14 November 2023

MAT 4800

We learned about holomorphic cochains before $\Rightarrow H^1(X,\mathcal{O}),\dots$

Now we use L^2-cochain.

<u>Def</u>: X a RS

Let $\mathcal{U}^* = \{U_i^*\}_{i\in I}$, finite I

$U_i^* \approx$ a disk in $\mathbb{C}$.

We don't assume that $\mathcal{U}^*$ is a covering of X.

$\mathcal{U} = (U_i)_{i\in I}$, assume $U_i \subseteq U_i^*$

$$C^0_{L^2}(\mathcal{U},\mathcal{O}) = \{ f \in C^0(\mathcal{U},\mathcal{O}) : f = (f_i)_{i\in I},\ f_i \in \mathcal{O}(U_i),$$
such that $\|f\|_{L^2(\mathcal{U})} = \sum_i \|f\|_{L^2(U_i)} \} < \infty$

$$C^1_{L^2}(\mathcal{U},\mathcal{O}) = \{ g \in C^1(\mathcal{U},\mathcal{O}) : g = (g_{ij}),\ g_{ij} \in \mathcal{O}(U_i\cap U_j)$$
$$\|g\|_{L^2(\mathcal{U})} = \sum_{i,j} \|g_{ij}\|_{L^2(U_i\cap U_j)} \}$$

<u>Corollary</u>: Let $\mathcal{V} \ll \mathcal{U}$

(meaning that $\mathcal{U} = (U_i)_{i \in I}$, $\mathcal{V} = (V_i)_{i \in I}$ ← same index

so that $V_i \subset\subset U_i$). $\boxed{\text{finite}} \leftarrow I$.

Then $\exists$ a <u>finite codimension</u> subspace H of $C^q(\mathcal{U}, \mathcal{O})$ such that $\forall f \in H$:

$$\|f\|_{L^2(\mathcal{V})} \leq \varepsilon \|f\|_{L^2(\mathcal{U})} . \square$$

We want to show that $H^1(X, \mathcal{O})$ is finite dimension. This implies that $\forall \mathcal{U}$ & $\mathcal{V} < \mathcal{U} \Rightarrow$ the image of $H^1(\mathcal{U}, \mathcal{O}) \to H^1(\mathcal{V}, \mathcal{O})$ is finite dimension.

So, first thing is to show that: if

$$\mathcal{V} \ll \mathcal{U} \Rightarrow H^1(\mathcal{U}, \mathcal{O}) \to H^1(\mathcal{V}, \mathcal{O})$$

is finite dimension.

The next 2 lemmas do this. (First prove for L^2, & then use that if $\mathcal{V} \ll \mathcal{U}$, & $f \in C^q(\mathcal{U}, \mathcal{O}) \Rightarrow f \in C^q_{L^2}(\mathcal{V}, \mathcal{O})$.

<u>Lemma 14.6</u>: Assume

$$\mathcal{W} \ll \mathcal{V} \ll \mathcal{U} \ll \mathcal{U}^*$$

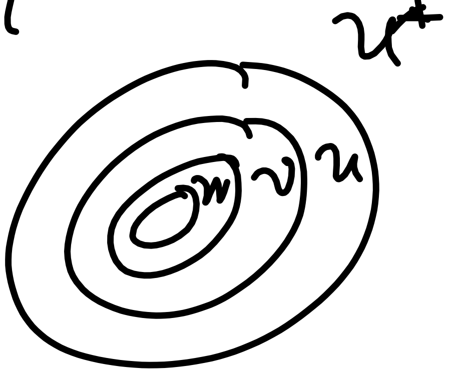
I: finite index set.

$\exists\, C > 0$ so that

$\forall\ \xi \in Z^1_{L^2}(\mathcal{V}, \theta)$, $\exists$

$\zeta \in Z^1_{L^2}(\mathcal{U}, \theta)$ & $\eta \in C^0_{L^2}(\mathcal{W}, \theta)$

so that: $\zeta = \xi + \delta\eta$ on $\mathcal{W}$.



$\Rightarrow \operatorname{Res} \zeta|_{\mathcal{W}}$

$= \operatorname{Res} \xi|_{\mathcal{W}}$

in $H^1(\mathcal{W}, \theta)$.

(ζ & ξ represents the same element in $H^1(\mathcal{W}, \theta)$.

