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Riemann surfaces

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The idea of a Riemann surface surfaced already in Riemann's inaugural dissertation from 1851. Functions defined by equations tend to be multivalued, as the old-timers expressed it. This occurs even for the simples case $w = z^2$ where the well known ambiguity in sign appears. For other equations equation, for instance $e^w = z$, the situation can be more severe. As we know, there are infinitely many branches of the logarithm. The Riemann surfaces were and are means to resolve this problem. They furnish places where multivalued functions become single valued! In their infancy the definitions of a Riemann surface, and there were a variety, reflected this point of view. The modern definition was strongly promoted by Felix Klein, and it is now ubiquitous in the literature; not only for defining Riemann surfaces, but is almost a universal device for defining geometric structures.

The idea is to use local coordinate charts and impose conditions on how they patch together. Doing calculations on such a space is a little like commanding a submarine. There is no help in looking out of the window on the real world, you are forced to navigate by the maps!

Of course, this idea goes far back in history at least to the Greeks. They understood that it is impossible to have one flat map covering the entire globe. One needs an atlas, that is a collection of maps.

To revert to a more serious tale, the Riemann sphere is an illustrative example. We habitually use two sets of coordinates to describe functions on it. Near the origin—in the southern part in the stereographic picture—we use the familiar coordinate z, but close to north pole—in the vicinity of the point at infinity— we use a coordinate w related to z by the equation $w = z^{-1}$.

The definition of a Riemann surface

With the example of the globe in mind, a Riemann surface has an underlying topological space X. By a *chart* in X, or we understand an open set U and a homeomorphism z_U from U onto an open subset $z_U(U)$ of $\mathbb{C}$. So the chart is the pair (U, z_U). The open set U will frequently be called *a coordinate neighbourhood*, or a *coordinate patch*. If the open set $z_U(U)$ happens to be a disk, we shall sometimes refer to the chart as a *coordinate disk*.

We call z_U a *coordinate* of the chart, so z_U is a map $z_U \colon U \to z_U(U) \subseteq \mathbb{C}$. In analogy with the commonplace real world, one may think of U as part of the terrain and the open subset $z_U(U)$ as the map[1]. The function z_U gives us the coordinates of the points in U, and the inverse function z_U^{-1} gives the points on X when the coordinates are known—the inverse coordinate function is sometimes called *a parametrization*.

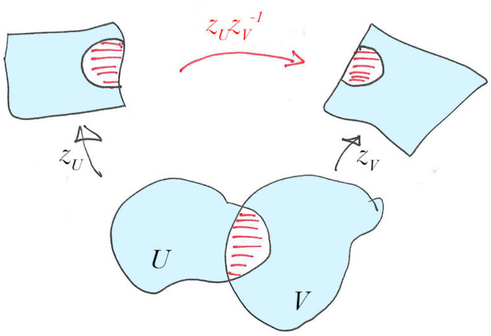
As an example consider the *Riemann sphere* $\hat{\mathbb{C}}$. It has the two open sets U_0 and U_∞, respectively the complement of $\{\infty\}$, that is the finite plane, and the complement of $\{0\}$. On the former one has the canonical coordinate z and on the latter one has the coordinate w given as $w = z^{-1}$ in the finite part of U_∞ and equal 0 at infinity.

(5.1) Given two charts $z_U \colon U \to z_U(U)$ and $z_V \colon V \to z_V(V)$ on X. They both survey the intersection $U \cap V$, and is of course of paramount interest to know which points of the two maps correspond to the same point in the terrain! The answer to that question is encoded in the so called *transition function*, that is the composition

$$z_U \circ z_V^{-1}|_{z_V(U\cap V)} \colon z_V(U \cap V) \to z_U(U \cap V).$$

Not to overload our notation we shall just write $z_U \circ Z_V$ for this function, with the tacit understanding it is defined on $z_V(U \cap V)$.

We say that two charts are *analytically compatible* if the corresponding transition function $z_U \circ z_V$ is holomorphic. This is perfectly meaningful, the transition function being a map between two open subsets of $\mathbb{C}$. As an example, on the intersection $U_0 \cap U_\infty$ in the Riemann sphere, the transition function $z_{U_\infty} \circ z_{U_0}^{-1}$ is the map $z \to z^{-1}$.



[1] In everyday language the map is frequently the piece of paper on which the map is printed, *i.e.*, the set $z_U(U)$. For us, as in the real real life, the map, or the chart, is the pair (U, z_U).

(5.2) By an *atlas* $\mathcal{U}$ on X we understand a collection of charts that together survey the whole topological space X, that is $\mathcal{U}$ is an open covering of X. The atlas is said to be an *analytic atlas* if additionally every two charts from the atlas are analytically compatible. Phrased differently, all the transition function arising in the atlas are holomorphic.

The set of analytical atlases on X are in the a natural way ordered by inclusion; one atlas is smaller than another if every chart in the former also is a chart in the latter. An analytic atlas is *maximal* if, well, it is maximal in this order. The existence of maximal atlases is an easy consequence of Zorn's lemma. If $\mathcal{U}_i$ is an increasing chain of analytical atlases, the union will be one, and by Zorn there is then a maximal one.

Defenition 5.1 *Let X be a connected, Hausdorff topological space. By an* analytic structure *on X, we understand a maximal analytical atlas on X. The pair of the space X and the maximal analytic atlas is called a* Riemann surface*.*

There are several comments to be made. First of all, it is common usage to let Riemann surfaces be connected by definition, mostly to avoid repeating the hypotheses that X be connected all the time. Some authors incorporate the hypothesis that X be second countable (that is, it has a countable basis[2] for the topology) but most do not, for the simple reason that universal covers of open plane sets are not *a priori* second countable—an illustrative example can be the complement of the Cantor set. It is however a relatively deep theorem of the Hungarian mathematician Tibor Radó (1895–1965) in 1925 that any Riemann surface is second countable. The third comment is that our definition works in any dimensions, one only has to replace charts in $\mathbb{C}$ by charts in $\mathbb{C}^n$.

(5.3) Let $\mathcal{U}$ be an analytic atlas on X and let V and W be two charts with coordinate functions z_V and z_W not necessarily belonging to the atlas $\mathcal{U}$. Assume that each one of them is analytically compatible with all charts from the atlas $\mathcal{U}$. Below we shall see that this implies that V and W are compatible as well, and and hence we can append them to $\mathcal{U}$ and get a bigger analytical atlas. And not stopping there, we can adjoin to $\mathcal{U}$ any chart being compatible with all charts in $\mathcal{U}$. In that way we get a gigantesque maximal atlas, and it is the unique maximal atlas containing $\mathcal{U}$.

Proposition 5.1 *Let X be a connected Hausdorff space. Every analytical atlas $\mathcal{U}$ on X. is contained in a unique maximal atlas, and consequently gives X a unique structure as a Riemann surface.*

Proof: After what we said just before the proposition, the poof is reduced to checking that if V and W are two charts both analytical compatible with all charts in $\mathcal{U}$ they

[2] There are many topological manifolds that are not second countable, even of dimension one! Hausdorff's so called "long line" is an example. In dimension two there are a great many examples, but none of them can be given the structure of a Riemann surface. However, in dimension two or more there are analytical spaces that do not have a countable basis for the topology. If you are interested in these outskirts of geometry, [?] is a nice reference.

are analytically compatible among themselves; that is, we must verify that $z_V \circ z_W^{-1}$ is holomorphic on $z_W(V \cap W)$. But for any chart U from $\mathcal{U}$ we obviously have the identity $z_V \circ z_W^{-1} = (z_V \circ z_U^{-1}) \circ (z_U \circ z_W^{-1})$ over $z_W(U \cap V \cap W)$, and as the coordinate neighbourhoods from $\mathcal{U}$ cover $V \cap W$, and being holomorphic is a local property, we are through. ❑

Two of the advantages with working with maximal atlases are that we are free to shrink coordinate neighbourhoods at will and that we can perform arbitrary biholomorphic coordinate changes. However, these maximal atlases are awfully large. In the complex plane for instance, the maximal analytical atlas consists of the pairs (U, ϕ) where U is *any* open subset and ϕ is *any* function biholomorphic in U! Luckily, results like proposition 5.1 above allows us to work with very small atlas when we work explicitly; for example on $\mathbb{C}$ we have the *canonical*[3] *atlas* with merely one chart, namely $(\mathbb{C}, \mathrm{id})$!

