MAT 4800
17 October 2023

Riemann - Hurwitz theorem:
X & Y be compact R.S.
Let $f: X \to Y$ be a non-constant holomorphic map of degree $N \geq 1$.
Let e_x $(x \in X)$ be the multiplicity of f at x. [So for all ~~of x~~, except a finite number of x, then $e_x = 1$.]

$$\Rightarrow \chi(X) = N \cdot \chi(Y) - \underbrace{\sum_{x \in X} (e_x - 1)}$$

(the summand is 0 for most x, except critical points where $e_x > 1$).

Example: There is no non-constant holomorphic map $f: X \to Y$ if genus (X) < genus of (Y).

$$\chi(X) = 2 - 2g(X)$$
$$\chi(Y) = 2 - 2g(Y)$$

$$\chi(X) = N\,\chi(Y) - \sum_{x\in X}(e_x - 1)$$

$$2 - 2g(X) = N(2 - 2g(Y)) - \sum_{x\in X}(e_x - 1)$$

$$2 + \underbrace{\sum_{x\in X}(e_x-1)}_{\geq 0} + 2Ng(Y) = 2N + 2g(X)$$

But $1 + Ng(Y) > 1 + g(X)$ if $g(Y) > g(X)$, contradiction $\square$

Example 2: If $g(X) > 1$, then every non-constant holomorphic map
$$f : X \to X$$
is an automorphism (i.e. $\deg(f) = 1$).

$$2 + 2Ng(X) \leq 2N + 2g(X)$$
$$\Rightarrow \quad 2g(X)(N-1) \leq 2(N-1)$$

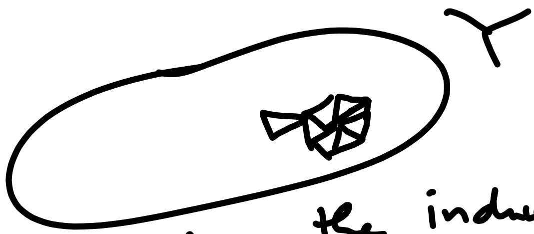
Since $g(X) > 1$ & $N \geq 1$, the above is possible iff $N = 1$. $\square$

Proof of RH:

Special case: if f has no critical points

So preimage of any point has exactly N points. $\Rightarrow f$ is a covering map

We triangulate X by small triangles so that $f^{-1}(\Delta)$ = disjoint union of N "triangle" & give a triangulation for X.



So roughly speaking the induced triangulation for X has N times points, N times edges, N times surfaces.

$\Rightarrow \chi(X) = N\chi(Y)$.

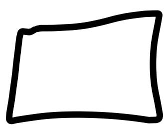
What if we have a critical point:

Let's look at what happens

At

 $\longrightarrow \mathbb{D}$

$z \mapsto z^k$

$\mathbb{D} \underset{\text{topologically}}{\sim}$ 

$\Rightarrow \chi(\mathbb{D}) = 4 - 4 + 1 = 1$

Check with this case: 1 critical point 0, where $e_0 = k$. For other x's: $e_x = 1$

RH is satisfied in this case:

$$\underset{\substack{\| \\ 1}}{\chi(\mathbb{D})} = \underset{\substack{\| \\ k}}{k\,\chi(\mathbb{D})} - \underset{\substack{\| \\ (k-1)}}{\sum_{x\in\mathbb{D}}(e_x - 1)}$$

Use the above 2 special cases & the additivity of Euler characteristic to conclude.

$$X \setminus f^{-1}(\text{critical values}) \longrightarrow Y \setminus \text{critical values}$$

is a covering map, so we can use the first special case.

$$\chi(X \setminus f^{-1}(\text{critical values}))$$

$$= N \quad \chi(Y \setminus \text{critical values})$$

By additivity:

$$\chi(X \setminus f^{-1}(\text{critical values}))$$

$$= \chi(X) - \# f^{-1}(\text{critical values})$$

§7. Holomorphic continuation.

$$\begin{tikzcd} & {|\mathcal{O}_X|} \arrow[d, "\pi"] \\ {[0,1]} \arrow[ur, dashed, "F"] \arrow[r, "u"'] \arrow[ur, phantom, "\circlearrowleft"] & X = RS \end{tikzcd}$$

if there is a lift F then F is called a holomorphic continuation of the germ $F(0)$ along the curve $u[0,1]$.