Moreover:

$$\max\{\, \|\zeta\|_{L^2(\mathcal{U})},\ \|\eta\|_{L^2(\mathcal{W})} \,\} \le C\, \|\xi\|_{L^2(\mathcal{V})}.$$

First, construct ζ & η so that

$\xi = \zeta + \delta\eta$ (without norm estimate)

The idea is to use cut-off function:

Choose ψ such that $\operatorname{supp}(\psi) \subseteq \mathcal{V}$,

$\psi|_{\mathcal{W}} = 1$.

So if we know something on $\mathcal{V}$:

$$f = g$$

$$\Rightarrow \quad \Psi f = \Psi g \text{ on } \mathcal{W}.$$

Ψf is <u>defined</u> on $\mathcal{U}$

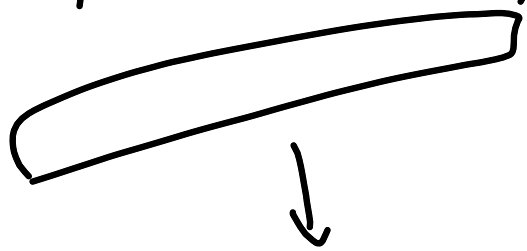
Then use Functional Analysis (Hahn-Banach theorem), given that L^2 is Hilbert space to have this:

This is this assignment:

$$(\xi, \eta) \longrightarrow \xi + \delta\eta$$

$$C_{L^2}(\mathcal{U}) \times C_{L^2}(\mathcal{W}) \longrightarrow C_{L^2}(\mathcal{V})$$

is linear, continuous & <u>surjective</u>.



ξ

preimage of ξ is a linear subspace called H_ξ. Then we choose $(\xi, \eta) \in H_\xi$ with smallest norm,

$$\left(\|\xi\|_{L^2(\mathcal{U})} + \|\eta\|_{L^2(\mathcal{W})} \right)$$

then this is bounded by $\|\xi\|_{L^2}$.

14.7 Lemma:

$$\mathcal{W} << \mathcal{V} << \mathcal{U} << \mathcal{U}^*$$

$\Rightarrow \exists$ finite dimensional vector space $S \subseteq Z^1(\mathcal{U}, \vartheta)$ such that: $\forall\, \xi \in Z^1(\mathcal{U}, \vartheta)$, $\exists\, \sigma \in S$ & $\eta \in C^0(\mathcal{W}, \vartheta)$ such that

$$\sigma = \xi + \delta\eta \text{ on } \mathcal{W}.$$

(Meaning $\sigma = \xi$ represents the same element in $H^1(\mathcal{W}, \vartheta)$. So

$$\text{image}\left(H^1(\mathcal{U}, \vartheta) \to H^1(\mathcal{W}, \vartheta)\right)$$

$$= \text{image}\left(\underbrace{S}_{\text{finite dim}} \to H^1(\mathcal{W}, \vartheta)\right)$$

$\Rightarrow$ has finite dim.)

Proof:
Can think about ξ like in $Z^1(\mathcal{V}, \vartheta)$ (by restriction.)

Lemma 14.6 gives:

$$\zeta_0 = \xi + \delta\eta_0 \text{ on } \mathcal{W}$$

where $\|\zeta\|_{L^2(\mathcal{U})}, \|\eta\|_{L^2(\mathcal{V})} \leq C\|\xi\|_{L^2(\mathcal{V})}$.

$$\zeta_0 = \xi_0 + \sigma_0$$

where $\xi_0 \in A_\varepsilon$, $\sigma_0 \in S_\varepsilon$

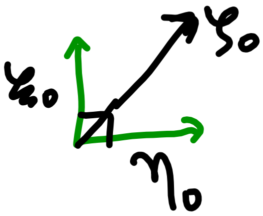
$$L^2 = A_\varepsilon \oplus S_\varepsilon$$

S_ε finite dimension

& $\forall f \in A_\varepsilon$:

$$\|f\|_{L^2(\vartheta)} \le \varepsilon \|f\|_{L^2(\mathcal{U})}.$$

(Lemma 14.3)



$$\Rightarrow \|\xi_0\|, \|\eta_0\| \le \|\zeta_0\|$$

$$\|\xi_0\|_{L^2(\mathcal{U})} \le \|\zeta_0\|_{L^2(\mathcal{U})}$$

& $$\|\xi_0\|_{L^2(\vartheta)} \le \varepsilon \|\xi_0\|_{L^2(\mathcal{U})} \le \varepsilon \|\zeta_0\|_{L^2(\mathcal{U})} \le C\varepsilon \|\zeta\|_{L^2(\vartheta)}$$

