1 November 2023
MAT 4800

(Important)
<u>Lemma 12.4</u>: If $\mathcal{V} < \mathcal{U}$ are open coverings then the map $\tau^{\mathcal{U}}_{\mathcal{V}} : H^1(\mathcal{U}, \mathcal{F}) \to H^1(\mathcal{V}, \mathcal{F})$

is injective.

<u>Proof</u>: $\tau^{\mathcal{U}}_{\mathcal{V}}$ is a linear map.
So $\tau^{\mathcal{U}}_{\mathcal{V}}$ is injective iff $\tau^{\mathcal{U}}_{\mathcal{V}}(g) = 0 \Rightarrow g = 0$.

$$g = (g_{ij}) \in Z^1(\mathcal{U}, \mathcal{F})$$

Assume that $\tau^{\mathcal{U}}_{\mathcal{V}}(g) \in B^1(\mathcal{V}, \mathcal{F})$, then we need to show that $g \in B^1(\mathcal{U}, \mathcal{F})$ itself.

Let $V_k, V_\ell \in \mathcal{V}$
& $U_i, U_j \in \mathcal{U}$ so that
$\boxed{g_{i,j} \in \mathcal{F}(V_i \cap V_j)}$ $V_k \subseteq U_i$, $V_\ell \subseteq U_j$

$$\tau^{\mathcal{U}}_{\mathcal{V}}(g)_{k,\ell} = g_{ij}\big|_{V_k \cap V_\ell}$$
$$= g_k - g_\ell \big|_{V_k \cap V_\ell}$$

$g_k \in \mathcal{F}(V_k)$, $g_\ell \in \mathcal{F}(V_\ell)$.

$\Rightarrow \forall\ m$:

$$g_k + g_{m,i} = g_\ell + g_{m,j}$$

on $U_m \cap V_k \cap V_\ell$.

Hence, $\forall\ m : \exists\ h_m \in \mathcal{F}(U_m)$

so that $h_m = g_k + g_{m,i}$ on $U_m \cap V_k$

$\Rightarrow$ ~~g~~ $g_{i,j} = h_i - h_j \Rightarrow g$ is a boundary. □

<u>Corollary</u>: $H^1(X, \mathcal{F}) = 0 \iff H^1(\mathcal{U}, \mathcal{F}) = 0\ \forall$ open covering $\mathcal{U}$ of $\mathcal{F}$.

<u>Proof</u>: Look at $H^1(\mathcal{U}, \mathcal{F}) \to H^1(X, \mathcal{F})$ & use injectivity.

<u>Corollary</u>: $H^1(X, \mathcal{F}) = 0$

$\iff \forall$ open covering $\mathcal{U}$ of X, there is $\mathcal{V} < \mathcal{U}$ so that

$$H^1(\mathcal{V}, \mathcal{F}) = 0. \quad \square$$

$H^1(X, \mathcal{F}) = 0$ is important when later we apply to S.E.S. (short exact sequence of sheaves) & one of the sheaf has ~~the~~ 0 cohomology.

S.E.S of sheaves $\Longrightarrow$ L.E.S. (long exact sequence) of cohomology

<u>Morphism of sheaves:</u>

Let $\mathcal{F}, \mathcal{G}$ be two sheaves of Abelian groups over X.

A morphism $\phi : \mathcal{F} \to \mathcal{G}$ is an assignment as follows:

* For every $U \subseteq X$ open, there is a group homeomorphism
$$\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$$
* These $\phi(U)$ are compatible with restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$, $\mathcal{G}(U) \to \mathcal{G}(V)$:

$V \subseteq U$ open sets

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\phi(U)"] \arrow[d, "\text{restriction map}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{G}(U) \arrow[d] \\ \mathcal{F}(V) \arrow[r, "\phi(V)"'] & \mathcal{G}(V) \end{tikzcd}$$

← restriction map

* Kernel of $\phi : \mathcal{F} \to \mathcal{G}$

For every $U \subseteq X$ open, define:
$$\operatorname{Ker}(\phi)(U) = \operatorname{Ker}(\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U))$$
Check that $\operatorname{Ker}(\phi)(U)$, $U \subseteq X$ open, gives a sheaf $\operatorname{Ker}(\phi)$ on X.

