{z - z_1} = \mathrm{Ind}(\gamma, z_0) - \mathrm{Ind}(\gamma, z_1) = 0$$

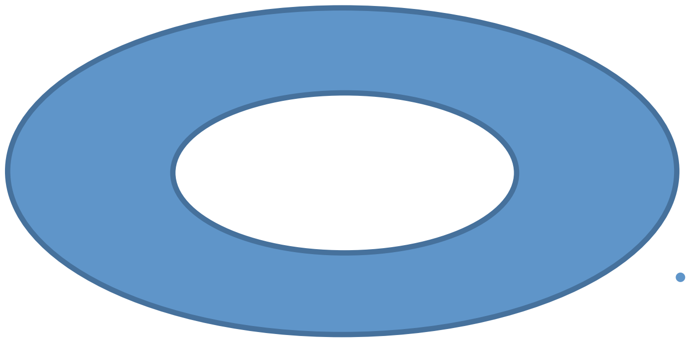
For z_0 in the unbounded component, $\frac{g'(z)}{g(z)} = \frac{1}{z-z_0}$, so

$$\int_\gamma \frac{g'(z)}{g(z)} \mathrm{d}z = \int_\gamma \frac{\mathrm{d}z}{z - z_0} = \mathrm{Ind}(\gamma, z_0) = 0$$

Pushing zeroes



Let $f(z) = \log \frac{z-z_0}{z-z_1} \in \mathcal{O}(K)$. Then $z - z_0 = \mathrm{e}^{f(z)}(z - z_1)$. Now, approximate f on K by $\tilde{f}(z) \in \mathcal{O}(\mathbf{C} \setminus \{z_1\})$, so $z - z_0 \sim \mathrm{e}^{\tilde{f}(z)}(z - z_1)$ on K.



Let $f(z) = \log(z - z_0) \in \mathcal{O}(K)$. Then $z - z_0 = \mathrm{e}^{f(z)}$. Approximate f on K by $\tilde{f} \in \mathcal{O}(\mathbf{C})$, so $z - z_0 \sim \mathrm{e}^{\tilde{f}(z)}$ on K. Thus we have approximated $z - z_0$ on K by a zero free entire function.

Theorem

If $K \subset \Omega$ is holomorphically convex, i.e., $\widehat{K}_{\mathcal{O}(\Omega)} = K$, then $\mathcal{O}^*(\Omega)\mid_K$ is dense in $\mathcal{O}^*(K)$.

Proof

Let $f \in \mathcal{O}^*(K)$ and let $\epsilon > 0$, $\epsilon < \min\{|f(z)|; z \in K\}$. Then there exists a rational function $R(z) = \frac{P(z)}{Q(z)} \in \mathcal{O}(\Omega)$ such that $|f - R|_K < \frac{1}{2}\epsilon$. P has no zeroes on K. Let $a_1, \dots, a_k$ be the zeroes of P in the bounded component of $\mathbf{C} \setminus K$, and let $a_{k+1}, \dots, a_m$ be the zeroes of P in the unbounded component of $\mathbf{C} \setminus K$, and pick b_j, $j = 1, \dots, k$, $b_j \notin \Omega$, in the same component as a_j. We may assume that

$$P(z) = \prod_{j=1}^{m} \left(z - a_j\right)^{m_j}$$

Then

$$g(z) = \sum_{j=1}^{k} m_j \log\left(\frac{z - a_j}{z - b_j}\right) + \sum_{j=k+1}^{m} m_j \log\left(z - a_j\right) \in \mathcal{O}(K)$$

and

$$\mathrm{e}^{g(z)} = \frac{P(z)}{\prod_{j=1}^{k}\left(z - b_j\right)^{m_j}} = \frac{P(z)}{P_0(z)}$$

We have $\min|Q(z)| = \delta > 0$. Let $M = \max_{z \in K}|P_0(z)|$, $N = \max_{z \in K}\left|\mathrm{e}^{g(z)}\right|$, and let $\mu > 0$ be given. If $h \in \mathcal{O}(\Omega)$, $|h - g|_K < \log(1 + \mu)$, then $\left|\mathrm{e}^{h-g} - 1\right|_K < \mu$. Hence for $z \in K$,

$$\left|R(z) - \frac{P_0(z)\mathrm{e}^{h(z)}}{Q(z)}\right| = \left|\frac{P_0(z)\mathrm{e}^{g(z)}}{Q(z)} - \frac{P_0(z)\mathrm{e}^{h(z)}}{Q(z)}\right| \le \frac{M}{\delta}\left|\mathrm{e}^{g(z)} - \mathrm{e}^{h(z)}\right| \le \frac{M}{\delta}\left|\mathrm{e}^{g(z)}\right|\left|1 - \mathrm{e}^{h(z)-g(z)}\right|$$
$$\le \frac{MN}{\delta} \cdot \mu < \frac{1}{2}\epsilon$$

when μ is sufficiently small. Therefore $R_0(z) = \frac{P_0(z)}{Q(z)}\mathrm{e}^{h(z)} \in \mathcal{O}^*(\Omega)$ is the required approximation.

Weierstrass' theorem

We shall prove a result on prescription of zeroes and poles. For this we need to study infinite products.

Let $\{a_n\} \subset \mathbf{C}$. We say that $\prod_{n=1}^{\infty} a_n$ is convergent if $p_N = \prod_{n=1}^{N} a_N$ is a convergent sequence, and we set

$$\prod_{n=1}^{\infty} a_n = \lim_{N\to\infty} p_N$$

If this limit is nonzero, it is clearly necessary that $\lim_{n\to\infty} a_n = 1$. We shall consider products

$$\prod_{n=1}^{\infty}(1+u_n) \text{ with } u_n \to 0$$

Sloppy calculation:

$$\log \prod^{N}(1+u_n) = \sum^{N} \log(1+u_n) \approx \sum^{N} u_n$$

Hence it follows that the convergence of $\prod(1+u_n)$ is related to the convergence of the series $\sum u_n$.

Correct calculation: Use the inequality $\log(1+x) \leq x$ to obtain

$$|p_N| \leq \prod^{N}(1+|u_n|)$$

$$\log|p_N| \leq \sum^{N} \log(1+|u_n|) \leq \sum^{N} |u_n|$$

$$|p_N| \leq \mathrm{e}^{\sum |u_n|}$$

Hence $\{p_N\}$ is bounded if $\sum^{\infty} |u_n| < \infty$.

