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MAT 4800

A way to construct holomorphic 1-form on an affine curve:

Given a smooth affine curve, i.e.:

$$X = \{(x,y) \in \mathbb{C}^2 : F(x,y) = 0\},$$

X is smooth (∇F is never $(0,0)$ on X).

By implicit derivation:

$$\Rightarrow \quad \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = 0.$$

So

$$\omega = \frac{\partial F}{\partial x} dx = - \frac{\partial F}{\partial y} dy$$

defines a holomorphic 1-form on X.

We can use $\frac{\partial F}{\partial x} dx$ when x is local coordinate, & $\frac{\partial F}{\partial y} dy$ when y is local coordinate.

Example: $E_{aff} = \{ y^2 - x^3 + 1 \}$

$$3x^2\, dx = 2y\, dy.$$

$\Rightarrow \omega =$

HW: What is the multiplicity of the zero of ω at the point $x = 0, y = \pm 1$?

LES (long exact sequence on cohomology) from SES on sheaves:

Assume that we have a SES of sheaves (on X):

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

then we have a LES on cohomology:

$$\begin{aligned} 0 \to H^0(X,\mathcal{F}) &\to H^0(X,\mathcal{G}) \to H^0(X,\mathcal{H}) \\ \to H^1(X,\mathcal{F}) &\to H^1(X,\mathcal{G}) \to H^1(X,\mathcal{H}) \\ \to H^2(X,\mathcal{F}) &\to H^2(X,\mathcal{G}) \to H^2(X,\mathcal{H}) \\ &\to \cdots \end{aligned}$$

$$H^0(X,\mathcal{F}) = \mathcal{F}(X)$$

The maps

$$H^i(X,\mathcal{F}) \to H^i(X,\mathcal{G})$$

(same degree i) are easy: just take a representation which is a collection of some local sections of $\mathcal{F}$, & then map by the morphism $\mathcal{F} \to \mathcal{G}$. Check that if a local section is δ-closed (or exact) then the same for image. This is because ~~[illegible]~~ the map is a morphism of Abelian groups.

So need to explain the maps:

$$H^i(X,\mathcal{H}) \to H^{i+1}(X,\mathcal{F})$$

(called : connecting maps).

We explain only for:

$$H^0(X,\mathcal{H}) \to H^1(X,\mathcal{F}).$$

Let $h \in H^0(X,\mathcal{H}) \overset{=}{} \mathcal{H}(X)$. Then it is represented by some global section h of $\mathcal{H}$.

Now : $\mathcal{G} \to \mathcal{H} \to 0$ is exact

$\Rightarrow$ $\forall x$: $\mathcal{G}_x \to \mathcal{H}_x \to 0$ is exact.

$\Rightarrow$ $\exists$ (very small) open covering $(U_i)_{i\in I}$ of X & $(g_i) \in C^0(\mathcal{U},\mathcal{G})$

so that : $g_i \longmapsto h|_{U_i}$.

$\Rightarrow$ $g_{ij} \in C^1(\mathcal{U},\mathcal{G})$

$\quad$ " $g_i|_{U_i\cap U_j} - g_j|_{U_i\cap U_j}$

is in the kernel of

$$\mathcal{G}(U_i\cap U_j) \to \mathcal{H}(U_i\cap U_j).$$

Now $\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$ is exact, so we can choose (U_i) smaller if necessary, &

$$\mathrm{Image}\left(\alpha\left(\mathcal{F}(U_i \cap U_j)\right)\right)$$
$$= \mathrm{Ker}\left(\beta\left(\mathcal{G}(U_i \cap U_j) \to \mathcal{H}(U_i \cap U_j)\right)\right)$$

In particular $g_{ij} \in \mathrm{Ker}(\beta)$

$\Rightarrow \exists\, f_{ij}$ so that

$$\alpha(f_{ij}) = g_{ij}.$$

Then we define image of h to be

$(f_{ij}) \in H^1(X, \mathcal{F})$.

(Need to check that (f_{ij}) is δ-closed.

$\Leftrightarrow \quad f_{ij} + f_{jk} - f_{ik} = 0$.

To see this, we use that

$0 \to \mathcal{F} \to \mathcal{G}$ is exact.