The Riemann sphere $\hat{\mathbb{C}}$ has as we saw a small atlas consisting of the two open sets U_0 and U_∞ with the coordinates z and w. On the intersection $U_0 \cap U_\infty$ the transition function is given as $w = z^{-1}$.

(5.4) When we are working in $\mathbb{C}$, disks are in use all the time. Similarly on a Riemann surface we shall frequently work with charts such that $z_U(U)$ is a disk, and for convenience we shall call such coordinate neighbourhoods for disks as well. If $z_U(U)$ is a disk about the origin and x is point in the disk with $z_U(x) = 0$ we say that U is *disk about x* or a disk *centered* at x. And of course we shall drop the index U pretty soon and only write z (or any other convenient letter) for the coordinate function.

(5.5) To analytic atlases are said to be *equivalent* if every chart in one is analytically compatible with every chart in the other. Two equivalent atlases are contained in the same maximal atlas, and hence they define the same structure as Riemann surface on X.

Other geometric structures

In the definiton one may impose other conditions on the transistion functions. For instance, the weaker condition that they C, gives us a structure of a smooth surface (or manifold how higher doimenson if the charts take values in $\mathbb{R}^n$) on X, and if addidinolly the Jaboian determinants of $z_U \circ z_V^{-1}$ all are positive, the smooth surface is orientable, and it becomes oriented once we make up our minds and choose one of the orientations of the plane.

Riemann surfaces are orientable because the jacobian of a biholomorphic map is positive. This follows by the Cauchy-Riemann equations, since

$$\det \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} = u_x^2 + v_y^2 > 0$$

[3] Once you have chosen your favorite model for the complex numbers, this is rally canonical. Be aware that the mapping id$\mathbb{C}$ is the function normally denoted by z in complex function theory.

where u and v are the real and the imaginary part fo the map.

One also strengthen the conditions on the transposition functions, and thus impose further constraints on the surfaces. For example, one can request the transition functions to be affine, that is of the form $z \mapsto az + b$ and one then speaks about an *affine structure* subordinate to the given analytic structure. Or one may ask that they are Möbius transformations. In that case the structure is called a *projective structure*.

As a final example, by a *real analytic structure* on a Riemann surface X, we understand an analytic atlas such that the coordinate domains $z_U(U)$ are symmetric about the real axis, and such if $f(z) = z_U \circ z_V^{-1}$ is a transition function, then $\overline{f(\overline{z})} = f(z)$. This last condition means that the Taylor development of f about real points have real coefficients.

PROBLEM 5.1. Show that X has a real structure if and only if it has an anti-holomorphic involution (Part of the exercise is to find out what this means!). ✷

PROBLEM 5.2. Let X be a Riemann surface with maximal atlas $\mathcal{U}$ with partchs (U, z_U). One defines the *conjugate* Riemann surface in the following way. The maximal atlas $\overline{\mathcal{U}}$ consists of the patches $(U, \overline{z_U})$ and the transitions functions are $\overline{z}_U \circ \overline{z}_V^{-1}$. Check that this is a Riemann surface. ✷

Holomorphic maps

The study of Riemann surfaces is to a great extend the study of maps between them, and if the maps are going tell us anything about the relation between the analytic structures on X and Y, these maps must be compatible with those structures. That is, they must be holomorphic in some sense. Being holomorphic is a local concept, so to tell what it means that a continuous map is holomorphic, is a local business, and charts are made for that.

(5.6) Assume that X and Y are two Riemann surfaces and that $f : X \to Y$ is a continuous map. Let V be a coordinate patch in Y and U one in X such that $f(U) \subseteq V$. Thence one may consider the map $z_V \circ f \circ z_U^{-1}$ which is a map from $z_U(U)$ to $z_V(V)$. Both these are open subsets of $\mathbb{C}$ so it is meaningful to require that $z_V \circ f \circ z_U^{-1}$ be holomorphic; and if there is a patch (V, z_V) in Y so that this is case, we say that f is *holomorphic in the patch* (U, z_U). This set up of coordinates patches round x and $y = f(x)$ adapted to f may be illustrated with a diagram like this

$$\begin{array}{ccccc} z_U(U) & \xleftarrow[z_U]{\simeq} & U & \hookrightarrow & X \\ \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f|_U} & & \downarrow{\scriptstyle f} \\ z_V(V) & \xleftarrow[z_V]{\simeq} & V & \hookrightarrow & Y, \end{array} \tag{5.1}$$

where $\tilde{f} = z_V \circ f \circ z_U^{-1}$.

The above definition is just an auxiliary definition, here comes the serious one:

Defenition 5.2 *Let X and Y be two Riemann surfaces and $f: X \to Y$ a continuous map between them. The map f is said to be* holomorphic *if it is holomorphic in every coordinate patch of the maximal analytic atlas on X.*

One says that f is *biholomorphic* or an *isomorphism* if f is bijective and the inverse is holomorphic. The composition of two holomorphic maps is holomorphic. Once you have grasped the definition this is quit clear, so it might be a good exercise to check in detail.

PROBLEM 5.3. Show that a Riemann surface X has a real structure if and only it is isomorphic to its conjugate surface $\overline{X}$. ✷

(5.7) Just like for defining analytic structures small atlases can be used to check that a map is holomorphic:

Proposition 5.2 *Let X and Y be two Riemann surfaces and $f: X \to Y$ a continuous map between them. If there is one analytic atlas $\mathcal{U}$ on X such that f is holomorphic in every patch of U, then f is holomorphic.*

PROOF: If $U' \subseteq U$, we have $z_V \circ f \circ z_{U'}^{-1} = z_V \circ f \circ z_U^{-1} \circ z_U \circ z_{U'}^{-1}$ ❏

(5.8) Local properties of traditional holomorphic functions we know from the beginning of the course, frequently have a counterpart for maps between Riemann surfaces. When being accustomed to the abstract definitions one transfers most local properties to Riemann surfaces with ease, once you have the standard set up on the retina it goes almost by itself, but we give detailed proofs at this stage of the course.

Transferring the "Open mapping theorem", gives us the following:

Proposition 5.3 *A non-constant holomorphic map between two Riemann surfaces is an open map.*

PROOF: This is just an exercise with the standard local set up, and of course, the substance comes from the open mapping theorem. Let A be open in X and let $y = f(x) \in f(A)$ be any point. As f is holomorphic near x, there is a patch (U, z_U) around x where f is holomorphic and we can, by shrinking U if necessary, assume that U is contained in A, thus we have the usual local set up like in 5.2:

$$\begin{array}{ccccc} z_U(U) & \xleftarrow[z_U]{\simeq} & U & \hookrightarrow & X \\ \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f|_U} & & \downarrow{\scriptstyle f} \\ z_V(V) & \xleftarrow[z_V]{\simeq} & V & \hookrightarrow & Y \end{array} \qquad (5.2)$$

where $\tilde{f} = z_V \circ f \circ z_U^{-1}$ and where $U \subseteq A$. By the Open mapping theorem we know that $\tilde{f}$ is an open map. Then $f|_U$(U) is open, which is what we need since $f(U) \subseteq A$. ❏

An important corollary is when X is compact;

Corollary 5.1 *Assume that f is a holomorphic map from a compact Riemann surface X to a Riemann surface Y. Then f is surjective and Y is compact.*

PROOF: On one hand the image $f(X)$ is closed X being compact, and on the other hand, after the proposition $f(X)$ is open. Hence $f(X)$ is a connected component of Y, and as Y by definition is connected, it follows that $f(X) = Y$. ❑

Proposition 5.4 *The fibres of a non-constant holomorphic map between Riemann surfaces are discrete.*

PROOF: Let $x \in X$ and let $y = f(x)$. It suffices to prove that x is isolated in $f^{-1}(y)$; that we have to find an open $U \subseteq X$ such that $U \cap f^{-1}(y) = \{x\}$. Again we resort to the standard set up with U a coordinate patch containing x. From before we know that the fibers of $\tilde{f}$ are discrete, so there is an open U' in $z_U(U)$ intersecting the fibre of $\tilde{f}$ in $z_U(x)$; and moving U' into X, we get our search for open set; *i.e.*, $z_U^{-1}(U') \cap f^{-1}(y) = \{x\}$. ❑

Tangent spaces and derivatives

The derivative of a map between two Riemann surfaces at point is not a number like we are used to when studying functions of one variable, but like most derivatives of functions of several variables it is a linear map, and since we are doing analysis over $\mathbb{C}$ it turns out to be complex linear map—the subtle point is naturally between which vector space. So to begin with, we must define the tangent space $T_{X,x}$ of a Riemann surface X at a point $x \in X$. The definition follows the now standard lines for defining tangent spaces in intrinsic geometry.

