October 11$^{\text{th}}$, 2018

MAT4800: Complex Analysis

Mandatory assignment 1

Submission deadline

Sunday October 28$^{\text{th}}$ 2018, 23:59 at Devilry (https://devilry.ifi.uio.no).

Instructions

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a **clear presentation** with all **necessary explanations**. Remember to include all relevant plots and figures. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Requirement to pass: 55%.

Problem 1. (20%) This question is about the complex projective space $\mathbb{P}^2$ of dimension 2. It is defined as the quotient $\mathbb{P}^2 = (\mathbb{C}^3\backslash\{(0,0,0)\})/\sim$, where $(0,0,0)$ is the origin in $\mathbb{C}^3$, and the equivalent relation is that $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$ if and only if there is $0 \neq \lambda \in \mathbb{C}$ so that $(x_1, y_1, z_1) = (\lambda x_2, \lambda y_2, \lambda z_2)$. Denote by $\pi : \mathbb{C}^3\backslash\{(0,0,0)\} \to \mathbb{P}^2$ the quotient map, and define the topology on $\mathbb{P}^2$ to be the quotient topology. This means that a set $A \subset \mathbb{P}^2$ is open iff $\pi^{-1}(A)$ is open in $\mathbb{C}^3\backslash\{(0,0,0)\}$. In the latter, $\mathbb{C}^3\backslash\{(0,0,0)\}$ is given its usual topology.

We denote by $[x : y : z] = \pi(x, y, z)$ the homogeneous coordinates on $\mathbb{P}^2$.

Recall that a polynomial $F(x, y, z)$ is homogeneous of degree d if for all $x, y, z, t \in \mathbb{C}$ we have $F(tx, ty, tz) = t^d F(x, y, z)$.

a) Show that the map π is continuous.

b) Is π a covering map? Why or why not?

c) Show that $\mathbb{P}^2$ is Hausdorff, that is for every two distinct points $p_1, p_2 \in \mathbb{P}^2$, there are open subsets $U_1, U_2 \subset \mathbb{P}^2$ so that $p_1 \in U_1$, $p_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$. (Hint: May use the results in question f) below.)

d) Give an example of a non-zero homogeneous polynomial of degree 3 in variables x, y, z. Give an example of a polynomial in variables x, y, z which is not homogeneous.

e) Let $F(x, y, z)$ be a non-zero homogeneous polynomial in variables x, y, z. Show that there is a unique subset $A \subset \mathbb{P}^2$ so that $\pi^{-1}(A) = \{(x, y, z) \in \mathbb{C}^3\backslash\{(0,0,0)\} :\ F(x, y, z) = 0\}$. Moreover, show that A is a closed subset of $\mathbb{P}^2$.

For simplicity, we usually denote such a set A as $A = \{[x : y : z] \in \mathbb{P}^2 :\ F(x, y, z) = 0\}$.

f) Let $[x : y : z]$ be homogeneous coordinates for $\mathbb{P}^2$. Let $U_1 = \{[x : y : z] \in \mathbb{P}^2 :\ z \neq 0\}$, $U_2 = \{[x : y : z] \in \mathbb{P}^2 :$ '$x \neq 0\}$ and $U_3 = \{[x : y : z] \in \mathbb{P}^2 :\ y \neq 0\}$. Given as usual the isomorphisms $(V_1, W_1) : U_1 \to \mathbb{C}^2$ where $V_1 = x/z$, $W_1 = y/z$; $(V_2, W_2) : U_2 \to \mathbb{C}^2$ where $V_2 = y/x$ and $W_2 = z/x$; and $(V_3, W_3) : U_2 \to \mathbb{C}^2$ where $V_3 = x/y$ and $W_3 = z/y$.

Describe V_2, W_2, V_3, W_3 as functions in V_1, W_1. For example, here $V_2 = W_1/V_1$.

