15 November 2023
MAT 4800

<u>Thm 14.12:</u> $Y \subset\subset X$ RS.

For every $a \in Y$, $\exists f \in \mathcal{M}(Y)$ so that f has a pole at a & f is holomorphic on $Y \setminus \{a\}$

<u>Example:</u> $X = \mathbb{C}$, $Y = \mathbb{D} = \{z : |z| < 1\}$
$a = 0$.

Then $f(z) = \frac{1}{z}$ is what we want.

In general, may be complicated to construct such functions.

<u>Proof:</u> From previous theorems

$$k = \dim\left(\mathrm{Im}\left(H^1(X, \mathcal{O}) \to H^1(Y, \mathcal{O})\right)\right)$$

is finite.

The idea is to use this & use the local constructions like in the example, to get a global one in $\mathcal{M}(Y)$ satisfying what we want.

Choose $U_1 = \mathbb{D}$, $a = 0 \in \mathbb{D}$.

Then let $U_2 = X \setminus \{a\}$.

$\mathcal{U} = (U_1, U_2)$ is an open covering of X.

The function z^{-j} ($j = 1, \dots, k+1$) are holomorphic on $U_1 \cap U_2 = \mathbb{D} \setminus \{0\}$.

So $z^{-j} \in C^1(\mathcal{U}, \mathcal{O})$. Actually $\xi_j = z^{-j} \in Z^1(\mathcal{U}, \mathcal{O})$. (HW: check that it is a ~~cycle~~ δ-closed cycle.)

Call $\tilde{\xi}_1, \dots, \tilde{\xi}_{k+1}$ be the restriction to Y (meaning, w.r.t. $\tilde{\mathcal{U}} = (U_1 \cap Y, U_2 \cap Y)$ an open covering of Y).

(their cohomology) They belong to $\operatorname{Im}(H^1(X, \mathcal{O}) \to H^1(Y, \mathcal{O}))$

which has dimension k.

It means that we can find $c_1, \dots, c_k \in \mathbb{C}$, not all 0, so that:

$$c_1 \xi_1 + \dots + c_k \xi_k + c_{k+1} \xi_{k+1} = 0$$

in $H^1(Y, \mathcal{O})$

$$\Rightarrow c_1 \xi + \dots + c_k \xi_k + c_{k+1} \xi_{k+1} = \delta \eta$$

$\eta \in C^0(\tilde{\mathcal{U}}, \mathcal{O})$.

$\eta = (f_1, f_2)$, $f_1 \in \mathcal{O}(U_1 \cap Y)$, $f_2 \in \mathcal{O}(U_2 \cap Y)$

& $\delta \eta = f_1 - f_2 |_{U_1 \cap U_2}$.

$$c_1\zeta_1 + c_2\zeta_2 + \dots + c_k\zeta_k + c_{k+1}\zeta_{k+1}$$
$$= f_1 - f_2 \quad \text{on } U_1 \cap U_2 \cap Y$$

Now define:

$$g = \begin{cases} -(c_1\zeta_1 + c_2\zeta_2 + \dots + c_k\zeta_k + c_{k+1}\zeta_{k+1}) + f_1 & \text{on } U_1 \cap Y \\ f_2 & \text{on } U_2 \cap Y \end{cases}$$

(Recall: z^{-j} is defined on $U_1 = \mathbb{D}$

$g \in \mathcal{M}(Y)$.

g has a pole of order at least 1 & at most $k+1$ at $a = 0$.

$g \in \mathcal{O}(Y \setminus \{a\})$.

<u>Corollary</u>: If $a \in X$ is a compact RS, & $k = \dim(H^1(X, \mathcal{O}))$, then there is a meromorphic function on X with pole at a of order between $[1, k+1]$, & is holomorphic otherwise.

<u>Example</u>: E = elliptic curve $\Rightarrow g = 1$.

Then $\exists$ $f : E \to \mathbb{P}^1$ holomorphic so that f has a pole of order between $[1, 2 = g+1]$, & f is holomorphic elsewhere.

f cannot have a pole of order 1. [otherwise $f^{-1}(\infty)$ = a point with multiplicity 1] $\Rightarrow f$ must have degree 1

$\Rightarrow E \simeq \mathbb{P}^1$ (a contradiction)

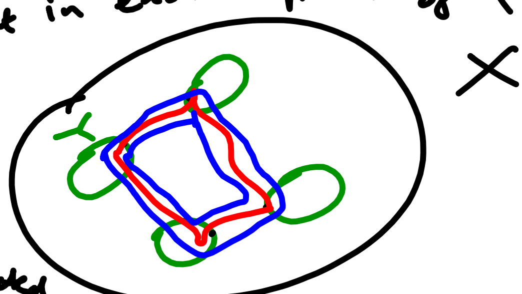
Weierstrass function $f: E \to \mathbb{P}^1$ has order 2.