$p_N - 1$ is a polynomial in $u_1, \dots, u_N$, without constant term. This gives

$$|p_N - 1| \leq \prod^{N}(1+|u_n|) - 1 \leq \mathrm{e}^{\sum |u_n|} - 1$$

Lemma 1

If $\{u_n(z)\}$ are bounded functions on a set E such that $\sum |u_n(z)|$ converges uniformly on E, then

$$f(z) = \prod^{\infty}\big(1+u_n(z)\big)$$

converges uniformly on E, and $f(z_0) = 0$ iff $u_n(z_0) = -1$ for some n.

Proof

It follows from $|p_N(z)| \leq \mathrm{e}^{\sum |u_n(z)|}$ that $\{p_N(z)\}$ is uniformly bounded on E, i.e., $|p_N(z)| \leq C$ for all $z \in E$. For $M > N$ we have

$$|p_M(z) - p_N(z)| = |p_N(z)| \left| \prod_{N+1}^{M} (1 + u_n(z)) - 1 \right| \leq C \left(\mathrm{e}^{\sum_{N+1}^{M} |u_n(z)|} - 1 \right) \to 0$$

as $N, M \to \infty$, which proves that $\{p_N(z)\}$ converges uniformly on E. The inequality also shows that

$$|p_M(z)| \geq |p_N(z)|(1 - \epsilon)$$

for N sufficiently large and $M > N$. Hence, the infinite product has a zero at z_0 iff some finite p_N does.

Theorem

If Ω is connected, $f_n \in \mathcal{O}(\Omega)$, no f_n is identically equal to zero and $\sum |1 - f_n(z)|$ converges uniformly on compacts in Ω, then $f(z) = \prod^{\infty} f_n(z)$ converges uniformly on compacts and $\operatorname{ord}_a(f) = \sum_{n=1}^{\infty} \operatorname{ord}_a(f_n)$.

Theorem Weierstrass

If $E \subset \Omega$ is discrete and for every $a \in E$ there is given an integer $k_a \in \mathbf{Z}$, then there is a holomorphic function $f \in \mathcal{O}^*(\Omega \setminus E)$ such that $(z - a)^{-k_a} f(z)$ is holomorphic and nonzero in a neighborhood of a for all $a \in E$.

Proof

Let $\{K_n\}$ be a holomorphically convex exhaustion of Ω and let $E_n = E \cap (K_n \setminus K_{n-1})$, $K_0 = \emptyset$. Let $g_n = \prod_{a \in E_n} (z - a)^{k_a}$. Then g_1 has the required property for $a \in E_1$. We would like to multiply by g_2, but the problem is convergence. Notice however that $g_2 \in \mathcal{O}^*(K_1)$, hence there is $h_2 \in \mathcal{O}^*(\Omega)$ such that $|g_2 h_2 - 1|_{K_1} < 2^{-2}$ and $g_1 \cdot (g_2 h_2)$ has the required property for $a \in E_1 \cup E_2$.

Inductively, we can find $h_n \in \mathcal{O}^*(\Omega)$ such that $|g_n h_n - 1|_{K_{n-1}} < 2^{-n}$. This implies that

$$f = g_1 \cdot \prod_{n=2}^{\infty} g_n h_n$$

has the required properties.

Exercise

The analogous version of Theorem 2 for Weierstrass' theorem is the following:

If $\{U_j\}_{j=1}^{\infty}$ is an open covering of Ω and $f_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ satisfy the cocycle condition $f_{ij} f_{jk} f_{ki} = 1$ in $U_i \cap U_j \cap U_k$ then there exist $f_i \in \mathcal{O}^*(U_i)$ such that $f_{ij} = \frac{f_i}{f_j}$ in $U_i \cap U_j$ for all i, j.

Show that this implies Weierstrass' theorem.

Theorem (Interpolation in a discrete set)

If $E \subset \Omega$ is discrete and for every $a \in E$ is given $\phi_a \in \mathcal{O}(D^*(a, r_a))$ and $k_a \geq 0$. Then there is $f \in \mathcal{O}(\Omega \setminus E)$ such that $f - \phi_a$ is holomorphic at a and $\operatorname{ord}_a(f - \phi_a) > k_a$ for all $a \in E$.

Proof

By Weierstrass' theorem there is $g \in \mathcal{O}(\Omega)$ such that $\mathrm{Z}(g) = E$ and $\mathrm{ord}_a g = k_a + 1$ for all $a \in E$. Then $\frac{\phi_a}{g} \in \mathcal{O}\big(D^*(a, r_a)\big)$ for all $a \in E$ and my Mittag-Leffler there is $h \in \mathcal{O}(D \setminus E)$ such that

$$h - \frac{\phi_a}{g} = \mathrm{O}(1) \text{ as } z \to a \text{ for all } a \in E$$

Then $h = \frac{\phi_a}{g} + \mathrm{O}(1)$ and $f = hg = \phi_a + \mathrm{O}\big(|z - a|^{k+1}\big)$ as $z \to a$.

Notice that h can have zeroes outside E.

If each ϕ_a is meromorphic then we can find such f without other zeroes:

Theorem

If $E \subset \Omega$ is discrete and for every $a \in E$ there is given $\phi_a \in \mathcal{O}\big(D^*(a, r_a)\big)$ such that $\mathrm{ord}_a \phi_a > -\infty$. Then there is $f \in \mathcal{M}(\Omega) \cap \mathcal{O}^*(\Omega \setminus E)$ such that $\mathrm{ord}_a(f - \phi_a) > k_a$ for all $a \in E$.