(5.9) Recall the ring $\mathcal{O}_{X,x}$ of *germs* of holomorphic functions near x. The elements are equivalence classes $[(\phi, U)]$ where U is an open neighbourhood of x and f a holomorphic function in U, two such pairs (ϕ, U) and (ψ, V) being equivalent if $W \subseteq U \cap V$ on which f and g coincides; that is $\phi|_U = \psi|_V$. One easily checks that this a ring with pointwise addition and multiplication as operations.

Choosing a coordinate patch U with coordinate z centered at x (recall that this means that $z(x) = 0$) one finds an isomorphism between $\mathcal{O}_{X,x}$ and the ring $\mathbb{C}\{z\}$ of powerseries in z with a positive radius of convergence. This i nothing more that the fact that any holomorphic function near the origin can be developed in a Taylor series and this series is unique.

The local ring is functorial. Given a holomorphic map $f\colon X \to Y$ and let $y = f(x)$. If $[\phi, U]$ is a germ of holomorphic function near y, the composition $\phi \circ f$ is holomorphic on $f^{-1}(U)$ and induces a germ $[\phi \circ f, f^{-1}(U)]$ near x. It is left to the zealous students to convince themselves that is a well defined and is a ring homomorphism.

The maximal ideal in $\mathcal{O}_{X,x}$ consisting of functions that vanish at x will be denoted by $\mathfrak{m}_x$.

(5.10) The *tangent space* $T_{X,x}$ is by definition the set of *point derivations* of $\mathcal{O}_{X,x}$, and point derivation $\tau\colon \mathcal{O}_{X,x} \to \mathbb{C}$ is a $\mathbb{C}$-linear map satisfying a product rule à la Leibnitz:

$$\tau(\alpha\beta) = \alpha(x)\tau(\beta) + \beta(x)\tau(\alpha).$$

It follows that $\tau(1) = 0$ (indeed, $1 \circ 1 = 1$!), and by linearity τ vanishes on constants. A point derivation vanishes as well on the square $\mathfrak{m}_x^2$ of the maximal ideal $\mathfrak{m}_x$; by Leibnitz's rule is obvious that if both $\alpha(x) = 0$ and $\beta(x) = 0$, it holds that $\tau(\alpha\beta) = 0$. Consequently every point derivation induces a map $\mathfrak{m}_x/\mathfrak{m}_x^2 \to \mathbb{C}$ and there is a map as in the following lemma. It is a good exercise to prove that it is an isomorphism.

Lemma 5.1 *There is a canonical isomorphism of complex vector spaces.* $T_{X,x} = \operatorname{Hom}_{\mathbb{C}}(\mathfrak{m}_x/\mathfrak{m}_x^2, \mathbb{C},)$*. In particular it holds that* $\dim_{\mathbb{C}} T_{X,x} = 1$.

PROOF: We have already define a map one way, so let us define a map the other way; that is, a map from $\operatorname{Hom}_{\mathbb{C}}(\mathfrak{m}_x/\mathfrak{m}_x^2, \mathbb{C},)$ to the tangent space $T_{X,x}$. Assume that $\phi\colon \mathfrak{m}_x/\mathfrak{m}_x^2 \to \mathbb{C}$ is a $\mathbb{C}$-linear map and let $\alpha \in \mathcal{O}_{X,x}$ be a germ. We are supposed to associate a point derivation, say τ_ϕ, to ϕ. The germ $\alpha - \alpha(x)$ obviously vanishes at x and belongs to $\mathfrak{m}_x$, so it is legitimate to put $\tau_\phi(\alpha) = \phi(\alpha - \alpha(x))$. One has the equality

$$(\alpha - \alpha(x))(\beta - \beta(x)) = (\alpha\beta - \alpha(x)\beta(x)) - \alpha(x)(\beta - \beta(x)) - \beta(x)(\beta - \beta(x)). \quad (5.3)$$

Since ϕ vanishes on $\mathfrak{m}_x^2$ and the left side of equation (5.3) above lies in $\mathfrak{m}_x^2$, we obtain

$$\tau_\phi(\alpha\beta) = \alpha(x)\tau_\phi(\beta) + \beta(x)\tau_\phi(\alpha),$$

that is Leibnitz's rule, and hence τ_ϕ is a point derivation. It is left as an exercise to show that one in this way obtains the inverse to the already defined map. ❑

(5.11) The map f^* induced a map, and that is the derivative of f at x, from $T_{X,x} \to T_{Y,y}$ simply by composition. That is we define the derivative $D_x\colon T_{X,x} \to T_{Y,y}$ by the assignment $D_x f(\tau) = \tau \circ f^*$. There is as always some checking to be done, but as always we leave that to the zealous students.

(5.12) A n important point is that the derivative is functorial. Id $f\colon X \to Y$ and $g\colon Y \to Z$ are two holomorphic maps with $f(x) = y$ and $\gamma(y) = z$ it holds true that

$$D_x f \circ g = D_x f \circ D_y g.$$

The formula boils down to the traditional chain rule after the mappings having been expressed in local coordinates. To become accustomed to the formalism of tangent space and derivatives in the intrinsic setting it is a good exercise to check this in detail

(5.13) The choice of a local coordinate z_U centered at the point x, *i.e.,* coordinates such that x corresponds to the origin, induces an isomorphism $\mathcal{O}_{X,x} \simeq \mathbb{C}\{z_U\}$, a germ corresponding to the Taylor series of a function representing the germ. In this correspondence the maximal ideal $\mathfrak{m}_x$ of functions vanishing at x corresponds to the ideal $(z_U)\mathbb{C}\{z_U\}$. Therefore $\mathfrak{m}_x/\mathfrak{m}_x^2$ is one dimensional with as basis the class of z_U, that we baptize dz_U. The basis of $T_{X,x}$ induced by the isomorphism in 5.3 and dual to dz_U is denoted by $\hat{dz_U}$.

(5.14) The usual set up of coordinates round x and $y = f(x)$ is as follows

$$\begin{array}{ccccc} z_U(U) & \xleftarrow[z_U]{\simeq} & U & \hookrightarrow & X \\ \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f|_U} & & \downarrow{\scriptstyle f} \\ z_V(V) & \xleftarrow[z_V]{\simeq} & V & \hookrightarrow & Y, \end{array}$$

where z_V is a local coordinate centered at the image point y of x valid in the vicinity V of y. On the open $z_U(U)$ set in $\mathbb{C}$ the map f materializes as a function $\tilde{f}$ holomorphic in $z_U(U)$, and the map $f^*\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ becomes the map $\mathbb{C}\{z_V\} \to \mathbb{C}\{z_U\}$ that sends z_V to $\tilde{f}(z_U)$.

We have the basis dz_V for $\mathfrak{m}_y/\mathfrak{m}_y^2$. and writing $\tilde{f}(z) = \tilde{f}'(0)z + z^2 g(z)$, we see that dz_V is sent to $\tilde{f}'(0)dz_U$ since the term $z^2 g(z)$ belongs to $\mathfrak{m}_x^2$.

