18 October 2023
MAT4800

Forster's conjecture:
If X is a open RS, then $\exists$ a holomorphic function $F : \mathbb{C}^2 \to \mathbb{C}$ so that $X = \{(x,y) \in \mathbb{C}^2 : F(x,y) = 0\}$ $\square$

Very open. In Oslo: Erlend Fornæss Wold.
Erik Løw

Analogy for compact RS:
Every compact RS is a curve in $\mathbb{P}^2$.

In general, not true for the compact analog.

Degree-genus formula: If X is a compact RS, $X \subseteq \mathbb{P}^2$ & defined by a polynomial of degree n, then:

$$g(X) = \frac{(n-1)(n-2)}{2}.$$

Example: $X = \{ [x : y : z] \subseteq \mathbb{P}^2 : x = 0 \}$

$X \cong \mathbb{P}^1$

degree $n = 1$

$$\Rightarrow g(X) = \frac{(1-1)(1-2)}{2} = 0.$$

Example 2: If $X \subseteq \mathbb{P}^2$,

$X = \{ [x : y : z] : y^2 z = x^3 + xz^2 \}$

elliptic curve

degree $n = 3$

$$\Rightarrow g(X) = \frac{(3-1)(3-2)}{2} = 1.$$

Example 3: $X = \{ [x : y : z] \subseteq \mathbb{P}^2 : x^2 + y^2 = z^2 \}$

check it is smooth.

degree $= 2 = n$

$$g(X) = \frac{(2-1)(2-2)}{2} = 0.$$

It is not obvious but here $X \approx \mathbb{P}^1$.

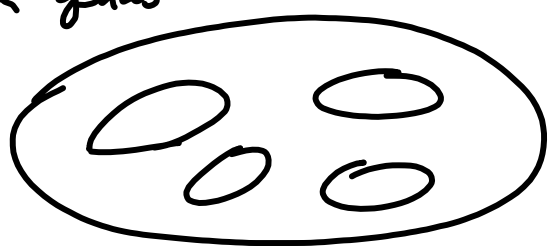
Example 3: If X compact RS, $X \subseteq \mathbb{P}^2$, then $g(X) \neq 4$.

Why?
$$\frac{(n-1)(n-2)}{2} = 4$$
$$\Rightarrow n^2 - 3n - 6 = 0.$$

No positive integer ~~solution~~ for n.

So no such curve.

On the other hand, there are compact RS with genus 4.



What is known?

Every ~~algebraic~~ compact RS X is a curve in $\mathbb{P}^3$.

Idea of proof: Use a positive line bundle L on X to get an embedding

$$X \hookrightarrow \mathbb{P}^N \text{ for } N \geq 3.$$

Then if $N > 3$, project to a generic hyperplane $H \subseteq \mathbb{P}^N$, to get $X \hookrightarrow \mathbb{P}^{N-1}$. (i.e. the projection is isomorphic when restricted to X.)

$$H \cong \mathbb{P}^{N-1}$$

* If X is an open RS, then X is a curve in $\mathbb{C}^3$.

* Elliptic curve:

→ Cryptography

→ Number theory (Fermat's last theorem

→

* Group structure on elliptic curve:

→ First way to see:

$$E = \mathbb{C}/\Lambda$$

So each point on E comes from a point on $\mathbb{C}$. On $\mathbb{C}$ we have addition ⇒ We can define an addition modulo Λ ⇒ get an addition on E.

very much like $11 = 4 \pmod 7$, $12 = 5 \pmod 7$ ⇒ $11 + 12 = 4 + 5 = 2 \pmod 7$.

→ Second way (more geometric):

- Elliptic curve is defined by a polynomial of degree 3.
- So, elliptic curve intersects any line in 3 points (multiplicities counted).

Ex: What is RS for $T^2 - x^2 = 0$, $x \in X = \mathbb{C}$?

Look at $Z = \{(x,y) \in \mathbb{C}^2 : y^2 - x^2 = 0\}$.

This is a singular curve.

$f(x,y) = y^2 - x^2$

$\nabla f = (-2x, 2y)$

$\nabla f = 0$ if $(x,y) = (0,0) \in Z$.

$x = 0$ is ~~an~~ in the discriminant for this.

So RS for $T^2 - x^2$ = The smooth part of Z + some points

How many points we need to add?

Recall how we construct a RS for an equation $F(T, x) = 0$:

. First define:

$$\tilde{Y} = \{ (x,y) \in \mathbb{C}^2 : x \notin \Delta, \; y^2 - x^2 = 0 \}$$

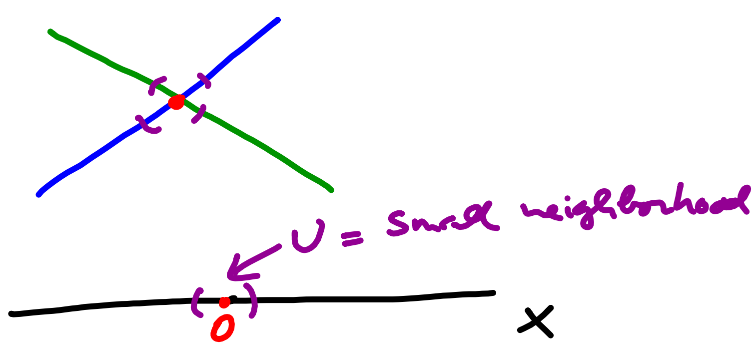
. For each $x \in \Delta$, choose a small open neighborhood U of x in X, & look at

$$\pi^{-1}(\underbrace{U\setminus\{x\}}_{\cong\, \mathbb{D}^*}) = \coprod_{i\in I} V_i$$

V_i will be connected components of $\pi^{-1}(U)$, & $\pi|_{V_i} : V_i \xrightarrow[\text{isomorphic}]{\sim} U\setminus\{x\}$.