Proof

$$E_0 = \{a : \phi_a \not\equiv 0\}$$

$$m_a = \mathrm{ord}_a \phi_a \text{ for } a \in E_0$$

By Weierstrass we can find $g \in \mathcal{M}(\Omega)$ such that

$$\mathrm{ord}_a g = m_a \text{ for } a \in E_0$$

$$\mathrm{ord}_a g > k_b \text{ for } b \in E \setminus E_0$$

$$g \in \mathcal{O}^*(\Omega \setminus E)$$

If $h \in \mathcal{O}(\Omega)$ and $f = g\mathrm{e}^{h(z)}$ then everything hold except possibly $\mathrm{ord}_a(f - \phi_a) > k_a$ for $a \in E_0$. How can we achieve this? Notice that $\frac{\phi_a}{g}$ is holomorphic and nonzero near a, so there is $h_a \in \mathcal{O}\big(D^*(a, r_a)\big)$ such that $\mathrm{e}^{h_a} = \frac{\phi_a}{g}$. Then

$$\begin{aligned}\mathrm{ord}_a\big(g\mathrm{e}^h - \phi_a\big) = \mathrm{ord}_a g\left(\mathrm{e}^h - \frac{\phi_a}{g}\right) = \mathrm{ord}_a g\big(\mathrm{e}^h - \mathrm{e}^{h_a}\big) = \mathrm{ord}_a g\mathrm{e}^{h_a}\big(\mathrm{e}^{h - h_a} - 1\big) \\ = m_a + \mathrm{ord_a}(h - h_a)\end{aligned}$$

By the preceding theorem, there is $h \in \mathcal{O}(\Omega)$ such that $\mathrm{ord}_a(h - h_a) > |m_a| + k_a$. This completes the proof.

Automorphisms of the disc

Definition

An automorphism of an open set $\Omega \subset \mathbf{C}$ is a biholomorphic map of Ω onto itself, i.e., a holomorphic map $f : \Omega \to \Omega$ which has a holomorphic inverse. The set of automorphisms on Ω is denoted by $\mathrm{Aut}(\Omega)$. This is a group.

$D = D(0,1) = \{|z| < 1\}$ is the unit disc, and $T = \{\lambda : |\lambda| = 1\}$.

Theorem (Schwarz lemma)

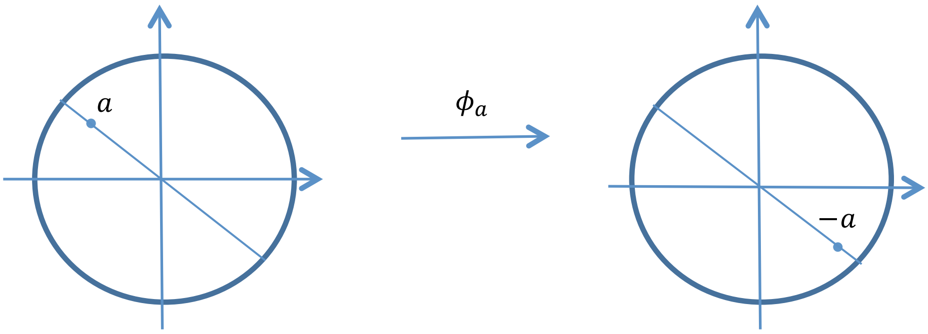
If $f \in \mathcal{O}(D)$, $|f(z)| \leq 1$ for all $z \in D$, and $f(0) = 0$, then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$.

Equality holds for some $z \in D$ iff $f(z) = \lambda z$ for some $|\lambda| = 1$.

Proof

Let $g(z) = \frac{f(z)}{z}$, $g(0) = f'(0)$. Then $g \in \mathcal{O}(D)$ and $\limsup_{z \to \zeta \in T} |g(z)| \leq 1$, hence the maximum modulus theorem implies that either $|g(z)| < 1$ for all $z \in D$ or $g(z) \equiv \lambda \in T$. In the first case $|f(z)| < |z|$ and $|f'(0)| < 1$, in the second case $f(z) = \lambda z$.

For $a \in D$, let $\phi_a(z) = \frac{z-a}{1-\bar{a}z}$. Then $\phi_a(a) = 0$ and $\phi_a(0) = -a$.



If $|z| = 1$ then

$$|\phi_a(z)| = \left|\frac{z-a}{(1-\bar{a}z)\bar{z}}\right| = \left|\frac{z-a}{\bar{z}-\bar{a}}\right| = 1$$

Hence $\phi_a : D \to D$. It is easy to see that $\phi_a^{-1} = \phi_{-a}$, and that ϕ_a is an automorphism.

Theorem

Every automorphism of D is of the form $\psi(z) = \lambda\phi_a(z)$ for some $\lambda \in T$.

Proof

If $\psi(0) = 0$ then $(\psi^{-1})'(0) \cdot \psi'(0) = 1$. Since $\psi, \psi^{-1} \in \mathrm{Aut}(D)$ and both are 0 at 0, their derivatives at zero must be ≤ 1 in absolute value. Strict inequality is impossible, so $|\psi'(0)| = 1$ and $\psi = \lambda z$ by the Schwarz lemma.

In general, if $\psi(a) = 0$, consider $\phi = \psi \circ \phi_{-a}$. Then $\phi \in \mathrm{Aut}(D)$, $\phi(0) = 0$, so $\phi(z) = \lambda z$, hence $\psi(z) = \lambda\phi_a(z)$.

Riemann mapping theorem

Theorem

If $\Omega \neq \mathbf{C}$ is simply connected (and connected), then Ω is biholomorphic to D.

We shall see that this follows from the fact that every $f \in \mathcal{O}(\Omega)$, f without zeros, has a holomorphic square root. This is true in a simply connected domain since f has a holomorphic logarithm. If $g = e^{\frac{1}{2}\log f}$, then $g^2 = f$.

$f : \Omega \to \mathbf{C}$ is biholomorphic onto its image iff f is injective.

The square root property is invariant under biholomorphism.

If $f : \Omega \to \Omega'$ is biholomorphic and has a holomorphic square root, then $\sqrt{f}$ is also biholomorphic. Also, if $w \in \mathrm{Im}(\sqrt{f})$, then $-w \notin \mathrm{Im}(\sqrt{f})$.

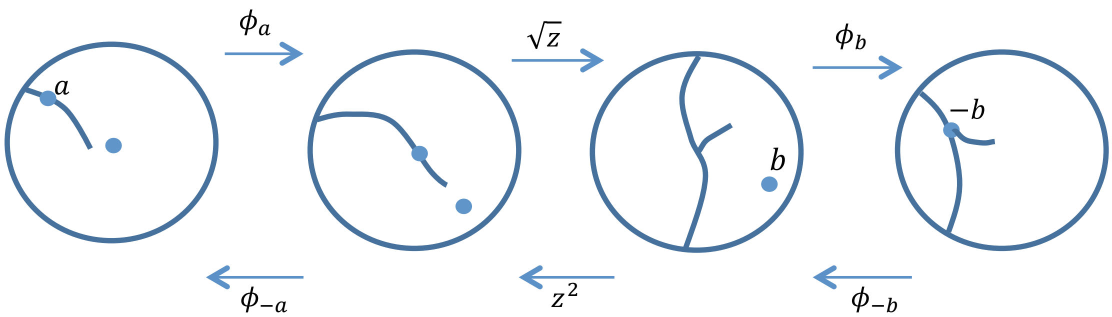
Proposition (Koebe)

If $0 \in \Omega \subset D$, $\Omega \neq D$ is connected and has the square root property, then there is a $H \in \mathcal{O}(\Omega)$ such that

(i) $H(0) = 0$, $H(\Omega) \subset D$,
(ii) H is injective,
(iii) $|H(z)| > |z|$ for all $z \in D$, $z \neq 0$.