<u>Corollary</u> (14.13) (Interpolation) Suppose X compact RS & $a_1, \dots, a_n$ are distinct points on X. Let $c_1, \dots, c_n \in \mathbb{C}$ (or $\mathbb{P}^1$). Then $\exists$ $f: X \to \mathbb{P}^1$ holomorphic, so that $f(a_i) = c_i$ $\forall i$.

<u>Proof</u>: Use Thm 14.12 & polynomial interpolation. $\square$

<u>Cor 14.14</u>: $Y \subset\subset X$ connected non-compact RS. Then $\exists$ $f \in \mathcal{O}(Y)$ so that f is not constant on any connected component of Y.

Proof: Choose a point in each component of Y.



Since X is connected $\Rightarrow$ path-connected, we can connect these points by curves (in Red). Then choose small open neighborhoods of these curves, we get $Y \subset\subset Y' \subset\subset X$, & Y' is connected.

Then apply Thm 14.12 to Y' & $a \in Y' \setminus Y$, we find $f' \in \mathcal{M}(Y')$ so that f' has a pole at a & f' is in $\mathcal{O}(Y' \setminus Y)$.

Since Y' is connected $\Rightarrow$ f' is not constant in any open subset of Y'.

$\Rightarrow$ f' is not constant on any component of Y.

$$f = f'|_Y \in \mathcal{O}(Y). \quad \square$$

The next results can be proven similarly.

<u>Thm 14.15</u> (stronger version of 14.12): $Y \subset\subset Y' \subset X \to$ non-compact RS.

then $\operatorname{Im}(H^1(Y', \mathcal{O}) \to H^1(Y, \mathcal{O})) = 0$!

<u>Rmk</u>: Non-compactness is used because we need Corollary 14.14. (Also, it is necessary: if X is compact & $Y \subset\subset X = Y' = Y$, ~~then~~ $Y = X$, $\operatorname{Im}(H^1(X,\mathcal{O}) \to H^1(X,\mathcal{O})) \neq 0$.

Compactness is needed in Cor 14.14 because we want a point $a \in Y' \setminus Y$. If X is compact & $Y = X$, there is no such a!!

Also, we know by maximum principle, if X is compact, then any holomorphic $f : X \to \mathbb{C}$ is constant.

* Elliptic curve:

1st way to define: $\mathbb{C}/\Lambda$

$\Lambda = \mathbb{Z}[e_1, e_2]$, $e_1, e_2 \in \mathbb{C}$, not $\mathbb{R}$-linear dependent.

2nd way to define : E = the closure in $\mathbb{P}^2$ of the affine curve

$$y^2 = x^3 + ax + b, \quad a, b \in \mathbb{C}$$

a, b must satisfy some conditions so that we get a smooth curve.

There is a way to go between (e_1, e_2) & (a, b).

Reference: Joseph Silverman, The arithmetic of elliptic curves

<u>Cor 14.16</u>: X = non-compact RS, $Y \subset\subset Y' \subset X$. For every $\omega \in \mathcal{E}^{0,1}(Y')$, $\exists f \in \mathcal{E}(Y)$ so that $d''f = \omega|_Y$.

$\mathcal{E}^{0,1}(Y) = \{ \omega$: 1-form on Y, locally $\omega = h\, d\bar{z} \}$

$$d'' f = \frac{\partial f}{\partial \bar{z}} d\bar{z}$$

$$\omega \in \mathcal{E}^{0,1}(Y) \Rightarrow d''\omega = 0$$

$$\left(d''\omega = \frac{\partial h}{\partial \bar{z}} \underbrace{d\bar{z} \wedge d\bar{z}}_{0} \right)$$

$\Rightarrow \omega$ is d''-closed on Y'.

Corollary says that then ω is d''-exact but only on Y. $\square$

<u>How to construct holomorphic 1-forms on RS</u>:

$X = RS$

Then $\pi : \tilde{X} \to X$ universal covering

$\tilde{X}$ is simply-connected RS.

$$\tilde{X} = \begin{cases} \mathbb{C} \\ \mathbb{D} \\ \mathbb{P}^1 \end{cases}$$

If ω is a holomorphic 1-form on X

$\Rightarrow$ $\tilde{\omega} = \pi^*(\omega)$ is a holomorphic 1-form on $\tilde{X}$.