Lemma 5.2 *In local coordinates the derivative $D_x f$ sends the basis element $\hat{dz_U}$ to $f'(0)\hat{dz_V}$.*

Local appearance of holomorphic maps

The first step of understanding a map is to understand its local behavior, so also with holomorphic maps. The fist result in that direction is a version of *the inverse function* theorem formulated in our setting.

Proposition 5.5 *Let $f\colon X \to Y$ be a holomorphic map between two Riemann surfaces and let $x \in X$ be a point. Assume that the derivative $D_x f$ does not vanish. Then there exists an open neighbourhood U about x such that $f|_U\colon U \to f(U)$ is an isomorphism (i.e., biholomorphic).*

Proof: The usual set up of coordinates round x and $y = f(x)$ is

$$\begin{array}{ccccc} z_U(U) & \xleftarrow[z_U]{\simeq} & U & \hookrightarrow & X \\ \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f|_U} & & \downarrow{\scriptstyle f} \\ z_V(V) & \xleftarrow[z_V]{\simeq} & V & \hookrightarrow & Y \end{array}$$

where $\tilde{f}$ is the representative of f in the local coordinates. By the lemma in the previous paragraph, $D_x f$ is just multiplication by $\tilde{f}'(0)$ in the basis dz_U and dz_V, hence $\tilde{f}'(0) \neq 0$, and from the earlier theory we know that thence $\tilde{f}$ is biholomorphic in a vicinity of 0, and shrinking U if necessary, the restriction $f|_U$ will be biholomorphic. ❑

Points where the derivative vanishes are said to be *ramification points* or *branch points* of the map f, and of course, it is *unramified* or *unbranched* at points where the derivative does not vanish. So, one may formulate the previous proposition by saying that a function is (locally) biholomorphic near points where it is unramified.

(5.15) Near a ramification point there is a local model for the behavior of f, depending on a number $\operatorname{ind}_x f$ called the *ramification index*; which is closely related to the vanishing multiplicity we know from before.

Proposition 5.6 *Let $x \in X$ be a point and let $f\colon X \to Y$ be a holomorphic map. Then there exist coordinate patches (U, z_U) and (V, z_V) around x and $f(x)$ respectively, with $f(U) \subseteq V$ such that $z_V \circ f \circ z_U^{-1}(z) = z^n$.*

In short the result says that locally and after appropriate changes of coordinates both near x and near y, the map f is given as the n-power map $z \to z^n$. But of course, behind this is the formally precise but rather clumsy formulation of the proposition.

The integer n does not depend on the chosen coordinate, and is *ramification index* hinted at, and is denote by $\operatorname{ind}_x f$.

Proof: Again we start with a standard set up with the patches centered at x and $f(x)$, that is $z_U(x) = 0$ as well as $z_V(f(x)) = 0$. See diagram (5.4) below. By xxx is part 1, there is a holomorphic function g in U such that $\tilde{f} = g^n$ with $g(0) = 0$ and $g'(0) \neq 0$. By shrinking U we may assume that g is biholomorphic in U, and therefore can be use as a coordinate! Hence we introduce the new patch $(U, g \circ z_U)$. For w lying in this patch, we find $\tilde{f}_1 = \tilde{f} \circ g^{-1}(w) = g(g^{-1}(w))^n = w^n$ and are through.

$$\begin{array}{ccccccc} g(z_U(U)) & \xleftarrow[g]{\simeq} & z_U(U) & \xleftarrow[z_U]{\simeq} & U & \hookrightarrow & X \\ & \searrow^{\tilde{f}_1} & \downarrow \tilde{f} & & \downarrow f|_U & & \downarrow f \\ & & z_V(V) & \xleftarrow[z_V]{\simeq} & V & \hookrightarrow & Y \end{array} \qquad (5.4)$$

❏

Problem 5.4. Show that $\tan\colon \mathbb{C} \to \widehat{\mathbb{C}}$ is unramified, but not surjective.Hence it is not a cover. Show that the image is $\widehat{\mathbb{C}} \setminus \{\pm i\}$, and show that $\tan\colon \mathbb{C} \to \widehat{\mathbb{C}}\{\pm i\}$ is a covering. ✷

Problem 5.5. Find the ramification points of the map $f(z) = \frac{1}{2}(z + z^{-1})$. ✷

Problem 5.6. Find the ramification points and the ramification indices of the $f(z) = z^n + z^{-m}$, n and m two natural numbers. ✷

Problem 5.7. Show that a holomorphic map between two compact Riemann surfaces is either constant or surjective. Show that if the map is not constant, the fibres are all finite. ✷

Some quotient surfaces

This section starts with two examples. The second one is important, elliptic curves being a central theme in several branches of mathematics. We end the section with a general quotient construction valid for a wide class of very nice actions.

In all these cases the quotient map serve as a parametrisation of the quotient surface X/G, except that points on X/G correspond to many values of the parameter—it is the task of the group to keep account of the different values. This makes it particularity easy to find coordinates, locally they are just the parameter values.

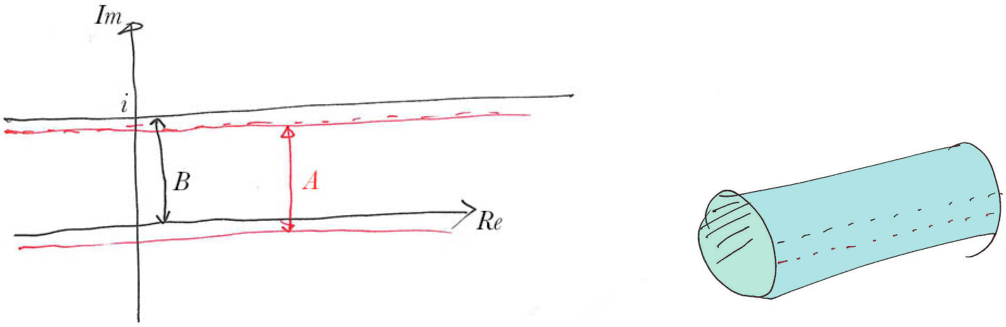
Example 5.1. The cylinder. On way of giving the cylinder an analytic structure is to consider it as the quotient of the plane by the action of the group generated by by the map $z \to z + i$. The topology on X is the quotient topology, the weakest topology making the quotient map $\pi\colon \mathbb{C} \to X$ continuous.

We shall put an analytic structure on X and this is an illustration of how the hocus-pocus with atlas and charts work, we shall do this in extreme detail. We shall specify an atlas with two charts. One is the infinite strip A between the real axis and the horizontal line $\operatorname{Im} z = 1$, or rather the image $\pi(A)$ in X. The quotient map π is a homeomorphism from A to $\pi(A)$, and the coordinate function on $\pi(A)$ is the inverse of this, we denote it by π_A^{-1}. That is the coordinate of $\pi(z)$ is z. The patch $\pi(A)$ covers most of the cylinder except the "seam", the image of the two boundary lines.

The second patch is *mutatis mutandis* constructed in the same way but from the different strip B between the horizontal lines $\operatorname{Im} z = 1 - t$ and $\operatorname{Im} z = -t$ where t is any real number between zero and one. The coordinate patch is the image $\pi(B)$ and the coordinate π_B^{-1}.

What happens then on the intersection $\pi(A) \cap \pi(B)$? What is the transition function? Is it holomorphic? First of all in A the inverse image $\pi_A^{-1}(\pi(A) \cap \pi(B))$ of the intersection is A with the line $\operatorname{Im} z = 1 - t$ removed since points on this line are not equivalent under the action to points in B.

So $\pi_A^{-1}(\pi(A) \cap \pi(B))$ has to components. The one where $0 \leq \operatorname{Im} z < 1 - t$ lies in B as well, and hence the transition function $\pi_B^{-1} \circ \pi$ is the identity. The other one, where $1 - t < \operatorname{Im} z < 1$, the composition $\pi_B^{-1} \circ \pi$ equals the translation $z \mapsto z - i$. In both cases the, the transition function is holomorphic and our two charts are analytically compatible. They constitute an analytic atlas and give the cylinder a complex structure.