We will choose ε small enough w.r.t. C so that later we will have a convergent geometric series.

$$\boxed{\xi}$$

$$\zeta_0 = \xi + \delta\eta_0 \qquad (14.6)$$

$$\zeta_0 = \boxed{\xi_0} + \sigma_0 \cap S \quad \text{finite dim} \qquad (14.3)$$

$$\zeta_1 = \xi_0 + \delta\eta_1 \qquad (14.6)$$

$$\zeta_1 = \boxed{\xi_1} + \sigma_1 \cap S \quad \text{finite dim}$$

$\vdots$

Also have bounds on norms of $\zeta_0, \zeta_1, \zeta_2 \ldots \xi_0, \xi_1, \xi_2, \ldots, \sigma_0, \sigma_1, \sigma_2, \ldots, \eta_1, \eta_2 \ldots$

So $\xi = \zeta_0 - \delta\eta_0 = \xi_0 + \sigma_0 - \delta\eta_0$
$= \zeta_1 - \delta\eta_1 + \sigma_0 - \delta\eta_0$

$$= \xi_1 + \sigma_1 - \delta\eta_1 + \sigma_0 - \delta\eta_0$$

$$= \zeta_2 - \delta\eta_2 + \sigma_1 - \delta\eta_1 + \sigma_0 - \delta\eta_0 \ldots$$

$\vdots$

$$= \zeta_k + (\sigma_{k-1} + \ldots + \sigma_0) - \delta(\eta_k + \ldots + \eta_0)$$

$$\|\sigma_k\| \leq 2^{-k} \|\xi\|$$

$$\|\eta_k\| \leq 2^{-k} \|\xi\|$$

$$\|\zeta_k\| \leq 2^{-k} \|\xi\|$$

if ε is small enough.

$\xi_k \to 0$ as $k \to \infty$

$\sigma_k + \dots + \sigma_0$ converges absolutely

& limit point σ is in S.

$\eta_k + \dots + \eta_0$ is also converge absolutely.

to some η.

$\Rightarrow \xi = \underset{\underset{S}{\uparrow}}{\sigma} - \underbrace{\delta\eta}_{\text{exact}}$ as wanted

S finite dimension. $\square$

Cor: image $(H^1(U,\mathcal{O}) \to H^1(W,\mathcal{O}))$

is finite dimensional.

Thm 14.9: X RS, $Y_1 \subset\subset Y_2 \subset X$

open. Then image of $H^1(Y_2,\mathcal{O}) \to$

$H^1(Y_1,\mathcal{O})$ is finite dimensional.

Proof:

Y_1 is relatively compact $\Rightarrow$ Y_1 can be covered by a finite set of simple disks.

$Y_1 \subset \bigcup_{i \in I} W_i$, I finite.

Then we can choose:

$$Y_1 \subseteq \mathcal{W} = (W_i) \ll \mathcal{V} \ll \mathcal{U} \ll \mathcal{U}^* \subseteq Y_2$$

We have by Leray's theorem:

$$H^1(U_i, \mathcal{O}) = H^1(W_i, \mathcal{O}) = 0 \ \forall i.$$

$$\Rightarrow H^1(Y_1, \mathcal{O}) = H^1(\mathcal{W}, \mathcal{O})$$

& ~~Image (H~~ $\text{Image}(H^1(Y_2, \mathcal{O}) \to H^1(Y_1, \mathcal{O})) = \text{Image}(H^1(\mathcal{U}, \mathcal{O}) \to H^1(Y_1, \mathcal{O}) = H^1(\mathcal{W}, \mathcal{O}))$ is finite dim (by 14.7). $\square$

Cor: If X is compact RS $\Rightarrow \dim(H^1(X, \mathcal{O})) < \infty$.

Proof: Choose $Y_1 = Y_2 = X$. Then $H^1(Y_1, \mathcal{O}) \to H^1(Y_2, \mathcal{O})$ is the identity map, and has finite dim image. $\square$

Def: If X is compact RS, then $g = \dim H^1(X, \mathcal{O})$ is the <u>genus</u> of X.