Proof

Pick $a \in D \setminus \Omega$.



Let $H = \phi_b \circ \sqrt{z} \circ \phi_a$. Then (i) and (ii) holds. H^{-1} is defined in all of D and is 2-1 (except at $-b$), therefore $|H^{-1}(w)| < |w|$ for all $w \neq 0$, so $|H(z)| > |z|$ for all $z \neq 0$.

Proof of Theorem

We know that Ω has the square root property.

Step 1. To map Ω biholomorphically onto a bounded domain.

Pick $a \in \mathbf{C} \setminus \Omega$ and $g \in \mathcal{O}(\Omega)$ such that $g^2(z) = z - a$. If $D(w,r) \subset g(\Omega)$ (which is open), then $D(-w,r) \cap g(\Omega) = \emptyset$ and

$$\psi(z) = \frac{1}{g(z) + w}$$

is biholomorphic in Ω and $|\psi(r)| < \frac{1}{r}$.

For small ϵ, $h(z) = \epsilon\big(\psi(z) - \psi(z_0)\big)$ is biholomorphic onto $0 \in \Omega_0 \subset D$. Observe that Ω_0 has the square root property.

Step 2. We shall produce a biholomorphic map $\Omega_0 \to D$ which is "maximal". Let

$$\mathcal{F} = \{f : \Omega_0 \to D\ ; f \text{ is holomorphic, injective, and } f(0) = 0\}$$

Let $z_0 \in \Omega_0$, $z_0 \neq 0$ and put

$$\alpha = \sup_{\mathrm{f} \in \mathcal{F}} |f(z_0)| \in (0,1]$$

and pick $f_n \in \mathcal{F}$ such that $\lim_{n\to\infty} |f_n(z_0)| = \alpha$. By Montel's theorem there is a convergent subsequence, i.e., we may assume $f_n \to f$ u.o.c. Since $f(0) = 0$ and $|f(z_0)| = \alpha > 0$, f is not constant. By corollary of Hurwitz theorem, f is injective, so f is a biholomorphism $f : \Omega_0 \to \Omega_1 = f(\Omega_0) \subset D$. We cannot have $\Omega_1 \neq D$, because by Koebe's theorem there is a $H : \Omega_1 \to D$ injective such that $\big|H\big(f(z_0)\big)\big| > |f(z_0)| = \alpha$, contradicting the definition of α.

It is instructive to read Theorem 1 of section 7.3 in Narasimhan.

Schwarz-Pick and Ahlfors lemma

$$\phi_a(z) = \frac{z-a}{1-\bar{a}z}$$

$$\phi_a'(z) = \frac{1-|a|^2}{(1-\bar{a}z)^2}$$

$$\phi_a'(0) = 1-|a|^2$$

$$\phi_a'(a) = \frac{1}{1-|a|^2}$$

If $f : D \to D$ is holomorphic and $z \in D$, let

$$g = \phi_{f(z)} \circ f \circ \phi_{-z}$$

Then $g(0) = 0$ and

$$g'(0) = \phi_{f(z)}'\big(f(z)\big) \cdot f'(z) \cdot \phi_{-z}'(0) = \frac{1}{1-|f(z)|^2} \cdot f'(z) \cdot (1-|z|^2)$$

We get

Theorem

If $f : D \to D$ is holomorphic, then

$$\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2}$$

Equality at one point implies that f is an automorphism.

Proof

The last statement follows from $g(w) = \lambda w$, so

$$f(w) = \phi_{-f(z)}\big(\lambda \phi_z(w)\big) \Rightarrow f = \phi_{-f(z)} \circ (\lambda \phi_z)$$

This formulation is equivalent to the Schwarz lemma. Pick gave an invariant definition of this:

Consider the (Kähler) metric

$$\mathrm{d}s_h^2 = \frac{\mathrm{d}z\mathrm{d}\bar{z}}{(1-|z|^2)^2}$$

on D, i.e., for a tangent vector $X \in \mathrm{T}_p D$, $p \in D$,

$$\mathrm{d}s_h^2(X) = \frac{|X|^2}{(1-|z|^2)^2}$$

Then

$$f^*\big(\mathrm{d}s_h^2\big) = \frac{|f'(z)|^2}{(1-|f(z)|^2)^2}\mathrm{d}z\mathrm{d}\bar{z} \leq \frac{\mathrm{d}z\mathrm{d}\bar{z}}{(1-|z|^2)^2} = \mathrm{d}s_h^2$$

i.e. $f^*\left(\mathrm{d}s_h^2\right) \leq \mathrm{d}s_h^2$

with equality at one point iff f is an automorphism.

We can define length of curves $\gamma : [a, b] \to D$ using the metric $\mathrm{d}s_h$:

$$\mathrm{L}(\gamma) = \int_a^b \mathrm{d}s_h\big(\gamma(t), \gamma'(t)\big)\mathrm{d}t$$

It follows that holomorphic functions decrease the length of curves,

$$\mathrm{L}(f \circ \gamma) \leq \mathrm{L}(\gamma)$$

and automorphisms preserve length.

This defines a distance on D by

$$\rho_h(z_1, z_2) = \inf \mathrm{L}(\gamma)\,, \qquad \gamma \text{ curve from } z_1 \text{ to } z_2$$

Holomorphic functions are distance decreasing, and automorphisms preserve distances. It follows that

$$\rho_h(z_1, z_2) = \rho_h\big(0, |\phi_{z_1}(z_2)|\big)$$

$$\rho(0, a) = \int_0^a \frac{\mathrm{d}t}{1 - t^2} = \frac{1}{2}\log\frac{1 + a}{1 - a}$$

so

$$\rho_h(z_1, z_2) = \frac{1}{2}\log\frac{1 + |\phi_{z_1}(z_2)|}{1 - |\phi_{z_1}(z_2)|}$$

Theorem

If $f : D \to D$ is holomorphic, then

(1) $f^*(\mathrm{d}s_h) \leq \mathrm{d}s_h$
(2) $\rho_h\big(f(z), f(w)\big) \leq \rho_h(z, w)$

Equality in one point in (1) or on one pair $z \neq w$ in (2) implies that f is an automorphism. We call $\mathrm{d}s_h$ the Poincaré metric and ρ_h the Poincaré distance.