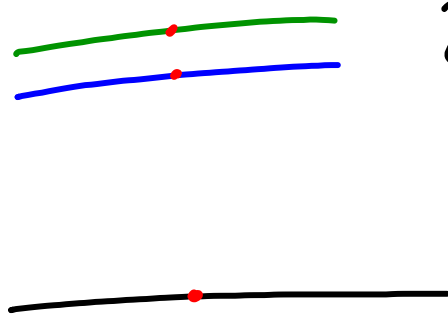
Add a point x_i to V_i for each i, then $V_i + \{x_i\} \cong (U\setminus\{x\}) + \{x\} \cong \mathbb{D}$.

So # points to add = # such connected components.



$\Rightarrow \pi^{-1}(U) =$ disjoint union of 2 punctured disks.

$\Rightarrow$ We need to add 2 points · & ·



What we get after adding 2 points to $Z \setminus \{(0,0)\}$

$\Rightarrow$ RS of $T^2 - x^2$, for $x \in \mathbb{C}$ is a disjoint union of 2 lines! (Not connected) They are in blowup of $\mathbb{C}^2$ at $(0,0)$, & not in $\mathbb{C}^2$!

Example: What is RS for
$$T^2 = x^4 - x^2, \quad x \in \mathbb{C} ?$$
$$Z = \{ (x,y) \in \mathbb{C}^2 : \ y^2 = x^4 - x^2 \}$$

Z is not smooth!
$$f(x,y) = y^2 - (x^4 - x^2)$$
$$\nabla f = (4x^3 - 2x, 2y)$$
$$\nabla f = 0 \iff y = 0, \quad x = 0 \text{ or } x = \pm\sqrt{\tfrac{1}{2}}.$$
Only $(0,0) \in Z$.

$$\Delta = \{ x \in \mathbb{C} : \ f(x,y) = 0 \text{ has multiple roots in } y \}$$
$$= \{ x \in \mathbb{C} : \ x^4 - x^2 = 0 \}$$
$$= \{ 0, 1, -1 \}.$$

Look at $x = 1$:

$\Rightarrow (1,0) \in Z$.

$\pi: Z \to X = \mathbb{C}$

$(x, y) \mapsto x \in X = \mathbb{C}$

Now if $U \underset{\mathbb{D}}{\overset{\subseteq}{ }}$ is a small open neighborhood of 1, then $\pi^{-1}(\underset{\mathbb{D}^*}{U \setminus \{1\}})$ has how many connected components?

Claim: $\pi^{-1}(U \setminus \{1\})$ is connected!

Let change coordinate: $w = x - 1 \Rightarrow$

$y^2 = x^4 - x^2 \Leftrightarrow y^2 = (w+1)^4 - (w+1)^2$

$= w^4 + 4w^3 + 6w^2 + 4w + 1 - w^2 - 2w - 1$

$y^2 = w\,h(w)$, where $h(0) \neq 0$.

We can change coordinate again:

$\hat{w} = w\,h(w)$ (this is only valid locally, & not globally)

$\Rightarrow y^2 = \hat{w}$

Now the map π is of this form:

$\pi : \{y^2 = \hat{w}\} \to \mathbb{C}$

$(\hat{w}, y) \mapsto \hat{w}$.

So if $U = \mathbb{D}(0, r)$

then $\pi^{-1}(U \setminus \{0\}) =$

$\mathbb{C}^2 \supseteq \{\underbrace{\hat{w} = y^2}_{\simeq \text{ a line}}\} \setminus \{(0,0)\}$. Hence connected!

Therefore, for $x = 1 \in \Delta$,
we only need to add one more point in
$Z \setminus \pi^{-1}(\Delta)$, & this point is actually
$(1,0) \in Z$ already. So we don't need
to change Z at this point.

Similarly, we do not need to
change the geometry of Z for $x = -1$
$\in \Delta$.
* So we need to look at $x = 0 \in \Delta$.

Let $\underset{\substack{\rotatebox{90}{$\subseteq$} \\ \mathbb{D}(0,r)}}{U} \subseteq \mathbb{C}$ be a small neighborhood
of $0 = x \in \Delta$.

$$y^2 = x^4 - x^2$$

Again, there is no way to write
$y = \pm \varphi(x)$ globally for $x \in \mathbb{C}$
(Basically, there is no holomorphic function which
is $\sqrt{x^4 - x^2}$.
But locally in U, there is!

$$x^4 - x^2 = x^2(-1 + x^2) = x^2 h(x),$$

$h(0) = -1 \neq 0$. ~~So by a~~ So $h(x) \neq 0$
anywhere in U, provided U is small enough.

