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MAT4800

Thm: Let X be a compact RS & $D \in \mathrm{Div}(X)$ a divisor with $\deg(D) < 0$. Then $H^0(X, \mathcal{O}_D) = 0$, meaning $\mathcal{O}_D$ has no global section.

Proof:

$$\mathcal{O}_D(X) = \{ f : X \to \mathbb{P}^1 \text{ holomorphic} \quad (f) \geq -D \}.$$

If $f \neq 0 \Rightarrow (f) = \sum_{x \in X} m_x [x]$, $D = \sum_{x \in X} n_x [x]$

$$(f) \geq -D$$

$$\Leftrightarrow \quad m_x \geq -n_x \quad \forall x$$

$$\Rightarrow \quad \underbrace{\sum m_x}_{\deg[f]} \geq -\underbrace{\sum n_x}_{\deg(D)}$$

X compact $\Rightarrow \deg[f] = 0$.

So $0 \geq -\deg(D) > 0$

a contradiction. $\square$

Skyscraper sheaf:

If $p \in X = RS$.

$$\mathbb{C}_p(\underset{\substack{\Downarrow \\ \text{open in } X}}{U}) = \begin{cases} \mathbb{C} & \text{if } p \in U \\ 0 & \text{if } p \notin U \end{cases}$$

<u>HW</u>: Check that it is a sheaf. $\square$

<u>Thm</u>: $H^0(X, \mathbb{C}_p) = \mathbb{C}$
$H^1(X, \mathbb{C}_p) = 0$.

(Later, skyscraper sheaf fits in a SES relevant to $\mathcal{O}_D$. & we can use LES of cohomology & the knowledge about cohomology of skyscraper sheaf to say about cohomology of $\mathcal{O}_D$.)

<u>Proof</u>:
$$H^0(X, \mathbb{C}_p) = \mathbb{C}_p(X) = \mathbb{C}$$
(because $p \in X$).

$H^1(X, \mathbb{C}_p) = 0$: Why?

$\alpha \in H^1(X, \mathbb{C}_p) \Rightarrow$ α is represented by a element of $C^1(\mathcal{U}, \mathbb{C}_p)$, where $\mathcal{U}$ is a very small open covering of X.

We can choose $\mathcal{U}$ so that $\mathcal{U} = (U_i)_{i \in I}$, & only 1 U_i contains p. (Example: choose $U_0 \ni p$, & replace U_i by $U_i \setminus \{p\}$.)

When computing H^1 cohomology, need only to look at $U_i \cap U_j$, where $i \neq j$.

then $\mathbb{C}_p(U_i \cap U_j) = 0$ for all $i \neq j$. $\Rightarrow \alpha = 0$ in cohomology.

Now we construct, for every divisor D on X, a SES of sheaves:

$$0 \to \mathcal{O}_D \hookrightarrow \mathcal{O}_{D+p} \xrightarrow{\beta} \mathbb{C}_p \to 0$$

First: $\mathcal{O}_D$ is a subsheaf of $\mathcal{O}_{D+p}$, because if $f \in \mathcal{M}(U)$ so that $[f]|_U \geq -D|_U \geq -(D+p)|_U$

$\Rightarrow$ ok

Now is the map $\mathcal{O}_{D+p} \to \mathbb{C}_p$:

So need to define for $U \subseteq X$ open a morphism $\mathcal{O}_{D+p}(U) \xrightarrow{\beta} \mathbb{C}_p(U)$

__If $p \notin U$:__ Then: $\mathbb{C}_p(U) = 0$ so β is the 0-map.

If $p \in U$ & $f \in \mathcal{O}_{D+p}(U)$, so we can have a Laurent expansion of f around $p = 0$ as follows:

$$f = \sum_{n=-k-1}^{\infty} c_n z^n$$

where $k = D(p)$. $\quad D = \sum_{x \in X} D(x)[x]$.

Define $\beta(f) = c_{-k-1} \in \mathbb{C} = \mathbb{C}_p(U)$

(Note that c_{-k-1} can be 0 if f has a pole at p of order $\leq k$.)