$\Rightarrow 0 \to \mathcal{F}(U_i \cap U_j \cap U_k) \to \mathcal{G}(U_i \cap U_j \cap U_k)$

is exact

$\Rightarrow$ if $\alpha(\tilde{f}) = 0 \Rightarrow \tilde{f} = 0$.

So only need to check that

$$\alpha(f_{ij} + f_{jk} - f_{ik}) = 0$$
$$= g_{ij} + g_{jk} - g_{ik}$$
$$\shortparallel$$
$$(g_i - g_j)|_{U_i \cap U_j} \quad . \quad \square$$

Recap: We use different exact sequences $0 \to \mathcal{F} \to \mathcal{G}$, $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$, $\mathcal{G} \to \mathcal{H} \to 0$ in different places of the proof.

Thm: Let $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$ be SES. If $H^1(X, \mathcal{G}) = 0$ then

$$H^1(X, \mathcal{F}) = \mathcal{H}(X) / \beta\mathcal{G}(X).$$
$$= H^0(X, \mathcal{H}) / \beta H^0(X, \mathcal{G}).$$

Proof: $0 \to \mathcal{F}(X) \to \mathcal{G}(X) \xrightarrow{\beta} \mathcal{H}(X)$
$\to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G}) = 0$

$\Rightarrow \quad \mathcal{H}(X) \to H^1(X, \mathcal{F})$ is surjective.

$\Rightarrow \quad H^1(X, F) \simeq \mathcal{H}(X) / \mathrm{Ker}(\mathcal{H}(X) \to H^1(X, F))$

(Kernel-image theorem)

$\mathrm{Ker}(\mathcal{H}(X) \to H^1(X, \mathcal{F}))$
$= \mathrm{Image}(\mathcal{G}(X) \to \mathcal{H}(X))$

Thm 15.14: (Dolbeault): $X = RS$.

Then: $H^1(X, \mathcal{O}) = \mathcal{E}^{0,1}(X) / d'' \mathcal{E}(X)$

$\mathcal{O} \to$ holomorphic functions; $d'' \to \bar\partial$

$H^1(X, \Omega) = \mathcal{E}^{(2)}(X) / d\mathcal{E}^{1,0}(X).$

$\Omega \to$ holomorphic 1-forms

sheaf of smooth

$\mathcal{E}^{0,1}$ = (0,1) - forms

$\mathcal{E}^{1,0}$ (locally : $\underbrace{f(z,\bar z)}_{\text{smooth}}\, d\bar z$

$\mathcal{E}$ ~~( )~~ = sheaf of smooth functions

$\mathcal{E}^{(2)}$ = sheaf of smooth 2 - forms

$$d'' g(z,\bar z) = \frac{\partial g}{\partial \bar z}\, d\bar z \in \mathcal{E}^{0,1}$$

$d = d' + d''$

$$d\big(g(z,\bar z)\, dz \big) = \left(\frac{\partial g}{\partial z}\, dz + \frac{\partial g}{\partial \bar z}\, d\bar z \right) \wedge dz$$

$$= \frac{\partial g}{\partial \bar z}\, d\bar z \wedge dz \in \mathcal{E}^{1,1}$$

HW: Show $0 \to \Omega \hookrightarrow \mathcal{E}^{1,0} \xrightarrow{d} \mathcal{E}^{(2)} \to 0$

$0 \to \mathcal{O} \hookrightarrow \mathcal{E} \xrightarrow{d''} \mathcal{E}^{0,1} \to 0$

Work on small open set U which is simply-connected.

§16. Riemann - Roch theorem

Divisors $\longleftrightarrow$ Sheaves $\longleftrightarrow$ Line bundles

$\Longrightarrow$ Cohomology of divisors

RR is a relation between cohomology of a divisor and its degree.

* Divisors: $X = RS$

A divisor on X is a formal finite sum:

$$D = \sum_{x \in X} n_x \cdot x$$

where $n_x \in \mathbb{Z}$, & $n_x = 0$ except if $x \in$ a finite set.

Example: $X = \mathbb{P}^1$ ($0, \infty \in \mathbb{P}^1$)

$D = [0] + 2[\infty]$

$D = [0] - 2[\infty]$

We can think about a divisor D as a function:

$$D : X \to \mathbb{Z}$$

where $D(x) = 0$ for except if $x \in$ in a finite set depending on D.

$n_x = D(x)$ here.