Moreover, (using $(g \circ h)^* \omega = h^*(\cancel{g}^* \omega)$)

if $\varphi \in \mathrm{Deck}(\tilde{X} \xrightarrow[\pi]{} X)$ then:

$$\varphi^* \tilde{\omega} = \tilde{\omega}.$$

$$\begin{tikzcd} \tilde{X} \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & \tilde{X} \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

$$\pi \circ \varphi = \pi$$

$$\Rightarrow \quad \tilde{\omega} = \pi^* \omega = (\pi \circ \varphi)^* (\omega) = \varphi^* \circ (\pi^* \omega) = \varphi^* (\tilde{\omega}).$$

$\tilde{\omega}$ is invariant by deck transformations.

Conversely, if $\tilde{\omega}$ is holomorphic 1-form on $\tilde{X}$ invariant under deck transformations then $\exists$ ω holomorphic 1-form on X so that $\tilde{\omega} = \pi^*(\omega)$.

We can define ω as follows:

let $p \in X$ & $\tilde{p} \in \pi^{-1}(p)$.

then π is covering map $\Rightarrow$ $\exists$ $p \in U \subseteq X$ & $\tilde{p} \in \tilde{U} \subseteq \tilde{X}$ so that $\pi|_{\tilde{U}} : \tilde{U} \to U$ is isomorphism.

then $\omega|_U \overset{\text{def}}{=} (\pi|_{\tilde{U}})_*(\tilde{\omega})$

$$= \left(\left(\pi|_{\tilde{U}} \right)^{-1} \right)^*(\tilde{\omega})$$

For $\mathbb{C}$: Any holomorphic 1-form has the formula $\omega = f(z)\,dz$, $f : \mathbb{C} \to \mathbb{C}$ holomorphic

For $\mathbb{D}$: $\omega = f(z)\,dz$, $f : \mathbb{D} \to \mathbb{C}$ holomorphic.

For $\mathbb{P}^1$: Claim if ω is a holomorphic 1-form on $\mathbb{P}^1 \Rightarrow \omega = 0$.

(hint - HW: 1 form on $\mathbb{P}^1$ as usual,

Assume ω holomorphic

let $\mathbb{P}^1 = U_1 \cup U_2$

$U_1 \simeq \mathbb{C}_z$, $U_2 \simeq \mathbb{C}_w$.

$$\omega_1 = \omega|_{U_1} = f(z)\,dz, \quad f : \mathbb{C} \to \mathbb{C} \text{ holomorphic}$$

$$\omega_2 = \omega|_{U_2} = f\left(\frac{1}{w}\right) d\left(\frac{1}{w}\right) = f\left(\frac{1}{w}\right) \frac{dw}{-w^2}.)$$

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Example: Holomorphic 1-forms on E = elliptic curve.

$\pi : \mathbb{C} = \tilde{E} \longrightarrow E = \mathbb{C}/\Lambda$

Deck transformation $= \{ \varphi_b : \mathbb{C} \to \mathbb{C}$ biholomorphic, $z \mapsto z + b,\ b \in \Lambda \}$.

$\tilde{\omega}$ holomorphic 1-form on $\mathbb{C}$

$\tilde{\omega} = f(z)\, dz$.

If $\tilde{\omega}$ is invariant by Deck transformations

$\Rightarrow \varphi_b^*(\tilde{\omega}) = \tilde{\omega} \quad \forall b \in \Lambda$.

$$\varphi_b^*(\tilde{\omega}) = \varphi_b^*(f(z)dz)$$
$$= f(\varphi_b(z))\, d(\varphi_b(z))$$
$$= f(z+b)\, dz$$

$\varphi_b^*(\tilde{\omega}) = \tilde{\omega} \Leftrightarrow f(z+b) = f(z) \quad \forall z \in \mathbb{C}. \ \forall b \in \Lambda.$

HW: $\Rightarrow f$ is a constant!

$\Rightarrow \omega$ is a holomorphic 1-form on E

$\Leftrightarrow \omega = \pi_*(\tilde{\omega})$ for $\tilde{\omega} = c\, dz$ on $\mathbb{C}$, c a constant

The space of holomorphic 1-forms on E has $\mathbb{C}$-dim $1 = g =$ genus of E.

The space of holomorphic 1-forms on $\mathbb{P}^1$ has $\mathbb{C}$-dim $0 = g =$ genus of $\mathbb{P}^1$.

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§15. Exact cohomology sequence (short)

We already know about exact sequence of sheaves & associated long exact sequence in cohomology. Here we study in detail.