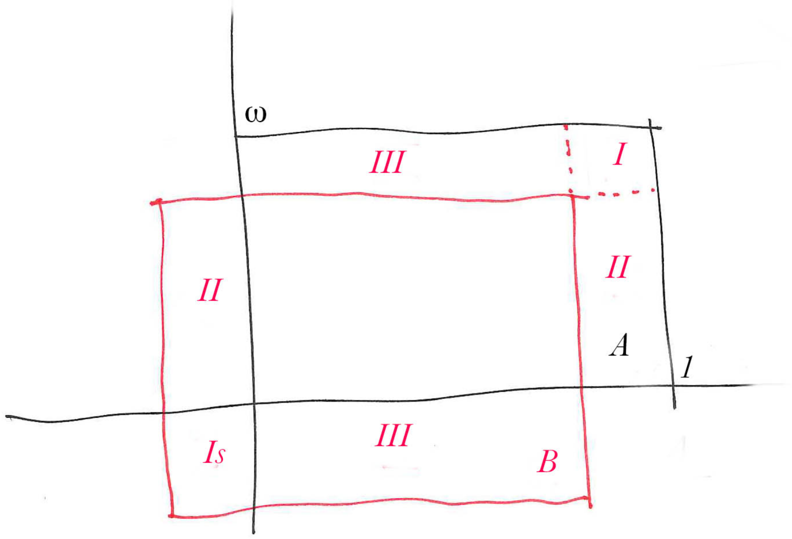


Figur 5.1: *The cylinder and the two coordinate patches.*

In fact, the cylinder is biholomorphic to the punctured plane $\mathbb{C}^*$. The biholomorphism is induced by the exponential function $e(z) = e^{2\pi z}$, that take values in $\mathbb{C}^*$. Clearly $e(z+i) = e(z)$, so e invariant under the group action, and therefore by the properties of the quotient space X, induces a continuous map $\tilde{e}\colon X \to \mathbb{C}^*$. It is easy to check using elementary properties of the exponential function (hence a task for zealous students) that $\tilde{e}$ is a homeomorphism. The only thing left, is to check that it is holomorphic, and this indeed comes for free: On the charts A and B the functions is be definition equal to $e^{2\pi z}$! The coordinate of a point $\pi(z)$ belonging to $\pi(A)$ (or $\pi(B)$) is z! ❄

Example 5.2. The next example is of the same flavour as the first, but the group action is more complicated—there will be two periods instead of just one –and the examples infinitely more interesting.

The Riemann surfaces will be compact with underlying topological space what topologists call a *torus*, a space homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$, which in bakeries is known as a doughnut. This is a genuine new surface—it is not biholomorphic to any open subset of the good old Riemann sphere $\widehat{\mathbb{C}}$— and it is known as an “elliptic curve”. These spaces entered the world of mathematics at a time when to compute the circumference of an ellipse (Very important question just after the discovery that the planets move in ellipses!) was the cutting edge of science, and the length-computation ended up with integrals related to bi-periodic functions, and as we shall see, bi-periodic functions lie behind the group action defining this Riemann surface.



Figur 5.2: *The atlas of the torus.*

Let Λ be the lattice $\Lambda = \{ n_1\omega + n_2 \mid n_1, n_2 \in \mathbb{Z} \}$ (in the figure we have for simplicity drawn ω as purely imaginary). It is an additive subgroup of the complex numbers $\mathbb{C}$, and we can form the quotient group $\mathbb{C}/\Lambda$. This is also a topological space when equipped with the quotient topology, and it is homeomorphic to the product $\mathbb{S}^1 \times \mathbb{S}^1$. We let $\pi\colon \mathbb{C} \to \mathbb{C}/\Lambda$ be the quotient map, it is an open map.

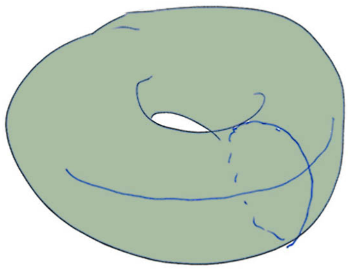
We intend to define an analytic structure on $\mathbb{C}/\Lambda$ in an analogous way as with the cylinder, by giving an atlas with two charts. The first is $A = \{ s + t\omega \mid 0 < s, t < 1 \}$, or rather the subset $\pi(A)$ of the torus. As no two points in A are congruent mod Λ, the set A maps invectively, and π being open, homeomorphically onto the open set $\pi(A)$ in $\mathbb{C}/\Lambda$. The coordinate of point $\pi(z)$ in $\pi(A)$ is simply z. The second chart is a small perturbation B of A, say $A = \{ s + t\omega \mid -\epsilon < s, t < 1 - \epsilon \}$, the image $\pi(B)$ is open and the coordinate of a point $p(z)$ is still z, but this time it must be chosen to lie B.

On the intersection of the two patches, the transition function is holomorphic. A quick (but incomplete) argument goes like this: Take a z in A whose image also lies in $\pi(B)$. Map it down to the torus and lift it back to a point w in B. Both z and the lift w lie in the same fibre of π, so w is a translate of z. Hence the transition functions are just translates, and we are tempted to say: which are holomorphic!

However, this is faulty since the difference $w - z$ can depend on z, and in fact it does!. One must assure oneself that this difference behaves holomorphically as a function of z. Luckily, the differences turn out to locally constant, *i.e.*, constant on the connected components of the intersection, and that will settle the case.

Contemplating figure 5.2 above, you easily convince yourself that this is true. The intersection manifests itself in A with four connected component, marked I, II, III

and IV in the figure, and the different translations are the as follows: The identity on IV, the map $z \to z - \omega$ on III, the map $z \to z - 1$ on II and finally the map $z \to z - i - 1$ in I. ❄



†

Problem 5.8. The quotient $\mathbb{C}/\Lambda$ is a group. Show that both the addition and the inversion maps are holomorphic. ✷

(5.16) Assume that G is a group acting holomorphically on a Riemann surface X. This means that all the action maps maps $x \mapsto g(x)$ are holomorphic, and of course the familiar axioms for an action must hold. If gh denotes the product of the two elements g and h in G, it holds true that $gh(x) = g(h(x))$, and $e(x) = x$ for all x where $e \in G$ is the unit. The set $Gx = \{ g(x) \mid g \in G \}$ is called the *orbit* of x, and the set $G(x) = \{ g \in G \mid g(x) = x \}$ of group elements that leave the point x fixed is called the *isotropy group* or *the stabiliser* of x.

The quotient X/G is as usual equipped with he quotient topology, a set in X/G being open if and only of its inverse image in X is open. This is equivalent to the quotient map $\pi\colon X \to X/G$ being open and continuos.

We concentrate on a class of particular nice actions called *free and proper*. They have following two properties.

- ☐ For any pair of points x and x' in X not in the same orbit, there are neighbourhoods U and U' of respectively x and x' with $U \cap gU'$ for all g.
- ☐ About every point $x \in X$ there is a neighbourhood disjoint from all its non-trivial translates; that is, there is an open U_x with $x \in U_x$ such that $gU_x \cap U_x = \emptyset$ for all $g \neq e$.

The first condition guarantees that the quotient X/G is a Hausdorff space. Indeed, if y and y' are two points in X/G, lift them to points x and and x' in X, and choose neighbourhoods U and U' as in the condition. Then $\pi(U)$ and $\pi U'$ are disjoint, if not there would be a point in U lying in the orbit a point in U', which is precisely what the condition excludes. And both $\pi(U)$ and $\pi(U')$ are open and one contains y and the other one y' so they are disjoint open neighbourhoods of respectively y and y'.

We proceed to define an analytic atlas on X/G. To begin with we chose one on X whose charts are (U, z_U) satisfy $gU \cap U\emptyset$ when $g \neq e$ (convince yourself that such an atlas may be found). The images $V = \pi(U)$ are open, and $\pi|_U$ are homeomorphisms onto U. The open patches of the atlas on X/G are the images V, and the coordinate functions w_V are given as let $w_V = z_U \circ \pi|_U^{-1}$. They take values in $z_U(U)$. We plan to show that this is an analytic atlas.

