10 October 2023
MAT 4800

Deck transformation:

Let $p: Y \to X$ be a covering map.

A deck transformation is a homeomorphism $f: Y \to Y$ which preserves fibers of p, that is $p \circ f = p$.

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

Remarks: If X & Y are RS, & $p: Y \to X$ is a covering map then p is holomorphic, & any $f \in \mathrm{Deck}(Y \xrightarrow[p]{} X)$ is also holomorphic.

So f is a holomorphic automorphism of Y.

$\mathrm{Deck}(Y \xrightarrow[p]{} X)$ is a group with the multiplication being the composition of the maps. $\square$

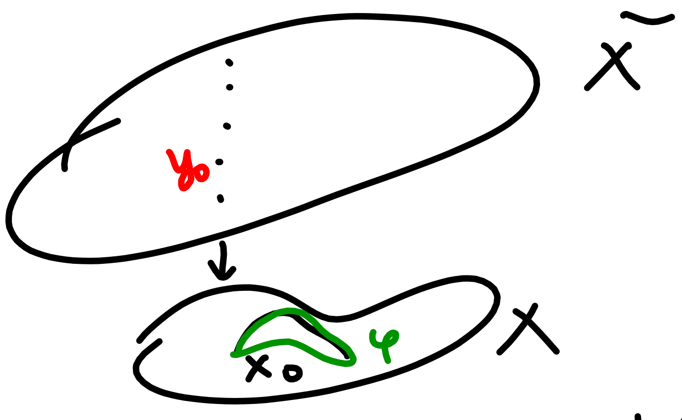
Theorem: let $p: \tilde{X} \to X$ be a universal covering. Then

$$\mathrm{Deck}(\tilde{X} \xrightarrow[p]{} X) \simeq \pi_1(X).$$

Proof: 1st step: Construct the maps from $\pi_1(X)$ to $\mathrm{Deck}(\tilde{X} \to X)$ & vice versa.

Choose a base point x_0.

We ~~start~~ assume that φ is an element of $\pi_1(X, x_0)$. Want to construct $F_\varphi \in \mathrm{Deck}(\tilde{X} \to X)$.

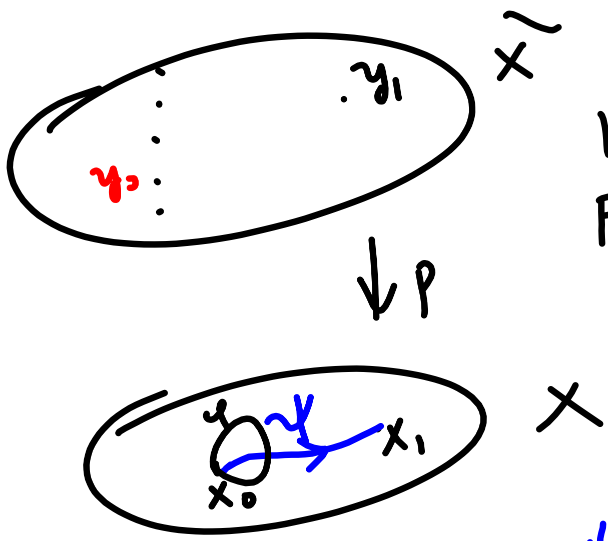


Choose any $y_0 \in p^{-1}(x_0)$.

We can lift φ to a unique $\tilde{\varphi}$, a curve in $\tilde{X}$ so that $\tilde{\varphi}(0) = y_0$

Define $F_\varphi(y_0) = \tilde{\varphi}(1)$

Check $F_\varphi(y_0) = y_0$ if φ is null-homotopic



What is $F_\varphi(y_1)$?

Define $x_1 = p(y_1) \in X$

Let ψ be any curve which connects $x_0 \to x_1$.

$$\varphi \longrightarrow \psi_\varphi = \psi \circ \varphi \circ \psi^{-1}$$

This is a closed curve ending at x_1.