The curvature of a metric $h\mathrm{d}z\mathrm{d}\bar{z}$ is defined by

$$K_h = -\frac{2}{h}\frac{\partial^2}{\partial z \partial \bar{z}}\log h = -\frac{1}{2h}\Delta(\log h)$$

For $h = \frac{1}{(1-|z|^2)^2}$ we get

$$
\begin{aligned}
K_h &= -2(1-|z|^2)^2 \frac{\partial^2}{\partial z \partial \bar{z}} \log(1-|z|^2)^{-2} = 4(1-|z|^2)^2 \frac{\partial^2}{\partial z \partial \bar{z}} \log(1-z\bar{z}) \\
&= 4(1-|z|^2)^2 \frac{\partial}{\partial z} \frac{-z}{1-z\bar{z}} = 4(1-|z|^2)^2 \cdot \frac{-1(1-z\bar{z}) - (-z)\cdot(-\bar{z})}{(1-z\bar{z})^2} \\
&= 4(1-|z|^2)^2 \cdot -\frac{1}{(1-z\bar{z})^2} = -4
\end{aligned}
$$

If $\mathrm{d}s_h = h\mathrm{d}z\mathrm{d}\bar{z}$ is metric on Ω and $f : U \to \Omega$ satisfies $f'(z) \neq 0$ everywhere, then

$$f^*(\mathrm{d}s_h^2) = |f'(z)|^2 h(f(z)) \mathrm{d}z\mathrm{d}\bar{z}$$

and

$$K_{f^*(\mathrm{d}s_h)}(z) = K_{\mathrm{d}s_h}(f(z))$$

Thus curvature is a conformal invariant.

The metric

$$\mathrm{d}s_a^2 = \frac{4a^2}{A} \frac{\mathrm{d}z\mathrm{d}\bar{z}}{(a^2-|z|^2)^2} \text{ on } D_a = \{|z| < a\}$$

has curvature $-A$. The previous theorem generalizes to

Theorem (Ahlfors lemma)

If M is a Riemann surface with metric $\mathrm{d}s_M^2$ with curvature $\leq -B$, where $B > 0$, and $f : D_a \to M$ is holomorphic, then

$$f^*(\mathrm{d}s_M^2) \leq \frac{A}{B} \mathrm{d}s_a^2$$

Proof

Define $u \geq 0$ on D_a by $f^*(\mathrm{d}s_M^2) = u\mathrm{d}s_a^2 = u(z)\frac{4a^2\mathrm{d}z\mathrm{d}\bar{z}}{A(a^2-|z|^2)^2}$. For $r \leq a$, u_r is defined by $f^*(\mathrm{d}s_M^2) = u_r\mathrm{d}s_r^2$ on D_r. So $u = u_a$ and

$$u_r(z) = u(z)\frac{a^2(r^2-|z|^2)}{r^2(a^2-|z|^2)}$$

So $u_r \to u$ when $r \to a$. It is therefore sufficient to prove that $u_r(z) \leq \frac{A}{B}$ for $z \in D_r$.

By the formula above, $u_r(z) = 0$ when $|z| = r$. If $u_r(z) \equiv 0$ we are done. Otherwise, u_r has a maximum at some $z_0 \in D_r$. Then f defines local coordinates around z_0, i.e., there is a neighborhood U of z_0 with $f'(z) \neq 0$ for $z \in U$ and we can compute the curvature of $\mathrm{d}s_M^2$ by computing it in U.

We have

$$f^*(\mathrm{d}s_M^2) = u_r\mathrm{d}s_r^2 = u_r(z)\frac{4r^2\mathrm{d}z\mathrm{d}\bar{z}}{A(r^2-|z|^2)^2} =: h(z)\mathrm{d}z\mathrm{d}\bar{z}$$

so

$$K_h = -\frac{2}{h}\frac{\partial^2}{\partial z\partial\bar{z}}\log h = -\frac{2}{h}\frac{\partial^2}{\partial z\partial\bar{z}}\left(\log u_r + \log\frac{4r^2}{A(r^2-|z|^2)^2}\right) = -\frac{2}{h}\left(\frac{\partial^2}{\partial z\partial\bar{z}}\log u_r + \frac{2r^2}{(r^2-|z|^2)^2}\right)$$
$$= -\frac{2}{h}\frac{\partial^2}{\partial z\partial\bar{z}}\log u_r - \frac{A}{u_r} \le -B$$

Hence $\frac{2}{h}\frac{\partial^2}{\partial z\partial\bar{z}}\log u_r \ge B - \frac{A}{u_r}$, but $\frac{\partial^2}{\partial z\partial\bar{z}}\log u_r(z_0) = \frac{1}{4}\Delta\log u_r(z_0) \le 0$ since z_0 is a maximum. This gives $u_r(z_0) \le \frac{A}{B}$.

Which M can have a metric with negative curvature?

1. $\mathbf{C}$ does not have such at metric.

   Proof

   If $\mathrm{d}s_{\mathbf{C}}^2$ is such a metric, let $f : D \to \mathbf{C}$ be defined by $f(z) = az$. Then

$$\left(f^*\mathrm{d}s_{\mathbf{C}}^2\right)(0) = |a|^2\mathrm{d}s_{\mathbf{C}}^2(0)$$

   Hence no such inequality can hold. The metric $(1+|z|^2)\mathrm{d}z\mathrm{d}\bar{z}$ has curvature $H = -\frac{2}{1+|z|^2}$ and is complete.