To this end let (V, w_V) and $(V', w_{V'})$ be two patches of the newly defined atlas on X/G. Our task is to show they are analytically compatible.

The part of $\pi^{-1}(V \cap V')$ lying in U is equal to the union of the different open sets $U \cap g(U')$ as g runs through G. These sets are open and pairwise disjoint since the sets $g(U')$ are , and therefore they form an open partition of $\pi^{-1}(V \cap V') \cap U$.

Now, there is only one partition of a locally connected set consisting of open and connected sets, namely the partition into connected components. The sets $U \cap g(U')$ are not necessarily connected, but it follows that they are unions of connected components of $\pi^{-1}(V \cap V') \cap U$.

It suffices to see that the transition function are holomorphic on each connected component of $z_U(\pi^{-1}(V \cap V') \cap U)$. But g^{-1} of course map maps $U \cap g(U')$ into the connected component $g^{-1}(U) \cap U'$ of $\pi^{-1}(V \cap V') \cap V'$, and the hence the transition function equals the restriction of $z_{U'} \circ g^{-1} \circ z_U$ on $z_U(U \cap g(U'))$.

PROBLEM 5.9. Let a be a positive real number and let η_a de defined by $\eta_a(z) = az$. The clearly η_a takes the upper half plane $\mathbb{H}$ into itself. Let G be the subgroup of $\mathrm{Aut}(\mathbb{H})$ generated by η_a. The aim of the exercise is to show that G acts on $\mathbb{H}$ in a proper and free manner, and that the resulting quotient $\mathbb{H}/G$ is biholomorphic to an annulus:

a) Show that $\liminf_{n\neq 0} |a^n - 1| \, (a^n + 1)^{-1} > 0$.

b) Let $z_0 \in \mathbb{H}$ and choose an ϵ with $0 < \epsilon < \liminf_{n\neq 0} |a^n - 1| \, (a^n + 1)^{-1} \, |z_0|$. Let U be the disk $|z - zo| < \epsilon$. Show that the disks $a^n U$ all are disjoint from U when $n \neq 0$. Conclude that the action is proper and free.

c) Show that the quotient $\mathbb{H}/G$ is a Riemann surface. Show that the function

$$f(z) = \exp(2\pi i \log z / \log a)$$

is invariant under the action of G and induces an isomorphism between $\mathbb{H}/G$ and the annulus $A = \{\, z \mid r < |z| < 1 \,\}$ where $r = \exp(-2\pi^2/\log a)$.

✷

Covering maps

Coverings play a prominent role in topology, and they have similar important role in theory of Riemann surfaces. May be they even have a more central place there due to the Uniformisation theorem. This fabulous theorem classifies all the simply connected Riemann surfaces up to biholomorphic equivalency, and amazingly, there are only equivalence classes them, namely the class of the complex plane $\mathbb{C}$, of the unit disk[4] $\mathbb{D}$ and of the Riemann sphere $\widehat{\mathbb{C}}$.

As we shall see, every Riemann surface has a universal cover which is a Riemann surface with a holomorphic covering map. Combining this with the Uniformisation theorem, one obtains the strong statement that any Riemann surface is biholomorphic to a free quotient of one of three on the list! This naturally has lead to an intense study of the subgroups of the automorphism groups of the three. Neither the plane nor the sphere have that many quotient, and most of the Riemann surfaces are quotients of the disk. The corresponding subgroups of $\mathrm{Aut}(\mathbb{D})$ form an extremely rich class of groups and can be very complicated.

It is also fascinating that the three classes of simply connected Riemann surfaces correspond to the three different versions of non-Euclidean geometry. The plane with the good old euclidean metric is a model for the good old geometry of Euclid and the other greeks, and the sphere naturally is a model for the spherical geometry. We already used the spherical metric when proving the Picard theorems. The renown french polymath Henri Poincaré put a complete metric on the disk, making it a model for the hyperbolic geometry, and naturally, that metric is called the *hyperbolic metric*.

(5.17) A *covering* map, or a *cover*, is a continuos map p between to topological spaces X and Y which fulfils the following requirement. Every point $y \in Y$ has an open neighbourhood U such that the inverse image decomposes as $p^{-1}(U) = \bigcup_\alpha U_\alpha$ where the U_α's are pairwise disjoint and are such that $p_{U,\alpha} = p|_{U_\alpha}$ is a homeomorphism between U_α and U. One says that the covering is *trivialized* over U; and in fact, it is trivial in the sense that there is an isomorphism $p^{-1}(U) \simeq U \times A$ such that p corresponds to the first projection, just send $u \in p^{-1}(U)$ to the pair $(p_{U,\alpha}(u), \alpha)$.

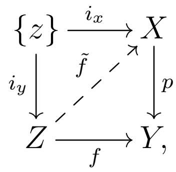
One usually assumes that Y is locally connected to have a nice theory. For us who only work with Riemann surfaces, this is not a restriction at all as points in a Riemann surface all have neighbourhoods being homeomorphic to disks. When the trivializing open set U is connected, the decomposition of the inverse image $p^{-1}(U) = \bigcup_{\alpha\in A} U_\alpha$ coincides with the decomposition of $p^{-1}(U)$ into the union of its connected components, which sometimes is useful.

(5.18) Covering maps have several good properties. For instance, there is a strong lifting theorem. Maps from simply connected spaces into Y can be lifted to X, that is one has the following theorem which we do not prove.

Proposition 5.7 *Assume that $p\colon X \to Y$ is a covering and that $f\colon Z \to Y$ is a continuous map where Z is simply connected. If z is a point in Z and x one in X such that $p(x) = f(z)$, there exists a unique continuous map $\tilde{f}\colon Z \to X$ with $\tilde{f}(z) = x$ and $f = p \circ \tilde{f}$.*

[4] or any Riemann surface biholomorphic to it. The upper half plane $\mathbb{H}$ is a very popular model.

For diagrammaholics, the proposition may be formulated with the help of the following diagram:



where i_x and i_y are the inclusion maps. One should read the diagrammatic message in the following way: The solid errors are given such that the solid square commutes, and the silent statement of the diagramm is that one can fill in a dotted arrow which makes the two triangular parts of the diagramm commutative.

(5.19) Coverings are as we saw locally homeomorphic to a product of an open set and a discrete space. And when the base Y is connected, this discrete space must up to homeomorphisms be the same everywhere; that is, the cardinality is constant over connected components of Y. One has:

Proposition 5.8 *If Y is connected is and $p\colon X \to Y$ is a cover, then the cardinality of the fibres $p^{-1}(y)$ is the same everywhere on Y.*

Proof: Let W_B be the set where $p^{-1}(y)$ is bijective to some given set B. Since p is locally trivial, W_B is open, and the same argument show that the complement $Y \setminus W_B$ is open as well (well, if the fibre is not bijective to B, it lies in some other W_C). It follows that $W_B = Y$ since Y is connected. ❑

In case all the fibres of p are finite, this can be phrased in a slightly different manner. Sending y to $\#p^{-1}(y)$ is a locally constant function on Y because p is locally trivial, and locally constant functions with integral values are constant on connected sets. The open sets U_α are frequently called *the sheets* or *the branches* over U, and if there are n of them, one speaks about an *n-sheeted covering*.