* <u>Image</u>: For each $U \subseteq X$ open,

$$\mathrm{Im}(\phi)(U) = \mathrm{Image}\big(\phi(U): \mathcal{F}(U) \to \mathcal{G}(U)\big)$$

$\{\mathrm{Im}(\phi)(U)\}_{U \text{ open} \subseteq X}$ is a presheaf on X. We can sheafify $\{\mathrm{Im}(\phi)(U)\}$ to get a sheaf $\mathcal{I}m(\phi)$.

Remark: For $U \subseteq X$ open:

$$\mathcal{I}m(\phi)(U) \neq \mathrm{Im}(\phi(U))\,!!$$

But for every $x \in X$:

$$\mathcal{I}m(\phi)_x = \mathrm{Im}(\phi)_x\,!!$$

HW:

$$\mathcal{I}m(\phi)_x = \mathrm{Image}(\phi_x : \mathcal{F}_x \to \mathcal{G}_x)$$

Remark: How to define the morphisms on stalks.

~~Let~~ Recall

$$\mathcal{F}_x = \coprod_{x \in \mathcal{U} \text{ open}} \mathcal{F}(\mathcal{U}) \Big/ \sim$$

An element of $\mathcal{F}_x$ is represented by a section $s \in \mathcal{F}(U)$ for some $x \in U$.

Then by definition we have:

$$\phi(s) \in \mathcal{G}(U) \longrightarrow \coprod_{x \in U} \mathcal{G}(U) \Big/ \sim = \mathcal{G}_x$$

To show that this gives us a map

$$\phi_x : \mathcal{F}_x \to \mathcal{G}_x \text{ , need to}$$

show that if $x \in V \subseteq U$, $s \in \mathcal{F}(U)$

& $\tau(s)$ = restriction of s to $\mathcal{F}(V)$

then $\phi(s) = \phi(\tau(s))$ in $\mathcal{G}_x$.

We only need to show that ~~there is~~

$$\tau(\phi(s)) = \phi(\tau(s)) \text{ in } \mathcal{G}(V)$$

this is just the commutativity of

$$\begin{tikzcd} s \in \mathcal{F}(U) \arrow[r, "\phi(U)"] \arrow[d, "\tau"'] & \mathcal{G}(U) \arrow[r] \arrow[d, "\tau"] & \mathcal{G}_x \\ \tau(s) \quad \mathcal{F}(V) \arrow[r, "\phi(V)"'] & \mathcal{G}(V) \arrow[r] & \mathcal{G}_x \end{tikzcd}$$

<u>Def</u>: exact sequence

A sequence of morphisms $\mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$ (sheaves over X)

is exact if $\forall x \in X$

the sequence of morphisms on stalks:

$$\mathcal{F}_x \xrightarrow{\phi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x$$

is exact. Which means:

$$\text{Image}(\phi_x) = \text{Kernel}(\psi_x) \ \forall x \in X.$$

* Short exact sequence (SES) of sheaves:

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is SES if

$$0 \to \mathcal{F}_x \xrightarrow{\phi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x \to 0 \text{ is}$$

a SES for all $x \in X$. Which means:

$0 \to \mathcal{F}_x \xrightarrow{\phi_x} \mathcal{G}_x$ is exact $\Leftrightarrow$ ϕ_x is injective

$\mathcal{F}_x \xrightarrow{\phi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x$ is exact $\Leftrightarrow$ $\mathrm{Im}(\phi_x) = \mathrm{Ker}(\psi_x)$

$\mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x \to 0$ is exact $\Leftrightarrow$ ψ_x is surjective.

<u>Lemma:</u> $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is ~~exact~~ SES

iff $\forall$ $U \subseteq X$ open:

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U) \to 0$$

is SES of Abelian groups. $\square$

<u>Explain about the use of the lemma:</u>

Given a sheaf, usually it is interesting to know about the global sections of the sheaf, which is <u>usually</u> difficult.