Define

$$F_\varphi(y_1) := F_{\psi_\varphi}(y_1).$$

(Another ψ' then ~~isom~~ homotopic:

$\psi_\varphi \underset{\text{homotopic}}{\sim} \psi'_\varphi$ by $F_\varphi(y_1) \Rightarrow$

$\Rightarrow$ same image

$F_\varphi(y_1)$ is well-defined.)

Now check F_φ is a group homomorphism (Lift of $\varphi \circ \varphi_1$

= lift of φ first, & then lift of φ_1 starting from the end point of lift of φ.

F_φ is a homomorphism:

Show that if φ is not null-homotopic then F_φ is not identity.

(This follows from the fact, used in the construction of the universal covering that image of a closed null-homotopic curve in $\tilde{X}$ is null-homotopic in X.)

We proved a stronger claim:

$\forall y_0$: $F_\varphi(y_0) \neq y_0$.

So $\pi_1(X, x_0) \to \mathrm{Deck}(\tilde{X} \xrightarrow{p} X)$ is injective.

<u>Inverse map:</u>

$$\mathrm{Deck}(\tilde{X} \xrightarrow{p} X) \to \pi_1(X, x_0).$$

Choose $f \in \mathrm{Deck}(\tilde{X} \xrightarrow{p} X)$

Let y_0 be a point in $p^{-1}(x_0)$.

$y_1 = f(y_0) \in p^{-1}(x_0)$

Choose any curve $\tilde{\varphi}$ in $\tilde{X}$ which connects $y_0 \to y_1$.

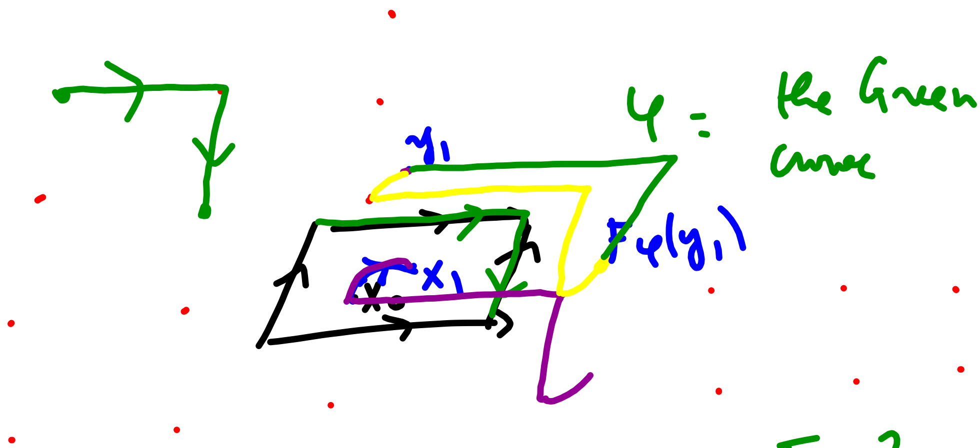
$\varphi = p(\tilde{\varphi})$ represents an element in $\pi_1(X, x_0)$.

Check that this does not depend on the choice of the curve $y_0 \to y_1$

(use that $\tilde{X}$ is simply connected.)

Check that these 2 maps are inverses to each other. $\square$

Example: Elliptic curve.



What is F_φ?

What is $F_\varphi(y_1)$?

Look at the purple curve!

Now show that $\pi_1(E) = \mathbb{Z} \oplus \mathbb{Z}$.

$\pi : \mathbb{C} \to E$ is covering map & $\mathbb{C}$ is simply connected

$\Rightarrow \pi : \mathbb{C} \to E$ is universal covering.

Now determine $\mathrm{Deck}(\mathbb{C} \xrightarrow[\pi]{} E) \subseteq \mathrm{Aut}(\mathbb{C})$.

Step 1: What is $\mathrm{Aut}(\mathbb{C})$?