Problem 5.10. Show that the exponential map $\exp\colon \mathbb{C} \to \mathbb{C}^*$ is a covering. Let $a \in \mathbb{C}^*$ describe the largest disk over which exp is trivial. ✷

Problem 5.11. Let $f(z) = \frac{1}{2}(z + z^{-1})$. Consider f as a map from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. Show that f induces a unbranched double covering (synonymous with a 2-sheeted covering) from $\hat{\mathbb{C}} \setminus \{\pm 1\}$ to $\hat{\mathbb{C}} \setminus \{\pm 1, 0\}$. ✷

Problem 5.12. The tangent function $\tan z$ takes values in $\hat{\mathbb{C}} \setminus \{\pm i\}$. Show that $\tan\colon \mathbb{C} \to \hat{\mathbb{C}} \setminus \{\pm i\}$ is a covering. Be explicit about trivializing opens. Hint: It might be usefull that $\arctan z = (2i)^{-1} \log(1 + iz)(1 - iz)^{-1}$. ✷

Problem 5.13. Show that a holomorphic covering between Riemann surfaces then has a derivative which vanishes nowhere. Is the converse true? ✷

(5.20) A *universal covering* of a topological space X is a covering $p\colon Y \to X$ such that the space Y is simply connected, recall that tis means hat Y is path connecetd and that $\pi_1(Y) = 0$. It is not difficult to see that such universal coverings are unique up to a homeomorphism respecting the covering maps. That is, if $p'\colon Y' \to X$ is another one, there is a homeomorphism $\phi\colon Y' \to Y$ with $p' = p \circ \phi$; or diagrammatically presented, there is a commutative diagram

$$\begin{array}{ccc} Y' & \xrightarrow{\phi} & Y \\ & {\scriptstyle p'}\searrow \quad \swarrow{\scriptstyle p} & \\ & X. & \end{array}$$

Not all topological spaces have a universal covering. The condition to have one is rather long (close to a breathing exercise): The pace X must be connected, locally path connected and semi-locally simply connected. But don't panic, Riemann surface all satisfies these conditions, as every point has a neighbourhood homeomorphic to a disk.

Problem 5.14. Let $A = \mathbb{C} \setminus \{\, 1/n \mid n \in \mathbb{N} \,\}$. Show A is not open and that that $0 \in A$. Show that any neighbourhood of 0 in A has loops that are not null-homotopic in A. Show that A does not have a universal covering. ✷

Problem 5.15. Let $: Y \to X$ be a universal cover. let $\mathrm{Aut_X}(Y)$ be the set of homeomorphisms $\phi\colon Y \to Y$ such that $p \circ \phi = p$, that is, the homeomorphism making the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\phi} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & X, & \end{array}$$

commutative. Show that $\mathrm{Aut_X}(Y)$ is a group under composition. Fix a point $x \in X$. Show that ϕ by restriction induces a self-map of the fibre $p^{-1}(x)$. Show that if this self-map is the identity, then $\phi = \mathrm{id}_Y$. Show that $\mathrm{Aut_X}(Y)$ is a naturally isomorphic to a subgroup of the symmetric group $\mathrm{Sym}(\phi^{-1}(x))$. Hint: Use the lifting theorem (theorem 5.7 on page 16). ✷

Problem 5.16. Show that the action of $\mathrm{Aut_X}(Y)$ is free and proper in the sense as in xxx. Show that it acts transitively on each fibre. ✷

Coverings of Riemann surfaces

Assume that X is a Riemann surface and that $p\colon Y \to X$ is a covering where Y for the moment is just a Hausdorff topological space. The analytic structure on Y is easily transported to Y in a canonical way so that the projection p becomes holomorphic. This is a very important result though it is almost trivial to prove.

Proposition 5.9 *Assume that X is a Riemann surface and that $p\colon Y \to X$ is a covering. Then there is unique analytic structure on Y such that p is holomorphic. In particular every Riemann surface has a universal cover that is a Riemann surface and the projection is holomorphic.*

Proof: Take any atlas $\mathcal{U}$ over X whose coordinate patches (U, z_U) are such that the opens U all trivialize p; that is, the inverse image $p^{-1}(U)$ decomposes in a disjoint union $\bigcup_{\alpha \in A} U_\alpha$ with each $\pi_{U,\alpha} : U_\alpha \to U$ being a homeomorphism. The atlas on X we search for, consists of all the U_α's for all the U's in $\mathcal{U}$ with the obvious choice of $z_U \circ p_{U,\alpha}$ for coordinate functions, and it turns out to be an analytic atlas. Indeed, on $U_\alpha \cap V_\beta$ one has

$$(z_U \circ p_{U,\alpha}) \circ (z_V \circ p_{V,\beta})^{-1} = z_U \circ p_{U,\alpha} \circ p_{V,\beta}^{-1} \circ z_V^{-1} = z_U \circ z_V^{-1}$$

since both $p_{U,\alpha}$ and $p_{V,\beta}$ are restrictions of same map p to $U_\alpha \cap U_\beta$.

It is obvious that the projection map p is holomorphic, contemplate the diagram below for a few seconds and you will be convinced:

$$\begin{array}{ccc} U_\alpha & \longrightarrow & z_U(U) \\ {\scriptstyle p_{U,\alpha}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ U & \xrightarrow[z_U]{} & z_U(U) \end{array}$$

❑

(5.21) Recall that if Z is any simply connected space a map $Z \to X$ can be lifted to a map $Z \to \tilde{X}$ which is unique once the image of one point in Z is given. When Z is another Riemann surface and f is holomorphic, the lift will be holomorphic as well. We even have slightly stronger statement:

Proposition 5.10 *Assume that $p\colon Y \to X$ is a covering between Riemann surfaces and that $f\colon Z \to Y$ is a continuous map such that $p \circ f$ is holomorphic, then f is holomorphic.*

Proof: Again the hart of the matter is to choose an atlas compatible with the given data. Start with an atlas on Y whose coordinate neighbourhoods trivialize the covering p. For each U and each $z \in f^-1(U)$ there is patch V on Z centered at z and contained in $f^{-1}(V)$. And as Z i locally connected we can find such V's that are connected. Then $\tilde{f}(V)$ is contained in one of the $U_{U,\alpha}$'s, and one has $f|_V = p_{U,\alpha} \circ \tilde{f}|_V$. As $p_{U,\alpha}$ is biholomorphic in U_α this gives $\tilde{f} = f|_V \circ p_{U,\alpha}^{-1}$ implying that $\tilde{f}$ is holomorphic in V, and hence in Z since the V's cover Z. ❑

Problem 5.17. Check that in proposition 5.10 above, it suffices to assume that p be a local homeomorphism. ✷

Problem 5.18. Let Λ be a lattice in $\mathbb{C}$. Show that the projection map $\mathbb{C} \to \mathbb{C}/\Lambda$ is a universal cover for the elliptic curve $\mathbb{C}/\Lambda$. ✷

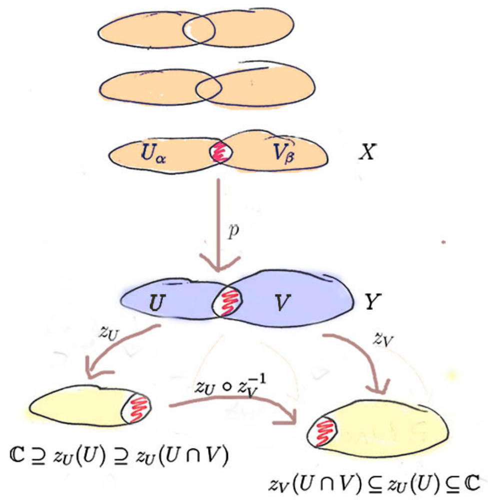
Problem 5.19. Let Λ be a lattice. A function is Λ-periodic if $f(z+\omega) = f(z)$ for all $\omega \in \Lambda$ and all $z \in \mathbb{C}$. Show that any holomorphic Λ-periodic function is constant. ✷

Example 5.3. We continue to explore the world of elliptic curves. In this example we study the holomorphic maps between two elliptic curves $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$, and shall show that they are essentially linear, that is induced by linear function $z \to az + b$ from $\mathbb{C} \to \mathrm{C}$.