2. $\mathbf{C}^* = \mathbf{C}\setminus\{0\}$ does not have such a metric, since $f(z) = \mathrm{e}^z$ is a covering $\mathbf{C}\to\mathbf{C}^*$, hence if $\mathbf{C}^*$ had a metric with negative curvature, so would $\mathbf{C}$. The metric $\frac{\mathrm{d}z\mathrm{d}\bar{z}}{\log(1+|z|^2)}$ has curvature $K = -\frac{2}{(1+|z|^2)^2}\left(\frac{|z|^2}{\log(1+|z|^2)} - 1\right) < 0$ and is complete.
3. The upper half plane $\mathbf{C}^+$ has such a metric since it is biholomorphic to D. A biholomorphic map is $f(z) = \frac{z-\mathrm{i}}{z+\mathrm{i}}$ with $f'(z) = \frac{2\mathrm{i}}{(z+\mathrm{i})^2}$ and

$$f^*\left(\frac{\mathrm{d}z\mathrm{d}\bar{z}}{(1-|z|^2)^2}\right) = \frac{|f'(z)|^2}{(1-|f(z)^2)^2}\mathrm{d}z\mathrm{d}\bar{z} = \frac{4}{|z+\mathrm{i}|^2\left(1-\left|\frac{z-\mathrm{i}}{z+\mathrm{i}}\right|^2\right)^2}\mathrm{d}z\mathrm{d}\bar{z}$$
$$= \frac{4}{(|z+\mathrm{i}|^2-|z-\mathrm{i}|^2)^2}\mathrm{d}z\mathrm{d}\bar{z} = \frac{4\mathrm{d}z\mathrm{d}\bar{z}}{\left((x^2+(y+1)^2)-(x^2+(y-1)^2)\right)^2}$$
$$= \frac{4\mathrm{d}z\mathrm{d}\bar{z}}{(4y)^2} = \frac{1}{4y^2}\mathrm{d}z\mathrm{d}\bar{z}$$
4. The punctured disc D^* has such a metric. We have a covering map $p\colon\mathbf{C}^+\to D^*$ given by $p(z) = \mathrm{e}^{\mathrm{i}z}$. This has local inverses $p^{-1}(w) = \frac{1}{\mathrm{i}}\log w$ and

$$(p^{-1})^*\left(\frac{\mathrm{d}z\mathrm{d}\bar{z}}{4y^2}\right) = \frac{|(p^{-1})'(w)|^2\mathrm{d}w\mathrm{d}\bar{w}}{4\left(\operatorname{Im}p^{-1}(w)\right)^2} = \frac{\mathrm{d}w\mathrm{d}\bar{w}}{4|w|^2(\log|w|)^2} = \frac{\mathrm{d}w\mathrm{d}\bar{w}}{|w|^2(\log|w|^2)^2} =: \mathrm{d}s_{D^*}^2$$

   This metric is also complete. If $0 < r < R < 1$, then

$$\rho_{D^*}(r,R) = \int_r^R \frac{\mathrm{d}t}{t(-\log t^2)} = -\frac{1}{2}\int_r^R\frac{\mathrm{d}t}{t\log t} = -\frac{1}{2}\log(-\log t)\,|_r^R$$
$$= \frac{1}{2}\left(\log\left(\log\frac{1}{r}\right) - \log\left(\log\frac{1}{R}\right)\right) \to \infty$$

   when $r\to 0$ or $R\to 1$. The circle $\gamma(t) = r\mathrm{e}^{\mathrm{i}t}$ has length

$$\ell(\gamma) = \int_0^{2\pi} \frac{r\mathrm{d}t}{r(-\log r^2)} = \frac{\frac{\pi}{2}}{\log\left(\frac{1}{r^2}\right)} \to 0$$

when $r \to 0$.

5. The doubly punctured plane $\mathbf{C} \setminus \{z_0, z_1\}$ has a metric $h(z)\mathrm{d}z\mathrm{d}\bar{z}$ with curvature bounded above by a negative constant.

Proof

We may assume $z_0 = 0, z_0 = 1$. We shall prove that

$$h(z) = \frac{(1+|z|^\alpha)}{|z|^\gamma} \cdot \frac{(1+|z-1|^\alpha)}{|z-1|^\gamma}$$

has the required property for suitable α and γ.

The expression for the Laplacian of a radial function $f(r)$ is

$$\Delta f(r) = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r}$$

(check this!). This gives

$$\Delta\left(\log\frac{(1+r^\alpha)}{r^\gamma}\right) = \Delta(\log(1+r^\alpha) - \gamma\log r) = \Delta(\log(1+r))$$

$$\frac{\partial}{\partial r}\log(1+r^\alpha) = \frac{\alpha r^{\alpha-1}}{1+r^\alpha}$$

$$\frac{\partial^2}{\partial r^2}\log(1+r^\alpha) = \alpha\frac{(\alpha-1)r^{\alpha-2}(1+r^\alpha) - r^{\alpha-1}\cdot\alpha r^{\alpha-1}}{(1+r^\alpha)^2} = \frac{\alpha r^{\alpha-2}}{(1+r^\alpha)^2}(\alpha-1-r^\alpha)$$

Hence

$$\begin{aligned}\Delta\log(1+r^\alpha) &= \frac{\alpha r^{\alpha-2}}{(1+r^\alpha)^2}(\alpha-1-r^\alpha) + \frac{1}{r}\cdot\frac{\alpha r^{\alpha-1}}{1+r^\alpha} = \frac{\alpha r^{\alpha-2}}{(1+r^\alpha)^2}\big((\alpha-1-r^\alpha)+(1+r^\alpha)\big)\\ &= \frac{\alpha^2 r^{\alpha-2}}{(1+r^\alpha)^2}\end{aligned}$$

This gives

$$K_h = -\frac{1}{2h}\Delta(\log h) = -\frac{\alpha^2}{2}\frac{|z|^\gamma|z-1|^\gamma}{(1+|z|^\alpha)(1+|z-1|^\alpha)}\left(\frac{|z|^{\alpha-2}}{(1+|z|^\alpha)^2} + \frac{|z-1|^{\alpha-2}}{(1+|z-1|^\alpha)^2}\right)$$

Hence $K_h(z) < 0$ for all $z \neq 0, 1$.