__Corollary ~~Thm~~:__ We have a LES.

$$0 \to H^0(X, \mathcal{O}_D) \to H^0(X, \mathcal{O}_{D+p}) \to \underset{\mathbb{C}}{\underset{\parallel}{H^0(X, \mathbb{C}_p)}} \to H^1(X, \mathcal{O}_D)$$

$$\to \underbrace{H^1(X, \mathcal{O}_{D+p}) \to \underset{0}{\underset{\parallel}{H^1(X, \mathbb{C}_p)}}} \to \dots$$

$$\Rightarrow \quad \mathbb{C} \to H^1(X, \mathcal{O}_D) \to H^1(X, \mathcal{O}_{D+p}) \to 0$$

So

$$H^1(X, \mathcal{O}_D) \to H^1(X, \mathcal{O}_{D+p}) \to 0$$

exact $\Rightarrow$ $H^1(X, \mathcal{O}_D) \twoheadrightarrow H^1(X, \mathcal{O}_{D+p})$

is surjective.

$$\mathbb{C} \to H^1(X, \mathcal{O}_D) \to H^1(X, \mathcal{O}_{D+p})$$

$\Rightarrow$ $\text{Image}(\mathbb{C} \to H^1(X, \mathcal{O}_D))$
$= \text{Ker}(H^1(X, \mathcal{O}_D) \to H^1(X, \mathcal{O}_{D+p}))$

$\text{Image}(\mathbb{C} \to H^1(X, \mathcal{O}_D))$ is either dimension 0 or 1 $\Rightarrow$ $\text{Ker}(H^1(X, \mathcal{O}_D) \to H^1(X, \mathcal{O}_{D+p}))$ is either 0 or 1.

Hence:

<u>Cor</u>: $\dim H^1(X, \mathcal{O}_D) = \dim H^1(X, \mathcal{O}_{D+p}) + \varepsilon$

where $\varepsilon = 0$ or 1.

<u>Thm 16.9</u> (RR): Suppose D is a divisor on a compact RS. X.

Then:

$$\dim H^0(X, \mathcal{O}_D) - \dim H^1(X, \mathcal{O}_D) = 1 - g + \deg(D).$$

Proof: Prove by induction on the number of points $x \in X$ where $D(x) \neq 0$.

- If $\# = 0 \Rightarrow D = 0$.

$\dim H^0(X, \mathcal{O}_D) = \dim H^0(X, \mathcal{O}) = 1$ (X : compact RS $\Rightarrow$ if $f: X \to \mathbb{P}^1$ holomorphic then $f = $ constant $\in \mathbb{C}$.)

$\dim H^1(X, \mathcal{O}) = g$ (by definition)!

$\deg(D) = 0$.

So it works!

- Assume it is true for $\# \leq k$. We will prove for $\# = k+1$.

Let D be such that $\# = k+1$. Then we can write $D = D' + mP$, where $\#$ for D' is k. ($m = D(P)$).

We can work by induction & hence can assume $m = 1$ & allow $D'(P) \neq 0$.

So we assume $D = D' + P$, & RR is true for D'. Look again at the SES:

$$0 \to \mathcal{O}_{D'} \to \mathcal{O}_{D = D' + P} \to \mathbb{C}_P \to 0$$

& LES:

$$0 \to H^0(\mathcal{O}_{D'}) \to H^0(\mathcal{O}_D) \to H^0(\mathbb{C}_P) = \mathbb{C} \to H^1(\mathcal{O}_{D'}) \to H^1(\mathcal{O}_D) \to H^1(\mathbb{C}_P) = 0.$$

An easy result in Linear algebra:
Assume that we have a LES of vector spaces

$$0 \to V_1 \to V_2 \to V_3 \to \dots \to V_m \to 0$$

then $$\sum_{i=1}^{m} (-1)^i \dim(V_i) = 0.$$

Apply to our sequence:

$$\Rightarrow \dim H^0(\mathcal{O}_{D'}) + 1 + \dim H^1(\mathcal{O}_D) = \dim H^0(\mathcal{O}_D) + \dim H^1(\mathcal{O}_{D'})$$

$$\Rightarrow \dim H^0(\mathcal{O}_{D'}) + 1 - \dim H^1(\mathcal{O}_{D'}) = \dim H^0(\mathcal{O}_D) - \dim H^1(\mathcal{O}_D).$$

RR is true for D':

$$\Rightarrow \dim H^0(\mathcal{O}_{D'}) - \dim(H^1(\mathcal{O}_{D'})$$
$$1 - g + \deg(D') = 1 - g + \deg(D) - 1$$
$$1 - g + \deg(D) = \dim H^0(\mathcal{O}_D) - \dim H^1(\mathcal{O}_D)$$

$\square$

Usually we are interested in knowing $H^0(\mathcal{O}_D) \Leftarrow$ global sections of $\mathcal{O}_D$.
whether it is not 0?

Cor: if $\deg(D) + 1 - g > 0$
$\Rightarrow \dim H^0(\mathcal{O}_D) > 0.$

(Also, later??, we may be able to estimate $\dim H^1(\mathcal{O}_D)$, by using duality, and hence can get better result.)

<u>Thm 16.11</u> (interpolation with pole):

X = compact RS, genus $= g$, & $a \in X$. Then $\exists$ $f : X \to \mathbb{P}^1$, such that $f \in \mathcal{O}(X \setminus \{a\})$ & f has exactly a pole of order $\leq g+1$ at a.