If $F(0) = F(1)$, then the germ $F(0)$ at $u(0) = u(1)$ can be extended to a holomorphic map in a neighborhood of the curve $u[0,1]$.

$F(0)$ $\downarrow$ convergent series

<u>Example:</u> Consider the germ $\sqrt{z}$ near $z_0 = 1$.

We cannot extend it to a neighborhood of the unit circle. What should we do? What we can do is this:

Let Y be the ~~connected~~ component of $|\mathcal{O}_X|$ which contains $\sqrt{z}$ near $z_0 = 1$.

Then we try to lift $\sqrt{z}$ to Y along ~~[0,1]~~ the unit circle. (So the lift above $u(1)$ will be -1.)

What about $\sqrt[3]{z}$?

Remark: This Y is the maximal analytic continuation of (X, x, f_x)

$f_x \Downarrow$ a germ at x

(Meaning: any lift of (X, x_0, f_{x_0}) is a subspace of Y).

§ 8. Holomorphic continuation of algebraic functions.

like $\sqrt{z}$, $\sqrt[3]{z-1}$, $\sqrt{z^3-1}$

$y = \sqrt{z^3-1}$
$\Rightarrow y^2 = z^3 - 1$
$\Downarrow$
elliptic curve.

In this section, we kind of looking at "minimal" liftings of holomorphic germs.

Thm 8.9: Let $P(T) = T^n + c_1 T^{n-1} + \ldots + c_n \in \mathcal{M}(X)[T]$.

(Meaning $c_1, \ldots, c_n$ are meromorphic functions on X.)

Then $\exists$ a ~~mero~~ RS Y & a meromorphic function $f \in \mathcal{M}(Y)$,

& an n-sheeted map
$$\pi : Y \to X$$
so that
$$(\pi^* P)(f) = 0.$$
$$(\pi^* P) \in \mathcal{M}(Y)[T]$$
defined by:
$$\pi^* P(y, T) = T^n + c_1(\pi(y))T^{n-1} + c_2(\pi(Y))T^{n-2} + \dots + c_n(\pi(Y)).$$
$\pi^* c(y) = c(\pi(y))$ is the pullback of $c(x)$.

$$\begin{tikzcd} Y \arrow[r, "\pi"] \arrow[rr, bend right, "\pi^* c"'] & X \arrow[r, "c"] & \mathbb{C} \end{tikzcd}$$

<u>Example:</u> Want to define $\sqrt{x^3 - x}$.

Define a new variable T so that
$$T^2 - (x^3 - x) = 0.$$
Then $P(T, x) = T^2 - (x^3 - x)$.

If Y & f is from Thm 8.9, then f "defines" $\sqrt{x^3 - x}$.

Thm 8.9 is stronger than inverse function theorem in 2 aspects: 1. it is global 2. it treats also critical points.

Example 2: if $p(T)$ is not a polynomial, may be not exist. ($p(T) = e^T - z$.)

Proof of Thm 8.9:

① Define discriminant
$X \supseteq \Delta = \{ x \in X : P(T) = 0$ has multiple roots in $T \}$

Example: $P(T) = T^2 - (x^3 - x)$.
What is Δ? $T^2 - c$ has multiple roots iff $c = 0$.
So $\Delta = \{ x \in \mathbb{C} : x^3 - x = 0 \}$.

$|\mathcal{O}_X| \supseteq Y' = \{ \varphi \in \pi^{-1}(X \setminus \Delta) : P(\varphi) = 0 \}$

$\pi : |\mathcal{O}_X| \to X$

Claim: $\pi|_{Y'} : Y' \to X$ is a covering map of degree n.

An element of $|\mathcal{O}_X|$ is a pair $\{ (x, \varphi) : \varphi \in \mathcal{O}_x \}$

P is of degree n & $x \notin \Delta$:

$\Rightarrow$ $\exists$ exactly n ~~diffe~~ distinct roots

called $y_1, \ldots, y_n$ of $P(T, x) = 0$.

(proof is similar to how we show existence of logarithm for nowhere zero holomorphic maps on a simply connected domain.