HW: $\mathrm{Aut}(\mathbb{C}) = \{ f : \mathbb{C} \to \mathbb{C},\ f(z) = az + b,\ a, b \text{ constants},\ a \neq 0 \}$

Hint: Show that ∞ is a pole of any $f \in \mathrm{Aut}(\mathbb{C}) \Rightarrow$ extend to a map $\mathbb{P}^1 \to \mathbb{P}^1$.

$\mathrm{Aut}(\mathbb{P}^1) =$ Mobius maps.

(Rmk: The same argument applies to $\mathbb{C} \setminus$ points.)

Step 2: $\mathrm{Deck}(\mathbb{C} \xrightarrow[\pi]{} E)$

$= \{ f : \mathbb{C} \to \mathbb{C},\ f(z) = az + b,\ a \neq 0 \ \&\ \pi \circ f = \pi \}$

$= \{ f : \mathbb{C} \to \mathbb{C},\ f(z) = az + b,\ a \neq 0,\ az + b - z \in \Lambda\ \forall z \}$

$\Lambda = \{ m\vec{e_1} + n\vec{e_2} : m, n \in \mathbb{Z} \}$

So $a = 1$, $b \in \Lambda$

$$\Rightarrow \quad \mathrm{Deck}(\mathbb{C} \xrightarrow[\pi]{} E) \cong \Lambda = \mathbb{Z} \oplus \mathbb{Z}. \quad \square$$

Example 2: $\pi : \mathbb{C} \to \mathbb{C}^*$

$z \to e^z$

universal covering.

$\mathrm{Deck}(\mathbb{C} \xrightarrow{\pi} \mathbb{C}^*)$?

$\mathrm{Aut}(\mathbb{C}) \ni f(z) = az + b$

$$\pi \circ f = \pi$$

$$e^{az+b} = e^z \quad \forall z$$

$$\Rightarrow e^b = 1 \quad (\text{Choose } z = 0)$$

$$e^{z(a-1)} = 1 \quad \forall z$$

$$\Rightarrow \begin{cases} a = 1 \\ e^b = 1 \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = 2\pi i m, \\ m \in \mathbb{Z}. \end{cases}$$

$$\mathrm{Deck}(\mathbb{C} \to \mathbb{C}^*) \simeq \mathbb{Z} \Rightarrow \pi_1(\mathbb{C}^*) = \mathbb{Z}. \quad \square$$

Example: $\mathrm{Aut}(\mathbb{C}^*) = ?$ $\mathbb{C}^* = \mathbb{C}\setminus 0$

$f \in \mathrm{Aut}(\mathbb{C}^*)$, then f has no essential singularities. (Otherwise, it cannot be injective, by Casorati-Weierstrass.) ☺.

$\Rightarrow$ extend to $\mathbb{P}^1 \to \mathbb{P}^1$ an automorphism of $\mathbb{P}^1$.

$$\mathrm{Aut}(\mathbb{P}^1) = \left\{ \frac{az+b}{cz+d} : ad-bc \neq 0 \right\}$$

So $f(z) = \dfrac{az+b}{cz+d}$

& $f(\mathbb{C}^*) = \mathbb{C}^*$.

$f(\{0,\infty\}) = \{0,\infty\}$.

- Case 1: $f(0)=0$, $f(\infty)=\infty$
  $\Rightarrow f(z) = az+b$, $a \neq 0$.
- Case 2: $f(0)=\infty$, $f(\infty)=0$.
  $\Rightarrow f(z) = \dfrac{1}{cz+d}$, $c \neq 0$. $d=0$.

. . . .

Euler characteristic.