Problem 2. (20%) Let $\pi : \mathbb{C}^3\backslash\{(0,0,0)\} \to \mathbb{P}^2$ be the projection as in the previous problem. Let $F(x, y, z) = zy^3 - x^4 - x^2z^2 + z^4$.

a) Show that $F(x, y, z)$ is homogeneous.

b) Let $X = \{[x : y : z] \in \mathbb{P}^2 :\ F(x, y, z) = 0\}$. Show that X is a Riemann surface.

c) Let $U = X\backslash\{[0 : 1 : 0]\}$ and define $\phi : U \to \mathbb{C}$ by the formula $\phi([x : y : z]) = x/z$. Show that ϕ is well-defined and is a holomorphic map.

d) Let $\mathbb{C}$ be embedded into $\mathbb{P}^1$ by the identification $w \in \mathbb{C} \to [w : 1] \in \mathbb{P}^1$. The map ϕ in c) can be viewed then as a map $\phi : U \to \mathbb{P}^1$. Show that there is a holomorphic map $\Phi : X \to \mathbb{P}^1$ which extends ϕ.

e) Is the map Φ a constant map? What is the degree of Φ?

f) What are the critical points and branch points of Φ? [Recall: Critical points of Φ are points where the derivative of Φ is 0. Branch points of Φ are images of critical points of Φ.]

g) Is Φ a covering map or not?

Problem 3. (20%) Recall that an elliptic curve is a quotient $E = \mathbb{C}/\Lambda$, where $\Lambda = \mathbb{Z}\tau_1 + \mathbb{Z}\tau_2$ is a lattice. Here τ_1, τ_2 are complex numbers which are independent over $\mathbb{R}$.

In this question, you are allowed to use that a subgroup G of $\mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^2$ must be one of the following forms: 1) $G = \{0\}$; 2) $G = \mathbb{Z}be_2$ where b is a positive integer; 3) $G = \mathbb{Z}(ae_1 + be_2)$ where a is a positive integer; and 4) $G = \mathbb{Z}(ae_1 + be_2) \oplus \mathbb{Z}ce_2$ where a, c are positive integer. Here $e_1 = (1, 0)$ and $e_2 = (0, 1)$ is the standard basis for $\mathbb{Z}^2$.
a) Show that all automorphisms ψ of $\mathbb{C}$ are of the form $\psi(z) = az + b$, where $a, b \in \mathbb{C}$ are constants and $a \neq 0$.
b) Show that the quotient map $\pi : \mathbb{C} \to E$ is a covering map. Deduce that π is the universal covering of E.
c) Compute the deck transformations $Deck(\mathbb{C}/E)$. Deduce that the fundamental group of E is $\mathbb{Z}^2$.
d) Determine all covering maps $Y \to E$ of finite degrees, where Y is connected.
e) Assume that $\varphi : Y \to E$ is a covering map, where Y is connected, and there is $p_0 \in E$ so that $\varphi^{-1}(p_0)$ has ≤ 3 elements. Show that Y must be an elliptic curve, and determine all of such Y. (Y may be different from E. Such a map φ is an isogeny.)

Problem 4. (20%) Let $X = \{[x : y : z] \in \mathbb{P}^2 : \ zy^2 - x^3 - z^3 = 0\}$.
a) Show that X is a Riemann surface.
b) Define $\mathcal{F}(\emptyset) = \{1\}$. For any $\emptyset \neq U \subset X$ an open set, define $\mathcal{F}(U)$ to be the set of all holomorphic functions from U to $\mathbb{C}\backslash\{0\}$. Define an Abelian group structure on $\mathcal{F}(U)$ by the following way: If $\phi_1, \phi_2 \in \mathcal{F}(U)$ then $\phi_1.\phi_2$ is the function $\phi : U \to \mathbb{C}\backslash\{0\}$ given by the formula $\phi(z) = \phi_1(z)\phi_2(z)$ for all $z \in U$. Show that with the usual restriction maps, $\mathcal{F}$ is a sheaf on X.
c) Determine $\mathcal{F}(X)$, the set of sections of $\mathcal{F}$ over X.
d) Let $U = X\backslash\{[0 : 1 : 0]\}$. Show that $\mathcal{F}(U)$ is strictly bigger than $\mathcal{F}(X)$.
e) Given $p \in X$, show that the stalk $\mathcal{F}_p$ is isomorphic to $\mathbb{C}[[t]]\backslash t\mathbb{C}[[t]]$. Here $\mathbb{C}[[t]]$ is the set of power series whose radius of convergence is > 0.

