10$^{\text{th}}$ October, 2022

MAT4800

Mandatory assignment 1 of 1

Submission deadline

Thursday 21$^{\text{st/nd/rd/th}}$ OCTOBER 2022, 14:30 in Canvas (canvas.uio.no).

Instructions

Note that you have **one attempt** to pass the assignment. This means that there are no second attempts.

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) no later than the same day as the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Guidelines for this assigment

This mandatory assignment contains standard material/results. This means that if you find it difficult to prove the results yourself, you may find help in the literature. It you need to consult the literature that is fine, but if so, include references, and make sure you present the proofs in your own way.

Working on the results in the assignment will be useful preparation for later in the course.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Let $\Omega \subseteq \mathbb{C}$ be a domain. We let $\mathcal{E}(\Omega)$ denote the space of smooth functions on Ω.

Recall that a smooth 1-form ω on Ω is on the form

$$\omega(z) = g(z)dx + h(z)dy \tag{1}$$

where $g, h \in \mathcal{E}(\Omega)$. With the notation $dz = dx + idy, d\bar{z} = dx - idy$ any such ω may be written uniquely

$$\omega = \tilde{g}(z)dz + \tilde{h}(z)d\bar{z}. \tag{2}$$

For $f \in \mathcal{E}(\Omega)$ we define

$$df(z) = f_x(z)dx + f_y(z)dy = f_z(z)dz + f_{\bar{z}}(z)d\bar{z}.$$

We say that ω as in (1) is *closed* if $h_x(z) - g_y(z) = 0$ for all $z \in \Omega$. This is equivalent to $\tilde{h}_z(z) - \tilde{g}_{\bar{z}}(z) = 0$ for all $z \in \Omega$.

Problem 1.

(i) Prove that if $\omega \in \mathcal{E}^1(\triangle_r(a))$ is closed, there exists an $f \in \mathcal{E}(\triangle_r(a))$ such that $\omega = df$.

(*Hint: In the unit disk $\triangle$, define $f(z) = \int_{l_z} \omega$, where l_z is the straight line segment between* 0 *and z. Find an application of Stoke's Theorem to prove the result (we remark that Stoke's Theorem is valid for domains with piecewise smooth boundary).*)

(ii) Conclude that if $g \in \mathcal{O}(\triangle_r(a))$ there exists $f \in \mathcal{O}(\triangle_r(a))$ such that $f' = g$.

For a domain $\Omega \subseteq \mathbb{C}$ we say that $H^1(\Omega, \mathbb{C}) = 0$ if the following holds: for any open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of Ω, and any collection of locally constant functions $f_{ij} : U_{ij} \to \mathbb{C}$ satisfying

$$f_{ij} + f_{jk} + f_{ki} = 0 \text{ on } U_{ijk},$$

for all $i, j, k \in I$, there are locally constant functions $f_i : U_i \to \mathbb{C}$ such that $f_i|_{U_{ij}} - f_j|_{U_{ij}} = f_{ij}$. (A function $f : U \to \mathbb{C}$, $U \subset \mathbb{C}$ open, is *locally constant* if for any point $z \in U$ there is an open neighborhood V of z such that f is constant on V. Equivalently, f is constant on each connected component of U.)

Problem 2.

(i) Prove that if $\Omega, \Omega' \subseteq \mathbb{C}$ are domains, and there exists a homeomorphism $\phi : \Omega \to \Omega'$, then $H^1(\Omega, \mathbb{C}) = 0$ if and only if $H^1(\Omega', \mathbb{C}) = 0$.

(ii) Prove that $H^1(\triangle, \mathbb{C}) = 0$.

(*Hint: Try to mimic the proof of Theorem 4 in Chapter 6 in Narasimhan's book.*)

(iii) Prove that for any simply connected domain $\Omega \subseteq \mathbb{C}$ we have that $H^1(\Omega, \mathbb{C}) = 0$.

(iv) Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain. Prove that if $\omega \in \mathcal{E}^1(\Omega)$ is closed, then there exists an $f \in \mathcal{E}(\Omega)$ such that $df = \omega$.

Problem 3. In this problem we assume that $\Omega \subseteq \mathbb{C}$ is a domain with $H^1(\Omega, \mathbb{C}) = 0$.

(i) Prove that for any $g \in \mathcal{O}(\Omega)$ there exists $f \in \mathcal{O}(\Omega)$ such that $f' = g$.

(ii) Prove that for any $g \in \mathcal{O}^*(\Omega)$ there exists $f \in \mathcal{O}(\Omega)$ such that $e^f = g$.

(iii) Prove that if $\Omega \neq \mathbb{C}$ then Ω is biholomorphic to $\triangle$.