Assuming $\gamma > 0$ and $0 < \alpha < 2$, we have for $z \to 0$:

$$K_h(z) \sim -\frac{\alpha^2}{2}\frac{|z|^{\gamma+\alpha-2}}{2} \to -\infty \qquad \text{if } \gamma+\alpha-2 < 0 \tag{1}$$

This also gives $K_h(z) \to -\infty$ when $z \to 1$. If $|z| \to \infty$, we have

$$K_h(z) \sim -\frac{\alpha^2}{2}\frac{|z|^{2\gamma+\alpha-2}}{|z|^{4\alpha}} = -\frac{\alpha^2}{2}|z|^{2\gamma-3\alpha-2} \to -\infty \qquad \text{if } 2\gamma - 3\alpha - 2 > 0 \qquad (2)$$

We see that $\gamma = 1.6$ and $\alpha = 0.2$ will satisfy both these inequalities. This implies that K_h is bounded above by a negative constant,

$$K_h(z) \leq -k$$

for all $z \neq 0, 1$.

This metric is actually sufficient to prove Picard's big theorem to follow. The metric is not complete, however. The points 0, 1, ∞ are all at finite distance and this cannot be fixed by using different α and γ. We shall add a function f to h to make it complete. This requires a result on the curvature K_{h+f}. The following lemma is used to do this:

Lemma

Let ϕ and ψ be two strictly positive C^2 functions in some open set in $\mathbf{C}$. Then

$$\phi\Delta(\log\phi) + \psi\Delta(\log\psi) \leq (\phi+\psi)\Delta(\log\phi+\psi)$$

Proof

A small computation gives

$$\phi\Delta(\log\phi) = \Delta\phi - \frac{4}{\phi}\left|\frac{\partial\phi}{\partial z}\right|^2$$

Another computation then gives

$$(\phi+\psi)\Delta(\log\phi+\psi) - \phi\Delta\log\phi - \psi\Delta\log\psi = \frac{4}{\phi\psi(\phi+\psi)}\left|\phi\frac{\partial\psi}{\partial z} - \psi\frac{\partial\phi}{\partial z}\right|^2 \geq 0$$

which proves the inequality.

In terms of curvatures, the inequality is given by

$$(\phi+\psi)^2 K_{\phi+\psi} \leq \phi^2 K_\phi + \psi^2 K_\psi$$

Hence, if we know that $K_\phi \leq -k_1$ and $K_\psi \leq -k_2$, we get

$$K_{\phi+\psi} \leq -\left(\frac{\phi^2}{(\phi+\psi)^2}k_1 + \frac{\psi^2}{(\phi+\psi)^2}k_2\right) = -\left(\frac{k_1k_2}{k_1+k_2} + \frac{(\phi k_1 - \psi k_2)^2}{(\phi+\psi)^2(k_1+k_2)}\right) \leq -\frac{k_1k_2}{k_1+k_2}$$

We shall now construct f. The metric will be given by $h + cf$ for some small constant c. Near 0, 1, and ∞, f will be the function of example 4. This means that $K_f = -4$ near these points, and completeness of $(h+cf)\mathrm{d}z\mathrm{d}\bar{z}$ follows immediately. To construct f, pick first a C^∞ cutoff function $\mu(z)$ such that $\mu \equiv 1$ in $\left\{|z| \leq \frac{1}{4}\right\}$ and $\mu \equiv 0$ in $\left\{|z| \geq \frac{1}{3}\right\}$. Then let

$$s(z) = \frac{\mu(z)}{|z|^2(\log|z|^2)^2}$$

f is then given by

$$f(z) = s(z) + s(z-1) + 1/|z|^4 s\left(\frac{1}{z}, \frac{1}{z}\right)$$

Notice that the metric $\frac{1}{|z|^4} s\left(\frac{1}{z}, \frac{1}{z}\right) \mathrm{d}z\mathrm{d}\bar{z}$ in $\{|z| > 4\}$ is the pullback of $s(z)\mathrm{d}z\mathrm{d}\bar{z}$ under the map $\frac{1}{z}$.

In $\Omega = \left\{|z| < \frac{1}{4} \text{ or } |z-1| < \frac{1}{4} \text{ or } |z| > 4\right\}$ we have $K_f = -4$ and $K_{cf} = -\frac{4}{c}$. The inequality above then gives

$$K_{h+cf} \leq -\frac{\frac{4}{c} \cdot k}{\frac{4}{c} + k} = -\frac{4k}{4+ck} < -\frac{4k}{4+k} < 0$$

In the compact set $\mathbf{C} \setminus \Omega$ we apply the first inequality with $\phi = (1-c)h$ and $\psi = c(h+f)$ to get

$$\begin{aligned}
K_{h+cf} &\leq \frac{1}{(h+cf)^2}\left((1-c)^2 h^2 K_{(1-c)h} + c^2(h+f)K_{c(h+f)}\right) \\
&= \frac{1}{(h+cf)^2}\left((1-c)h^2 K_h + c(h+f)K_{h+f}\right) \\
&\leq \frac{1}{(h+cf)^2}\left(-(1-c)h^2 k + c(h+f)K_{h+f}\right) \to -k
\end{aligned}$$

uniformly as $c \to 0$ by compactness. Hence for small c, K_{h+cf} is bounded above everywhere by a negative constant. This completes the construction.

Comment

The modular function $\lambda(z)$ is a covering map $\lambda : \mathbf{C}^+ \to \mathbf{C} \setminus \{0,1\}$ whose covering transformations all preserve the metric $\frac{1}{4y^2}\mathrm{d}z\mathrm{d}\bar{z}$. Hence, as in example 4, we may push this metric down to $\mathbf{C} \setminus \{0,1\}$ to obtain a complete metric with constant negative curvature -4. The construction of the modular function is quite complicated.

We also get Ahlfors lemma for maps from D^*. (We have put $A = 1$.)

Theorem (Ahlfors lemma for D^*)

If M is a Riemann surface with metric $\mathrm{d}s_M^2$ with curvature $\leq -B$, with $B > 0$, and $f : D^* \to M$ is holomorphic, then

$$f^*(\mathrm{d}s_M^2) \leq \frac{4}{B}\mathrm{d}s_{D^*}^2$$

Proof

We have $\mathrm{d}s_{D^*}^2 = (p^{-1})^*\mathrm{d}s_D^2$. The map $f \circ p : D \to M$ is holomorphic, so by the Ahlfors lemma for D we have

$$(f \circ p)^*(\mathrm{d}s_M^2) = p^*\left(f^*(\mathrm{d}s_M^2)\right) \leq \frac{4}{B}\mathrm{d}s_D^2$$

which gives

$$f^*(\mathrm{d}s_M^2) = (p^{-1})^*\left(p^*\left(f^*(\mathrm{d}s_M^2)\right)\right) \le (p^{-1})^*\left(\frac{4}{B}\mathrm{d}s_D^2\right) = \frac{4}{B}\mathrm{d}s_{D^*}^2$$

Theorem

Suppose $\Omega \subset \mathbf{C}$ has a metric with curvature $\le -B$. Then

(a) There is no nonconstant holomorphic map $f : \mathbf{C} \to \Omega$.
(b) No holomorphic function $f : D^* \to \Omega$ can have an essential singularity at 0.