Then $\pi : Y' \to X \setminus \Delta$ is n-sheeted covering map.

Define f' to be $f' : Y' \to \mathbb{C}$

$$f'(y') := \varphi(\pi(y')).$$

(Basically

$$y' = (x, \varphi) \Rightarrow f'(y') = \varphi(x))$$

So $(\pi^* P)(f') = 0$ on Y'.

What to do with $x_0 \in \Delta$?

Choose a small neighborhood $\mathbb{D}$ of x_0 in X.

If $\mathbb{D}$ is very small, then

$Y' \supseteq \pi^{-1}(\mathbb{D}\setminus x_0)$ is a disjoint union of

$\coprod_{i\in I} U_i$, $|I| < \infty$

so that $\pi : U_i \to \mathbb{D}^*$ is a covering map of degree $< n$.

$\Rightarrow$ it must be biholomorphic to a map $Y' \supseteq \mathbb{D}^* \to \mathbb{D}^* \subseteq X$, $z \mapsto z^k$ for $k \geq 1$.

Now $Y = Y'$ unioned with the centers of the above $\mathbb{D}^* \subseteq Y'$.

$\Rightarrow Y$ becomes a RS also.

and π extends to a branched covering map $\pi : Y \to X$ of degree n. $\square$

<u>In our case</u>, more explicit description:

$X = \mathbb{C}$

~~If~~ $P(T,x)$ ~~be~~ in $\mathbb{C}[x][T]$

Claim:

$Y = \{(x, y) \in \mathbb{C}^2 : P(y, x) = 0\}$.
(or a blowup. A smooth model if $P(y,x)=0$ is not smooth.)

<u>Example</u>:

$$P(T, x) = T^3 - (x^2+1)T^2 - xT + x^5 + 3x^3$$

$$Y' = \{(x, y) \in \mathbb{C}^2 : x \notin \Delta_P : P(y, x) = 0\}$$

discriminant

smooth (so don't need blowup)

$$Y = \{(x, y) \in \mathbb{C}^2 : P(y, x) = 0\}$$

$\pi : Y \to X$ is simply $(x, y) \to x$

What is the relation between critical points of π & the discriminant Δ?

Answer: $\Delta = \pi(\text{critical points of } \pi)$

$= \text{critical values of } \pi$.

$\Rightarrow \boxed{\text{Critical points} \subseteq \pi^{-1}(\Delta)}$

<u>Revisit example</u>:

$$Y = \{(x, y) \in \mathbb{C}^2 : y^2 = x^3 - 1\}$$

$$\pi : Y \to X$$
$$(x, y) \mapsto x$$

What are critical points?

Discriminant $\Delta = \{x \in \mathbb{C} : \underbrace{x^3 - 1}_{=0}\}$

$\Rightarrow$ Critical points $\subseteq \pi^{-1}(\Delta)$

$= \{(0, \zeta_1), (0, \zeta_2), (0, 1)\}$

$\zeta_1^3 = \zeta_2^3 = 1$

We can easily check that each of the above 3 points are actually critical points.

Example:

$Y = \{(x,y) \in \mathbb{C}^2 :$
$y^2 - xy + x = 0\}$

is Y a RS?

$\nabla f = \{(2y - x, -y + 1)$

Yes!

$\pi : Y \longrightarrow X$

$(x,y) \mapsto x$

Critical points of π?

$\Delta =$ discriminant

$= \{ x \in \mathbb{C} : \quad y^2 - xy + x = 0$

has multiple roots in $y \}$

$y^2 + ay + b = 0$ has multiple roots iff $a^2 = 4b$.

$a = -x, \; b = x \Rightarrow \quad x^2 = 4x \Rightarrow$

$x = 0$ or $x = 4$.

$\Delta = \{0, 4\}$.

$\pi^{-1}(\Delta) = \{(0,0), (4,2)\}$.

HW: Show that $\pi^{-1}(\Delta)$ is actually the critical points of π.

<u>Example:</u>

$$Y = \{ (x,y) \in \mathbb{C}^2 : y^3 - y = x^2 \}$$

$$\pi : Y \to X$$
$$(x,y) \mapsto x$$

What is the discriminant?

What are the critical points of π?