Before we learned that if $X \subseteq \mathbb{P}^2$ is smooth compact RS, & X has degree n then:

$$g = \frac{(n-1)(n-2)}{2}.$$

Example: E = Elliptic curve is defined by an equation of degree n=3 in $\mathbb{P}^2$.

$$\Rightarrow g(E) = \frac{(3-1)(3-2)}{2} = 1.$$

$$\Rightarrow \dim H^1(E, \mathcal{O}) = \cancel{2}\ 1.$$

Another way to compute is to use Riemann - Hurwitz theorem:

Assume X & Y are compact RS. Assume we know g(Y), & $\exists$ a non-constant holomorphic map $f: X \to Y$.

then $\chi(X) = n\,\chi(Y) - \sum_{p \in X}(e_p - 1)$

n = degree of f. $\boxed{\chi = 2 - 2g}$

Thm: X compact RS $\Rightarrow \exists$ a non-constant holomorphic map $f: X \to \mathbb{P}^1$.

Example:

if $X \subseteq \mathbb{P}^2$, then just use a linear projection (like $(x,y) \to x$)

To see what is the local degree of this map, need to check if x or y is a local coordinate for X. If x is local coordinate then the map is $x \to x$, so local degree is 1. If y is local coordinate, then the map must be written similar to $y \to y^k$ & now k can be ≥ 2. In the latter case, to see what could be k, one can do using

Implicit Differentiation to compute:

$$\frac{dx}{dy}$$

$$\frac{d^2x}{dy^2}$$

$$\vdots$$

until one gets non-zero.

Example: look at

$\mathbb{C}^2 \supseteq X = \{ y^5 = x^3 + x + 1 \}$

& $\pi : X \to \mathbb{C}$

$(x,y) \mapsto x$.

$$f = \cancel{y^3}\, x^3 + x + 1 - y^5$$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (3x^2+1, -5y^4)$$

$$\nabla f = (0,0) \iff \begin{cases} 3x^2+1=0 \\ y=0 \end{cases}$$

These points are not on X.

$\Rightarrow$ X is smooth.

Critical point of $\pi : X \to \mathbb{C}$, $(x,y) \mapsto x$

$$\nabla f = (3x^2+1, -5y^4)$$

x is local coordinate for X if $-5y^4 \neq 0$. In that case, π is a local isomorphism $x \to x$, & hence local degree is 1.

if $y = 0$ then x ~~cannot~~ is not a local coordinate, $x = h(y)$. The map $\pi(x,y) = x$, is now $y \to h(y) = x$.

$y=0$
$\Rightarrow x^3+x+1=0$
$\Rightarrow$ 3 points

$$X = \{ y^5 = x^3 + x + 1 \}$$

Implicit differentiation:

$\Rightarrow$
$$\frac{\partial}{\partial y}(x^3+x+1) = 5y^4$$

$$\left(x' = \frac{dx}{dy} \right) \qquad 3x^2 x' + x' = 5y^4.$$

At $y=0$ & $x^3+x+1=0 \Rightarrow 3x^2+1 \neq 0$
$\Rightarrow x' = 0$. These 3 points
$(y=0,\ x^3+x+1=0)$ are critical points!

$\Rightarrow k \geq 2$. Next:

Again Implicit Differentiate $3x^2 x' + x' = 5y^4$
$$\Rightarrow 6x \cdot (x')^2 + 3x^2 \cdot x'' + x'' = 20y^3$$

We know at $(y=0,\ x^3+x+1=0)$
then $x' = 0$ & $3x^2+1 \neq 0$
$$\Rightarrow x'' = 0.$$

$k \geq 3$.

Similarly:
$$(\ldots)x' + (\ldots)x'' + (3x^2+1)x''' = 60y^2$$
$$\Rightarrow x''' = 0 \Rightarrow k \geq 4.$$

HW: What is k?

$$(\ldots)x' + (\ldots)x'' + (\ldots)x''' + (3x^2+1)\,x'''' = 120y$$

again 0

$$(\ldots)x' + (\ldots)x'' + (\ldots)x''' + \cancel{13}(\ldots)x'''' + (\ldots)x''''' = 120 \neq 0$$

$$x''''' \neq 0.$$