Proof

(a) Restricting to a disc of radius a (with $A = 1$), the Schwarz lemma gives
$$f^*\left(\mathrm{d}s_\Omega^2\right) \le \frac{1}{B}\mathrm{d}s_a^2 = \frac{1}{B}\frac{4a^2}{(a^2 - |z|^2)^2}\mathrm{d}z\mathrm{d}\bar{z} \to 0$$
when $a \to 0$. Since $f^*\left(\mathrm{d}s_\Omega^2\right) = |f'(z)|^2 h\left(f(z)\right)\mathrm{d}z\mathrm{d}\bar{z}$, this gives $f'(z) = 0$, so f is constant.

To prove (b), we use the following

Lemma

If $f \in \mathcal{O}(D^*)$ has an essential singularity at 0, then $f(D^*)$ is dense in $\mathbf{C}$.

Proof

If not, there is $a \in \mathbf{C}$ and $\delta > 0$ such that $|f(z) - a| \ge \delta$ for all $z \in D^*$. But then $g(z) = \frac{1}{f(z)-a}$ satisfies $|g(z)| \le \frac{1}{\delta}$, hence has a removable singularity at 0. But then $f(z) = \frac{1}{g(z)} + a$ either has a pole or a removable singularity at 0.

To prove (b), notice that if $f\ D^* \to \Omega$ has an essential singularity at 0, then $f(D_r^*)$ is dense in $\mathbf{C}$ for all $r > 0$, hence there is a sequence $z_n \to 0$ such that $f(z_n) \to p \in \Omega$. If ρ is the metric defined by $\mathrm{d}s_\Omega$, i.e.,

$$\rho(z, w) = \inf\left\{\int_0^1 \mathrm{d}s_\Omega\left(\gamma'(t)\right)\mathrm{d}t : \gamma: [0,1] \to \Omega, \gamma(0) = z, \gamma(1) = w\right\}$$

and $\bar{B}(p, r) \subset \Omega$, then $\inf\{\rho(p, z): |p - z| = r\} = \delta > 0$. If $\rho\left(p, f(z_n)\right) < \frac{1}{2}\delta$ and γ is a curve of length $\le \frac{1}{2}\delta$ starting at $f(z_n)$, then $\gamma \subset B(p, r)$, hence $|\gamma(t)| \le |p| + r = C$ for all t.

We may assume that $r_n = |z_n|$ decrease strictly to zero. Since $f(z_n) \to p$ there is N such that $\rho\left(p, f(z_n)\right) < \frac{1}{2}\delta$ for $n \ge N$.

Let γ_n be the circle $|z| = r_n$. Then

$$\mathrm{L}(f \circ \gamma_n) \le \frac{1}{\sqrt{B}}\mathrm{L}(\gamma_n) \le \frac{2\pi}{\sqrt{B}\log\frac{1}{r_n^2}} \to 0$$

when $n \to \infty$. Hence for large n, $\mathrm{L}(f \circ \gamma_n) \le \frac{1}{2}\delta$. This implies that $|f(z)| \le C$ for all z with $|z| = r_n$.

This means that $|f(z)| \le C$ for all z in the annuli $A_n = \{r_{n+1} \le |z| \le r_n\}$ and therefore in a punctured disc D_r. Hence f has a removable singularity at 0.

Theorem

(a) Picard's small theorem: A nonconstant entire function cannot omit more than one value.

(b) Picard's big theorem: If a holomorphic function has an essential singularity at a, then f takes all complex values except possibly one in any punctured disc around a.

Proof

(a) If f omits two values z_0 and z_1 then $f : \mathbf{C} \to \Omega = \mathbf{C} \setminus \{z_0, z_1\}$. Since Ω has a metric with curvature $\leq -B$, this follows from 1.4 (a).

(b) Follows in the same way from 1.4 (b).

We will now use the complete metric on $\mathbf{C} \setminus \{z_0, z_1\}$ constructed in example 5 above.

Theorem (Schottky's Theorem)

Given $R_0 > 0$ and $r < 1$, then there is a constant $M = M(R_0, r)$ such that if $f : D \to \mathbf{C} \setminus \{z_0, z_1\}$ is holomorphic and $|f(0)| \leq R_0$, then $|f(z)| \leq M$ for all z with $|z| \leq r$.

Proof

Let γ be the curve $\gamma(t) = tz$. By Ahlfors lemma, $\mathrm{L}(f \circ \gamma) \leq \frac{1}{\sqrt{B}} \mathrm{L}(\gamma) = \frac{1}{2} \log \frac{1+|z|}{1-|z|} \leq \frac{1}{2} \log \frac{1+r}{1-r}$. It follows that $f(z)$ must be bounded since $\mathrm{d}_\Omega(f(0), w) \to \infty$ as $|w| \to \infty$.

It follows that $f(z)$ must also stay away from z_0 and z_1, i.e., $|f(z) - z_0| \geq M_0$ and $|f(z) - z_1| \geq M_1$.

The same proof can be used to prove bounds on maps $f : D^* \to \mathbf{C} \setminus \{z_0, z_1\}$ on either annular regions or circles. Here is the circle version:

Theorem (Schottky's Theorem in D^*)

Given $R_0 > 0$ and $r < 1$, there is a constant M such that if $F : D^* \to \mathbf{C} \setminus \{z_0, z_1\}$ is holomorphic and $f(z) \leq R_0$ for some z with $|z| \leq r$, then $|f(\zeta)| \leq M$ for all ζ with $|\zeta| = |z|$.

Proof

We use the curve $\gamma(t) = z\mathrm{e}^{\mathrm{i}t}$, $0 \leq t \leq 2\pi$, whose length is

$$\frac{\pi}{2 \log \left(\frac{1}{|z|^2} \right)} \leq \frac{\pi}{2 \log \left(\frac{1}{r^2} \right)}$$

and Ahlfors lemma for D^*.