Working with stalks is usually easier (because stalk is Abelian group). So checking exactness using stalks is usually easy.

Lemma says that if we can check exactness on stalks then we have exactness of globale sections & maybe we use that fact to find global sections.

Ex (Exponential exact sequence)

$$0 \to \mathbb{Z}_X \xrightarrow[2\pi i]{} \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

$\mathcal{O}_X$ = sheaf of holomorphic functions

$\Rightarrow \mathcal{O}_X(U) = \{ f : U \to \mathbb{C} \text{ holomorphic} \}$

$\mathcal{O}_X^*$ = sheaf of nowhere zero holomorphic functions

$\Rightarrow \mathcal{O}_X^*(U) = \{ f : U \to \mathbb{C}^* \text{ holomorphic} \}$

<u>Constant sheaf:</u> A constant sheaf $\mathcal{A}$ on X is defined as follows:

First, we define a presheaf $\widehat{\mathcal{A}}$ on X so that:

* $\forall\ U \subseteq X$ open $(U \neq \emptyset)$

$\Rightarrow \widehat{\mathcal{A}}(U) = A \leftarrow$ a fixed Abelian group

* Restriction map is identity map

$$A = \widehat{\mathcal{A}}(U) \longrightarrow \widehat{\mathcal{A}}(V) = A$$
$$a \mapsto a$$

Usually, this is not a sheaf! Example:

$X = \{a, b\}$ where every subset is open set & $A = \mathbb{Z}$.

$$\begin{tikzcd} & \{a\} \arrow[dr] & \\ \phi \arrow[ur] \arrow[rr] \arrow[dr] & & \{a,b\} \\ & \{b\} \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} & \hat{A}(\{a\}) = \mathbb{Z} \arrow[dl, "0"'] & \\ 0 = \hat{A}(\phi) & & \hat{A}(\{a,b\}) = \mathbb{Z} \arrow[ul, "id"'] \arrow[ll, "0"] \arrow[dl, "id"] \\ & \hat{A}(\{b\}) = \mathbb{Z} \arrow[ul, "0"] & \end{tikzcd}$$

$m \in \hat{A}(\{a\})$, $n \in \hat{A}(\{b\})$

$m \neq n \Rightarrow$ There is no global section $s \in \hat{A}\{a,b\}$ which restricts to m & $n \Rightarrow$ not a sheaf.

<u>What is the sheafification?</u> A = sheafification of $\hat{A}$

$$\begin{tikzcd} & A(\{a\}) = \mathbb{Z} = \hat{A}(\{a\}) \arrow[dl, "0"'] & \\ 0 = A(\{\phi\}) & & A(\{a,b\}) \arrow[ul, "\tau"'] \arrow[ll] \arrow[dl, "\tau"] \\ & A(\{b\}) = \mathbb{Z} \arrow[ul] & \end{tikzcd}$$

$A(\{b\}) = \hat{A}(\{b\})$, $A(\{a,b\}) = \mathbb{Z} \oplus \mathbb{Z}$, $\mathbb{Z} \oplus \mathbb{Z} \to \hat{A}(\{a\})$, $\hat{A}(\{b\})$

τ is not identity map ($\mathbb{Z} \oplus \mathbb{Z} \neq \mathbb{Z}$) but projection.

<u>HW</u>: $\mathbb{Z}_X(U) = \mathbb{Z}^{\#\text{ connect components of } U}$

$U \subseteq X$ open & $\mathbb{Z}_X$ = constant sheaf $\mathbb{Z}$ on X.

We will show that

$$0 \to \mathbb{Z}_X \xrightarrow[2\pi i]{} \mathcal{O}_X \xrightarrow[\exp]{} \mathcal{O}_X^* \to 0$$

Suffice to show that on stalks.

$$(\mathbb{Z}_X)_p = \mathbb{Z}$$

$$(\mathcal{O}_X)_p = \text{?} \left(\text{~~Taylor~~ power series with a positive radius of convergence} \right)$$

$$(\mathcal{O}_X^*)_p = \ldots + \text{the constant term is non zero}$$

$$= \sum_{n=0}^{\infty} a_n z^n, \ a_0 \neq 0.$$

Can think about ~~an element~~ a section of $\mathbb{Z}_X(U)$ as a function $U \to \mathbb{C}$, which is locally a constant. Then it is a holomorphic function.