More examples of sheaf homomorphisms.

$*$ $d : \mathcal{E} \to \mathcal{E}^{(1)}$. $(d = d' + d'')$

$$df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z} = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$f(x,y) = xy = \left(\frac{z+\bar{z}}{2}\right)\left(\frac{z-\bar{z}}{2i}\right)$$

$$= \frac{1}{4i}\left(z^2 - \bar{z}^2\right)$$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = y\,dx + x\,dy$$

$$= \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z} = \frac{1}{4i}(2z)\,dz + \frac{1}{4i}(-2\bar{z})\,d\bar{z}$$

$$d : \underset{\substack{\uparrow \\ \text{1-forms}}}{\mathcal{E}^{(1)}} \to \mathcal{E}^{(2)}$$

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* $\mathcal{O} \hookrightarrow \mathcal{E}_X$

$\mathbb{C}_X \hookrightarrow \mathcal{E}_X$

$\mathbb{Z} \hookrightarrow \mathcal{O}_X$

$\Omega \hookrightarrow \mathcal{E}^{1,0}$ → sheaf of smooth 1-forms of type(1,0) → locally $f(z,\bar{z})\,dz$ (smooth function)

Ω → sheaf of holomorphic 1-forms → Locally: $f(z)\,dz$ (holomorphic)

* $\mathcal{O} = \operatorname{Ker}(\mathcal{E} \xrightarrow{d''} \mathcal{E}^{0,1})$ $\frac{\partial}{\partial \bar{z}} = 0$

(Cauchy - Riemann equations)

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* $\Omega = \operatorname{Ker}(\mathcal{E}^{1,0} \xrightarrow{d} \mathcal{E}^{(2)})$

$\mathcal{E}^{1,0}$ = locally $\omega = f(z,\bar{z})\,dz$

$$d\omega = \left(\frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}\right) \wedge dz$$

$$= \frac{\partial f}{\partial \bar{z}}\, d\bar{z} \wedge dz$$

$$\Rightarrow d\omega = 0 \iff \frac{\partial f}{\partial \bar{z}} = 0 \iff f \in \mathcal{O}$$

$$\iff \omega \in \Omega$$

* $\mathbb{Z}_X = \operatorname{Ker}(\mathcal{O} \xrightarrow{\exp} \mathcal{O}^*)$

$f \mapsto e^{2\pi i f}$

locally $e^{2\pi i f} \equiv 1 \iff 2\pi i f \in 2\pi i\,\mathbb{Z}$

f holomorphic & f has discrete image $\Rightarrow$ f must be constant.

$* \quad 0 \to \mathcal{O} \hookrightarrow \mathcal{E} \xrightarrow{d''} \mathcal{E}^{0,1} \to 0$

is an exact sequence. Need to look at: $0 \to \mathcal{O} \to \mathcal{E}$, $\mathcal{O} \to \mathcal{E} \to \mathcal{E}^{0,1}$, $\mathcal{E} \to \mathcal{E}^{0,1} \to 0$

To check this, need to check for a small open set $U \in X$, which we can assume to be simply connected. Can assume $U = \mathbb{D} \subseteq \mathbb{C}$.

$\underline{\mathcal{O} \to \mathcal{E} \to \mathcal{E}^{0,1}}$: What is kernel of $\left(\mathcal{E}(U) \xrightarrow{d''} \mathcal{E}^{0,1}(U)\right)$

$U = \mathbb{D}$.

$f \in \mathcal{E}(U) \Rightarrow d''f = \frac{\partial f}{\partial \bar{z}} d\bar{z}$

$d''f = 0 \Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0 \Leftrightarrow f \in \mathcal{O}(U)$.

So $\mathcal{O} = \operatorname{Ker}\left(\mathcal{E} \to \mathcal{E}^{0,1}\right)$.

$\underline{\mathcal{E} \xrightarrow{d''} \mathcal{E}^{0,1} \to 0}$:

Let $\omega \in \mathcal{E}^{0,1}(U)$

$\Rightarrow \omega = f(z, \bar{z})\, d\bar{z}$, where

$* \quad f(z, \bar{z}) \in \mathcal{E}(U)$, $U = \mathbb{D}$.

By Dolbeault's lemma: $\exists\, g \in \mathcal{E}(U)$ so that $\frac{\partial g}{\partial \bar{z}} = f(z, \bar{z})$.

$\Rightarrow \quad d'' g = \omega$.

$\Rightarrow \omega \in$ Image of $\left(\mathcal{E} \xrightarrow{d''} \mathcal{E}^{0,1}\right)$.

$\square$