Problem 5. (20%) This question is about differential 1-forms on elliptic curves. Let $E = \{[x : y : z] \in \mathbb{P}^2 : \ zy^2 = x^3 + 4z^3\}$. Let U_1, U_2, U_3 be open sets of $\mathbb{P}^2$ as in Problem 1 f), and let $V_1, W_1, V_2, W_2, V_3, W_3$ be as in that problem.
a) Show that $E \cap U_1 = \{(V_1, W_1) \in \mathbb{C}^2 : \ W_1^2 - V_1^3 - 4 = 0\}$. Show that the 1-form $\omega = dV_1/W_1$ is a holomorphic 1-form on $E \cap U_1$. [Hint: Near $W_1 = 0$, use another description of ω as $h_1(V_1, W_1)dW_1$, where $h_1(V_1, W_1)$ is holomorphic near $W_1 = 0$ in $E \cap U_1$.]
b) We can write $E \cap U_2 = \{f_2(V_2, W_2) = 0\}$ where f_2 is a polynomial. Determine f_2. Deduce that if $(V_2, W_2) \in E \cap U_2$ then $V_2W_2 \neq 0$.
c) On $E \cap U_2 \cap U_1$ we can write $\omega = h_2(V_2, W_2)dW_2$, where $h_2(V_2, W_2) = p_2(V_2, W_2)/V_2W_2$, and $p_2(V_2, W_2)$ is a polynomial. Determine $p_2(V_2, W_2)$. Deduce that ω extends to a holomorphic 1-form on $U_1 \cup U_2$.
d) We can write $E \cap U_3 = \{f_3(V_3, W_3) = 0\}$ where f_3 is a polynomial. Determine f_3. Deduce that $(0, 0) \in E \cap U_3$ and that $\lim_{(V_3,W_3)\in E\cap U_3,\ (V_3,W_3)\to(0,0)} V_3^3/W_3 = 1$.
e) On $E \cap U_3 \cap U_1$, we can write $\omega = dV_3 + h_3(V_3, W_3)dV_3$, where $h_3(V_3, W_3) = p_3(V_3, W_3)/q_3(V_3, W_3)$, $p_3(V_3, W_3)$ is a polynomial and $q_3(V_3, W_3) = W_3(1 - 12W_3^2)$. Determine $p_3(V_3, W_3)$. [Hint: Use implicit differentiation, based on the

equation of $E \cap U_3$ determined in $E \cap U_3$ in question d) above.] Deduce that ω can be extended to a holomorphic form on an open neighbourhood in $E \cap U_3$ of the point $E \cap U_3$.
f) Show that if $(V_3, W_3) \in E \cap U_3$, and $(V_3, W_3) \neq (0, 0)$ then $(V_3, W_3) \in E \cap (U_1 \cup U_2)$. Deduce that ω can be extended to a holomorphic 1-form on all of E.

Problem 6. (5%) This is a bonus question. We need to determine a day for the final exam. There is a choice between either Friday (30 November 2018) or Monday (10 October 2018).

Write all time intervals on these days (from 9:00 to 18:00) for which you are **unavailable** together with a reasonable reason.

After collecting all of your answers on this question, I will decide on a final exam date, which then **cannot be** changed!

If you do not give an answer to this question, then I will assume that every time interval will work for you.