The map $\mathbb{Z}_X \xrightarrow{2\pi i} \mathcal{O}_X$ just multiply everything by $2\pi i$.

The map $\mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^*$ is just $f \to e^f$.

The stalks, WTS exactness:

$$0 \to \mathbb{Z} \xrightarrow[2\pi i]{} (\mathcal{O}_X)_p \to (\mathcal{O}_X^*)_p \to 0$$

By lemma, suffices to show that for every open $p \in U$, there is $p \in V \subseteq U$

so that
$$0 \to \cancel{\mathcal{F}(U)}\ \mathbb{Z}_X(V) \to \mathcal{O}_X(V) \to \mathcal{O}_X^*(V) \to 0$$
is exact.

We choose $V = \mathbb{D}(p, r) = \{|z-p| < r\}$

$0 \to \mathbb{Z} \xrightarrow[2\pi i]{} \mathcal{O}_X(V)$: easy to see exactness ($\mathbb{Z} \to \mathcal{O}_X(V)$ injective)

$\mathcal{O}_X(V) \xrightarrow[\exp]{} \mathcal{O}_X^*(V) \to 0$: Why exact?

$\Leftrightarrow \mathcal{O}_X(V) \xrightarrow[\exp]{} \mathcal{O}_X^*(V)$ surjective

Since V is simply connected $\Rightarrow$ by existence of logarithm : if $g \in \mathcal{O}_X^*(V)$

$\Rightarrow \exists f \in \mathcal{O}_X(V)$ so that
$$g = e^f.$$
$\Leftrightarrow \mathcal{O}_X(V) \xrightarrow[\exp]{} \mathcal{O}_X^*(V)$ is surjective.

How about :
$$\mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}_X(V) \xrightarrow{\exp} \mathcal{O}_X^*(V)$$

exactness?
$\Leftrightarrow \operatorname{Ker}\left(\mathcal{O}_X(V) \xrightarrow{\exp} \mathcal{O}_X^*(V)\right) = 2\pi i\, \mathbb{Z} = \operatorname{Im}\left(\mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}_X(V)\right)$

So if $f \in \mathrm{Ker}(\mathcal{O}_X(V) \xrightarrow{\exp} \mathcal{O}_X^*(V))$

$\Leftrightarrow \quad e^f = 1$ on V.

We need to show that $f =$ constant $2\pi i n$ for some $n \in \mathbb{Z}$.

$f : V \to \mathbb{C}$ holomorphic

$e^f = 1 \Leftrightarrow \forall z : f(z) \in \underbrace{2\pi i \mathbb{Z}}_{\text{discrete}}$

$\Rightarrow f$ must be a constant

(example we can use the open mapping theorem: if f is not constant $\Rightarrow$ image of f is an open subset of $\mathbb{C}$, & hence cannot belong to $2\pi i \mathbb{Z}$.)

So we have exactness of:

$$0 \to \mathbb{Z}_X \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0.$$

* LES of cohomology for SES of sheaves

let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a SES of sheaves on X. Then there exists a LES of cohomology groups:

$$0 \to \underset{\substack{\| \\ H^0(X,\mathcal{F})}}{\mathcal{F}(X)} \to \mathcal{G}(X) \to \mathcal{H}(X)$$

$$\to H^1(X,\mathcal{F}) \to H^1(X,\mathcal{G}) \to H^1(X,\mathcal{H})$$

$$\to H^2(X,\mathcal{F}) \to \cdots$$

Most interesting case is when:

$$H^1(X, \mathcal{F}) = 0.$$

then we have an ~~exact~~ SES:

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to 0.$$

Example: if $X = \mathbb{C}$,

$$0 \to \mathbb{Z}_X \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$