Let the $f\colon \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ be holomorphic. The salient point is that $p'\colon \mathbb{C} \to \mathbb{C}/\Lambda'$ is the universal cover of $\mathbb{C}/\Lambda'$, so that any holomorphic map from a simply connected Riemann surfaces into $\mathbb{C}/\Lambda'$ lifts to a holomorphic map into $\mathbb{C}$ by proposition 5.10. We apply this to the map $f \circ p$ and obtains a holomorphic function $F\colon \mathbb{C} \to \mathbb{C}$ that fits into the commutative diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{F} & \mathbb{C} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ \mathbb{C}/\Lambda & \xrightarrow[f]{} & \mathbb{C}/\Lambda'. \end{array}$$

Fix for a moment a member ω of the lattice Λ and consider the difference $F(z+\omega) - F(z)$. As a function of z it takes values in the discrete subset Λ' of $\mathbb{C}$. It is obviously continuous (even holomorphic), and hence it must be constant. Taking derivatives shows that $F'(z+\omega) = F'(z)$, so that the derivative is Λ-periodic, and from problem 5.19 we conclude that $F'(z)$ is constant. Hence $F(z) = az + b$. ❄



Figur 5.3: *Charts on a covering surface.*

Proper maps

Recall that a proper map between two topological spaces is a continuous map whose inverse images of compact sets are compact. A continuous map whose source space is compact, is automatically proper, and of course. Notice that the target space can be decisive for the map being proper or not; for instance, homeomorphisms are proper, but open embeddings[5] are usually not.

(5.22) Any proper, holomorphic maps between Riemann surfaces must have finite fibres. The fibres are discrete by proposition 5.4 on page 7 and as f is proper, they are compact as well.

Proper maps are always closed whether holomorphic or not. To see this, let b be a point in the closure of the image $f(A)$ of a closed a set $A \subseteq X$, and let $\{a_\nu\}$ be a sequence in A such that $\{f(a_\nu)\}$ converges to b. The subset $B = \{\, f(a_n) \mid n \in \mathbb{N}\,\} \cup \{b\}$ of Y is compact. Hence the inverse $f^{-1}(B)$ is also compact because f is assumed to be proper. As $\{a_n\} \subseteq f^{-1}(B) \cap A$, there is a subsequence of $\{a_\nu\}$ converging to a point a in A, and by continuity, $f(a) = b$. We have thus proven

Proposition 5.11 *A proper, holomorphic map between two Riemann surfaces is closed and have finite fibres.*

[5] An open embedding is a map whose image is open and which is homeomorpic onto its immage.

PROBLEM 5.20. Give an example of a smooth map between Riemann surfaces whose fibres are not all finite. Give an example of an open imbedding that is not proper. Give an example of an open imbedding (of topological spaces) that is proper, but not a homeomorphism. ✷

PROBLEM 5.21. Show that the composition of two proper maps is proper. ✷

PROBLEM 5.22. Assume that $f: X \to Y$ is proper and that $A \subseteq X$ is a closed, discrete set. Show that $f(A)$ is discrete. ✷

PROBLEM 5.23. If $f: X \to Y$ is proper and $A \subseteq Y$ is closed, show that the restriction $f|_{X \setminus f^{-1}(A)}: X \setminus f^{-1}(A) \to Y \setminus A$ is proper. ✷

(5.23) Every covering map is a local homeomorphism by definition, but the converse is not true. A cheap example being an open immersion; that is, the inclusion map of an open set U in a space X. If U is not a component of X any point in the boundary of U will not have a trivializing neigbourhood. If you want a surjective example, there is an equally cheap one. Take any covering with more than two points in the fibres and remove one point from one of the fibres.

If the map in addition to being a covering also is a proper map, it will be a covering:

Lemma 5.3 *A proper, local homeomorphism $f: X \to Y$ is a covering map.*

PROOF: Take any point $y \in Y$. The fibre $f^{-1}(y)$ is finite because f is proper. Round each point x in the fibre there is an open U_x which f maps homeomorphically onto an open V_x in Y. By shrinking these sets we may assume they are pairwise disjoint, *i.e.*, replace U_x with $U_x \setminus \bigcup_{x' \neq x} U_{x'}$ and notice that $x \notin U_{x'}$ if $x' \neq x$ since f is injective on $U_{x'}$.

The finite intersection $V = \bigcap_{x \in f^{-1}(y)} V_x$ is an open set containing y, and clearly the different sets $f^{-1}(V) \cap U_x$ for $x \in f^{-1}(y)$ are open, disjoint sets mapping homeomorphically onto V. ❑

The degree of a proper holomrphic maps

This section is about proper maps between Riemann surfaces and the cardinality of their fibres. Their fibres are finite, and case the map is a cover, all fibres have the same number of points as saw in prop xxx above. The theme of this paragraph is to extend this result to maps having branch points, however the branch points counted with a multiplicity which turns out to be equal to the ramification index $\operatorname{ind}_x f$.

Proposition 5.12 *Let $: X \to Y$ be a proper,holomorphic map between two Riemann surfaces. Then the number $\sum_{f(x)=y} \operatorname{ind}_x f$ is independent of the point $y \in Y$ and is called the degree of f. If f is not branched in any point in $f^{-1}(y)$, it holds true that $\#f^{-1}(y) = \deg f$.*

So let $f\colon X \to Y$ be a proper map. The points in X where the derivative $D_xF\colon T_{X,x} \to T_{Y,f(x)}$ vanishes are isolated points; indeed, locally in charts (U, z_U) of X and (V, z_V) on Y the function f is represented by the holomorphic function $\tilde{f} = z_V \circ f \circ z_U$, and the derivative $\tilde{f}'$ represents D_xf for $x \in U$. We know that $\tilde{f}'$ is holomorphic and hence has isolated zeros.

Hence the set $B = \{\, x \mid D_xf \,\}$ is a closed, discrete set in X called the *branch locus* or *ramification locus* of f. The image $f(B_f)$ is closed and discrete as well, our map f being proper, and on the open set $W = Xf^{-1}(f(B_f))$ the map f is unramified. Hence is a local homeomorphism there and since the restriction $f|_W\colon W \to Y \setminus f(B_f)$ is proper, it is covering by lemma 5.3 above.

Proposition 5.13 *Let $: X \to Y$ be a proper,holomorphic map between two Riemann surfaces. Then the number $\sum_{f(x)=y} \operatorname{ind}_x f$ is independent of the point $y \in Y$ and is called the* degree *of f. If f is not branched in any point in $f^{-1}(y)$, it holds true that $\#f^{-1}(y) = \deg f$.*

PROOF: Let B be the branch locus of f and put $W = f^{-1}(f(B))$. Then $f|_W\colon W \to Y \setminus f(B)$ is a covering map. Moreover $f(B)$ being a discrete set, the complement $Y \setminus f(B)$ is connected, and by 5.8 on page 17 the number of points in the fibres $f^{-1}(y)$ is the same for all $y \in Y \setminus f(B)$.

So we pass to examining the situation round a fibre containing branch points. Let $f^{-1}(y) = \{x_1, \ldots, x_r\}$ and let $n_i = \operatorname{ind}_{x_i} f$, of course some of these can be one. By the local description of branch points (proposition 5.6 on page 10) we can find coordinate patches U_i with coordinate z_i round each x_i and V_i round y such that in the patch U_i one has $f(z_i) = z_i^{n_i}$.

Shrinking the U_i if necessary, they can be assumed to disjoint, and replacing V with the intersection $\bigcap_i V_i$, we can assume that $V = f(U_i)$ for all i. With this in place the inverse image $f^{-1}(V)$ decomposes as the union $\bigcup_i U_i$. Now, there are points y' in V such that the map f is unbranched over y', thence $\#f^{-1}(y')$ decomposes as $\sum_i \#(f^{-1}(y') \cap U_i)$. Clearly this sum equals $\sum_i n_i$, indeed, f is represented as $f(z_i) = z_i^{n_i}$ on the patches U_i and equations $z_i^{n_i} = \epsilon$ has n_i solutions. On the other hand all unbranched fibres have the same number of points, so we are through. ❏

PROBLEM 5.24. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two proper holomorphic maps between Riemann surfaces. Show that the composition $g \circ f$ is proper and that one has $\deg g \circ f = \deg f \deg g$. ✷

PROBLEM 5.25. Let $\Lambda \subseteq \mathbb{C}$ be a lattice. Show that for each integer n the map $z \to nz$ induces a proper map $[n]\colon \mathbb{C}/L \to \mathbb{C}/\Lambda$. Show that $[n]$ is unramified and determine its degree. HINT: Compute the derivative of $[n]$. ✷