Let X be a compact R.S.
& X is represented by triangles.
Then Euler characteristic of X is

$$\chi(X) = \underset{\substack{\text{vertices}\\(\dim 0)}}{V} - \underset{\substack{\text{edges}\\(\dim 1)}}{E} + \underset{\substack{\text{faces}\\(\dim 2)}}{F}$$

$$\chi(X) = 2 - 2g$$

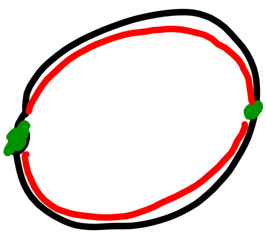
g = genus of X

Examples:

* 

$V = 2$, $E = 1$, $F = 0$

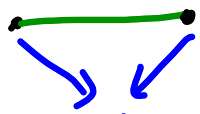
$\Rightarrow \chi = 1$

* circle 

$V = 2$, $E = 2$, $F = 0$

$\Rightarrow \chi = 0$

Another way

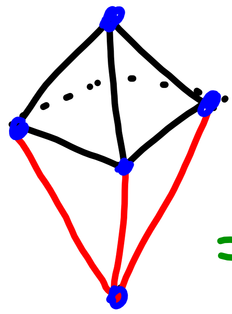


identify these 2 vertices we get a circle

$V = 1$, $E = 1$, $F = 0$,

$\chi = 0$.

* Riemann Sphere



$V = 5$

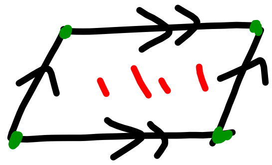
$E = 9$

$F = 6$

$\Rightarrow \chi = 2 = 2 - 2g$

$\Rightarrow 2g = 0 \Rightarrow g = 0$

* elliptic curve



$V = 1$

$E = 2$

$F = 1$

$\Rightarrow \chi = 0 = 2 - 2g$

$\Rightarrow 2g = 2 \Rightarrow g = 1.$

<u>HW</u>: look at theorem on classification of compact orientable surfaces by Euler characteristic.

(<u>Note</u>: Analytically, genus is not enough to determine the isomorphism class of a compact RS. Example: elliptic curves)

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§ 6. Sheaves

Sheaf is a way to work with local objects. (An application for this course is complex analytic extension, for example of $\sqrt{z}$)

<u>Presheaf</u>: X : topological space

A presheaf $\mathcal{F}$ of <u>Abelian groups</u> on X is a data:

- For each open set $U \subseteq X$, an Abelian group called $\mathcal{F}(U)$.
- If V is an open subset of U, then we have a <u>restriction map</u>:

$$\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V).$$

$\to$ group homomorphism

- Restriction map has some properties:

$$\rho^U_U = \text{identity.}$$

$$W \subseteq V \subseteq U$$

$$\Rightarrow \rho^V_W \circ \rho^U_V = \rho^U_W$$

(restriction U to W is the same as restrict U to V first & then V to W.)

If $f \in \mathcal{F}(U)$, & $V \subseteq U$ open, we write $f|_V$ for $\rho^U_V(f)$.

$X = \{a, b\}$, with discrete topology.

An open set of X is ϕ, $\{a\}$, $\{b\}$, $\{a, b\}$.

Define a presheaf $\mathcal{F}$ on X as follows:

$$\mathcal{F}(\phi) = 0$$

$$\mathcal{F}(\{a\}) = \mathcal{F}(\{b\}) = \mathbb{Z} \quad \mathcal{F}(\{a,b\}) = \mathbb{Z}$$

$$\begin{tikzcd}
 & \mathcal{F}(\{a\}) = \mathbb{Z} \arrow[dl, "0\text{-map}"'] & \\
0 = \mathcal{F}(\phi) & & \mathcal{F}(\{a,b\}) = \mathbb{Z} \arrow[ul] \arrow[ll, "0\text{-map}"'] \arrow[dl] \\
 & \mathcal{F}(\{b\}) = \mathbb{Z} \arrow[ul, "0\text{-map}"] &
\end{tikzcd}$$

HW: How many such presheaves $\mathcal{F}$ can you have? (Answer: $\mathbb{Z} \times \mathbb{Z}$)