* Divisor coming from a meromorphic function $f : X \to \mathbb{P}^1$ holomorphic

$$\Rightarrow (f) = \sum_{f(x)=0} n_x \cdot [x] - \sum_{f(x)=\infty} n_x \cdot [x]$$

where n_x = multiplicity of x.

Example: $X = \mathbb{C}$,

$$f(x) = \frac{x(x-2)}{(x-1)^3}$$

$\Rightarrow$ Roots are $0, 2$, pole is 1 (multiplicity $= 3$)

$$\Rightarrow (f) = [0] + [2] - 3[1].$$

Degree of a divisor:

$$D = \sum_{x \in X} n_x [x]$$

$$\Rightarrow \deg(D) = \sum_{x \in X} n_x.$$

Example: $X = \mathbb{P}^1$

$$D = [0] - 2[\infty]$$

$$\Rightarrow \deg(D) = 1 - 2 = -1.$$

Lemma: If X is compact RS, & $f : X \to \mathbb{P}^1$ meromorphic function $\Rightarrow \deg(f) = 0$.

Proof: We know before that

zeros of f = # poles of f

$$\deg(f) = \sum \text{zeros of } (f) - \sum \text{poles of } f = 0. \quad \square$$

Linear equivalence:

$$D_1 \underset{\text{l.e.}}{\sim} D_2$$

$\Leftrightarrow \exists\, f : X \to \mathbb{P}^1$ meromorphic so that $D_1 - D_2 = (f)$.

If $D_1 \sim 0 \Rightarrow D_1 = (f)$ for some f. We call such D_1 a principal divisor.

Q: X = compact RS.
If D is principal $\Rightarrow \deg(D) = 0$.
Conversely, is it true that if $\deg(D) = 0$ then D is principal?
Not true! Example if X = elliptic curve.

Divisor of meromorphic 1-form:

Locally $\omega = f\,dx$.

If $y \in$ the open set where ω has the above form, then define $n_y = \begin{cases} \text{multiplicity of } f \text{ at } y \text{ if } y \text{ is a zero or a pole} \end{cases}$

Then

$$(\omega) = \sum_{\substack{y \in X \\ f(y)=0}} n_y . [y] - \sum_{\substack{y \in X \\ f(y)=\infty}} n_y . [y]$$

f only defined locally

(Same like for divisors of meromorphic functions, except that here we need to work locally to be able to write $\omega = f\,dx$, x is local coordinate.)

<u>Sheaf associated to a divisor:</u>

$$D = \sum_{x \in X} n_x . x \quad \text{a divisor.}$$

We define a sheaf as follows:

$$\mathcal{M}(U) \supseteq \mathcal{O}_D(U) = \{ f : U \to \mathbb{P}^1 \text{ meromorphic},\ (f) + D \geq 0 \text{ on } U \}.$$

(U: open set $\subseteq X$)

<u>Example:</u> $X = \mathbb{P}^1$

$$D = 2[0] - 3[\infty]$$

$f \in \mathcal{O}_D(U) \Leftrightarrow f \in \mathcal{M}(U)$ & f has a pole of order at most 2 at 0, & a zero of multiplicity <u>at least</u> 3 at ∞.

We know $\mathcal{M}$ is a sheaf
sheaf of meromorphic function
of Abelian groups, with the $+$ binary operation the usual sum of 2 functions.

Now $\mathcal{O}_D \subseteq \mathcal{M}$, so to show that $\mathcal{O}_D$ is a sheaf, it suffices to show that $\mathcal{O}_D$ is a subgroup of $\mathcal{M}$.

If we have a divisor $D = \sum n_x \cdot x$, then we write $D \geq 0 \iff n_x \geq 0 \ \forall x$.

Convenience: if f is the 0-function, we don't define (0), but we can think about (0) like $(+\infty)\sum[x]$.

Line bundle & sheaves: (Read more references on the course's website).

A holomorphic line bundle on $X = RS$, is a data including:

+ An open covering (U_i) of X.

+ For each U_i, the set

$$U_i \times \mathbb{C}$$

+ For every 2, U_i, U_j an identities between $U_i \times \mathbb{C}$ & $U_j \times \mathbb{C}$ on $(U_i \cap U_j) \times \mathbb{C}$. Meaning (a transition function):

$$\varphi_{i,j} : (U_i \cap U_j) \times \mathbb{C} \to (U_i \cap U_j) \times \mathbb{C}$$

$$(z, t) \to (z, \varphi_{i,j}(z)t)$$

$\varphi_{i,j}$ holomorphic.