(we have a lot of holomorphic functions from $X \to \mathbb{P}^1$).

<u>Proof</u>: Choose $D = (g+1)[a]$

$\deg(D) = g+1$.

So $\dim H^0(\mathcal{O}_D) \geqslant 1 - g + \deg(D)$
$= 1 - g + g + 1 \geqslant 2$.

$\Rightarrow$ $\exists$ $f \in \mathcal{O}_D(X)$, $f \neq$ constant.

So if $x \neq a$: $[f]|_x \geqslant -D(x) = 0$
$\Rightarrow$ f is holomorphic at x.

if $x = a$: $[f]|_a \geqslant -D(a) = -(g+1)$

$\Rightarrow$ f has a pole of order at most $g+1$.
(or maybe f also holomorphic at a).

Cor: if f is as in the proof of the above theorem then: $f^{-1}(\infty) = a$ & order $\leq g+1 \Rightarrow \deg(f) \leq g+1$.

Cor: If f is as above & X has genus $g = 0 \Rightarrow \deg(f) \leq 1$ & $f: X \to \mathbb{P}^1$ not const $\Rightarrow \deg(f) = 1 \Rightarrow X \cong \mathbb{P}^1$. $\square$

Cor: For every $a \in X$ = compact RS, & any k. There is $f \in \mathcal{M}(X)$ non constant, $f \in \mathcal{O}(X \setminus \{a\})$, f has a pole of order at least k at a.

We only need to find m so that

$$\dim H^0(\mathcal{O}_{ma}) > \dim H^0(\mathcal{O}_{ka})$$

To do this just need to choose m such that $m + 1 - g > \dim H^0(\mathcal{O}_{ka})$.

Rmk: The same if we want poles at a finite number of points $a_1, \dots, a_\ell$.

Important topics in the course:

$\to$ Power series for holomorphic functions in a disk.

→ Classification of singularities: 3 types (Removable, pole, or essential).

→ ~~Covering maps~~ Degree & critical points of a holomorphic map. Local version ($z \to z^k$)

→ Existence of logarithm for $f \in \mathcal{O}^*(U)$, U = a disk.

→ Covering maps

→ Construction of universal covering, deck transformation & relation to π_1.

→ Affine & projective curves, how to check smoothness, ...

→ Differential forms, holomorphic 1-form

→ Open mapping theorem & maximum principle.

→ Stokes theorem (Cauchy's integral formula)

→ Cauchy - Riemann equation.

→ Simply connected RS ($\mathbb{C}$, $\mathbb{D}$, $\mathbb{P}^1$) & their automorphism groups. (All are restriction of $\mathrm{Aut}(\mathbb{P}^1)$)

→ Presheaf, Sheaf, how to get Sheafification of a presheaf

→ Čech cohomology of a sheaf.

→ Good properties of a simply connected domain in $\mathbb{C}$ (e.g. a disk): Can solve $\bar{\partial}$ - equation, exact $\overset{d}{=}$ closed (Poincaré lemma), Leray covering for H^1

→ Some common sheaves:
~~Sheaf~~ Constant sheaf, Skyscraper sheaf

- $\mathcal{E}$
- $\mathcal{O}$, $\mathcal{O}^*$
- $\mathcal{E}^{(1)}$, $\mathcal{E}^{1,0}$, $\mathcal{E}^{0,1}$, Ω
- $\mathcal{E}^{(2)}$, $\mathcal{E}^{1,1}$, ...

→ Some common operators: $d = \partial + \bar{\partial}$, $d' = \partial$, $d'' = \bar{\partial}$

+ Sheaf morphisms, SES of sheaves & LES of cohomology, transition map $H^0(X, \mathcal{H}) \to H^1(X, \mathcal{F})$ in a LES

+ Finiteness of cohomology for compact RS.

+ RR theorem & applications to existence of certain maps, genus.

+ Divisor & associated sheaf.

+ Forms: pullback, zeros & poles, residue (residue theorem).

+ Euler characteristic v.s. genus, Riemann-Hurwitz theorem.

+ How to construct maps, forms ... by using the universal covering (look at those invariant by the deck transformation).

<u>HW</u>: $\pi : \mathbb{C} \to E \leftarrow$ elliptic curve.

~~w~~ dz on $\mathbb{C}$ invariant by deck transformation $z \to z+b$, $b \in \Lambda$.

So dz gives a holomorphic 1-form ω on E. What is $\pi^*(\omega)$? (Hint: $\pi^*(\omega) = dz$.)

($H^0(E, \Omega) \simeq \mathbb{C}$, by this construction.)