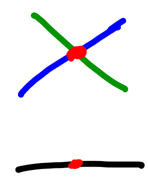
So by existence of logarithm,

$\exists \ g \in \mathcal{O}(U)$ so that
$$g(x)^2 = h(x).$$

in U: $\quad y^2 = x^2 h(x) = x^2 g(x)$

$$\Rightarrow \left(y - \underbrace{x g(x)}_{\text{change coordinate } \hat{\omega} = xg(x)}\right)\left(y + x g(x)\right) = 0.$$

Which means that locally (in U) the set Z looks like the one in the 1st example:
$y^2 = \hat{\omega}^2$



$\Rightarrow$ Need to add 2 points into $Z \setminus \{(0,0)\}$ to get RS for $y^2 = x^4 - x^2$.

Puiseux series: similar to what we saw in the previous example.

Let's consider a very simple case

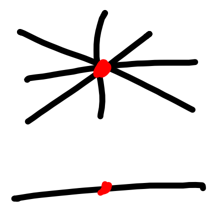
$$y^m = x^n h(x), \text{ where } h(0) \neq 0.$$

Puiseux series let us know about the behaviour near $(0,0)$.

If m is a division of n, ~~the~~ (Example $m = 4$, $n = 12$), we can do like before, after changing coordinates $\hat{\omega}^n = \cancel{xg(x)}\ x^n h(x)$ [existence of logarithm) then:

$$y^m = \hat{\omega}^n$$

& near $(0,0)$, this looks like m lines crossing at the same point.



What if m is not a division of n?

Example $y^3 = x^5 h(x)$

$$\Rightarrow \quad y^3 = \hat{\omega}^5$$

$\Rightarrow$ Puiseux series is $\xi - \hat{\omega}^5$

(so y is kind of $\sqrt[3]{\xi} \to$ not defined around 0)

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§9. Differential forms (on RS)

First, work with differential forms on (an open subset of) $\mathbb{C}$

$\mathbb{C} \approx \mathbb{R}^2$

$\underbrace{T}_{\text{tangent bundle}}\mathbb{C}$ ~~is~~ is generated by 2

vector (fields) $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$

can think about like unit vector on x-axis

similarly

Differential forms is dual to vectors.

$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ generate a vector space isomorphic to $\mathbb{R}^2$. Call it V.

The dual $V^* = \{ f : V \to \mathbb{R}$ linear map$\}$

$V^* \simeq \mathbb{R}^2$

Then let dx & $dy \in V^*$ defined by:

$$dx\left(\frac{\partial}{\partial x}\right) = 1, \quad dx\left(\frac{\partial}{\partial y}\right) = 0$$

$$dy\left(\frac{\partial}{\partial x}\right) = 0, \quad dy\left(\frac{\partial}{\partial y}\right) = 1$$

dx, dy is dual to $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$.

dx, dy generate V^*.

<u>Vector field on $\mathbb{R}^2$</u>: Something of the form

$$v(x,y) = u_1(x,y)\frac{\partial}{\partial x} + u_2(x,y)\frac{\partial}{\partial y}$$

Flow of a vector field:

Let $v(x,y)$ be a vector field.

Then a flow is a function:

$$\varphi : [0,1] \to \mathbb{R}^2$$

so that $\frac{d\varphi}{dt} = v(\varphi(t)) \ \forall t \in [0,1]$

A flow is determined by initial value $\varphi(0)$.

Show on picture how to get a flow.

On $\mathbb{C}$: $\frac{\partial}{\partial z}$ & $\frac{\partial}{\partial \bar{z}}$?

$$z = x + iy$$
$$\bar{z} = x - iy$$

If we have a function $f(x,y)$

$$\Rightarrow \ df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

Apply to $f(x,y) = z = x + iy$:

$$dz = dx + i\,dy$$
$$d\bar{z} = dx - i\,dy$$

Now $\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}$ is dual to $dz, d\bar{z}$,

i.e: $dz\left(\frac{\partial}{\partial z}\right) = 1, \ dz\left(\frac{\partial}{\partial \bar{z}}\right) = 0,$

$d\bar{z}\left(\frac{\partial}{\partial z}\right) = 0, \ d\bar{z}\left(\frac{\partial}{\partial \bar{z}}\right) = 1.$

Also ~~d~~ $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ generate V

$\Rightarrow \quad \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \in V \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}^2$

is ~~gen~~ linear combination of $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ with complex coefficients.

$$\frac{\partial}{\partial z} = \alpha_1 \frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \bar{z}} = \beta_1 \frac{\partial}{\partial x} + \beta_2 \frac{\partial}{\partial y}.$$

$\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$.

HW: Determine $\alpha_1, \alpha_2, \beta_1, \beta_2$ s.t. $dz, d\bar{z}$ is dual to $\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}$?

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u_1, u_2 : smooth (C^1, C^2, C^∞, analytic ...)

Show on picture what a vector field looks like.