We write $\mathcal{L} = (X, (\varphi_{ij}))$.

$$\mathcal{L} \xrightarrow{\pi} X.$$

if $(z, t) \in U_i \times \mathbb{C}$ then $\pi(z,t) = z$.

& for each $x \in X$: $\pi^{-1}(X) = \mathcal{L}_x \simeq \mathbb{C}$ (a line).

And what the transition does is as follows: if $x \in U_i \cap U_j \Rightarrow$

$$\pi^{-1}(x) \simeq \mathbb{C} \subseteq (U_i \times \mathbb{C}) \cap (U_j \times \mathbb{C})$$

is represented by two lines in $U_i \times \mathbb{C}$ & $U_j \times \mathbb{C}$,

& we need to have linear transfunction:

$$U_i \times \mathbb{C} \qquad\qquad U_j \times \mathbb{C}$$
$$\text{\rotatebox{90}{$\subseteq$}} \qquad\qquad\qquad \text{\rotatebox{90}{$\subseteq$}}$$
$$\pi^{-1}(x) \longrightarrow \pi^{-1}(x),$$

which is given by the non-zero number
$$\varphi_{i,j}(x) \in \mathbb{C}^*.$$

(If this identification is not linear.) then we have a more general object, i.e fiber bundle.)

So, we can think about $\varphi_{i,j} : U_i \cap U_j \to \mathbb{C}^*$ holomorphic, i.e. $\varphi_{i,j} \in \mathcal{O}^*(U_i \cap U_j)$ which is a group. We can do more

generally by replacing $\mathcal{O}^*(U_i \cap U_j)$ with a group.

$$\mathbb{C}^* = GL_1$$

GL_2 gives us a vector bundle.

What conditions do we need for $(\varphi_{i,j})$:

$\varphi_{i,i} = id = 1$ (because we identify $U_i \times \mathbb{C}$ with itself).

$$\varphi_{i,j} = \frac{1}{\varphi_{j,i}} = \varphi_{i,j}^{-1}$$

$\varphi_{i,j}$: $U_i \times \mathbb{C} \to U_j \times \mathbb{C}$, $\varphi_{j,i} : U_j \times \mathbb{C} \to U_i \times \mathbb{C}$

$\varphi_{i,j} \cdot \varphi_{j,k} \cdot \varphi_{k,i} = 1$

(This is the same as a closed element in $C^1(\mathcal{U}, \mathcal{F})$, except we use here multiplication instead of addition.

But if U is simply connected, then there is a way to move between multiplication & addition, by:

$f \in \mathcal{O}^*(U) \Rightarrow f = e^g$ for some $g \in \mathcal{O}(U)$

g is well-defined up to modulo $2\pi i \mathbb{Z}$

So $f_1 f_2$ " $\Leftrightarrow$ " $g_1 + g_2$, $f_1 = e^{g_1}$, $f_2 = e^{g_2}$

$(\varphi_{ij}) \in C^1(\mathcal{U}, \mathcal{O}^*)$

& is "closed". So it "represents" an element in $H^1(X, \mathcal{O}^*)$.

<u>Division $\Rightarrow$ line bundle:</u>

if $D = \sum_x n_x [x]$ a division,

then we can choose an open covering $\mathcal{U} = (U_i)_{i \in I}$ of X & $f_i \in \mathcal{M}(U_i)$ so that $(f_i)\big|_{U_i} = D\big|_{U_i}$.

(Just choose U_i = a disk. For example, if

$p_1, p_2 \in U_i$:

$$D|_{U_i} = [p_1] - 2[p_2]$$

then define $f_i = \dfrac{z - p_1}{(z - p_2)^2}$

$\Rightarrow (f_i) = D|_{U_i}$.

& so then

$\varphi_{i,j} = f_i / f_j \in \mathcal{O}^*(U_i \cap U_j)$

$\varphi_{i,i} = 1, \quad \varphi_{i,j} = \dfrac{1}{\varphi_{j,i}},$

$\varphi_{i,j}\varphi_{j,k}\varphi_{k,i} = 1.$

$(\varphi_{i,j})$ defines a line bundle on X.