24 Oct 2023
MAT 4800

dz & $d\bar{z}$

$$\underbrace{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}}_{V = \mathbb{R}^2}, \quad \underbrace{dx, dy}_{V^*}$$

$$dz, d\bar{z} \in V^* \otimes \mathbb{C}$$

$$z = x + iy \Rightarrow dz = dx + i\,dy$$
$$\bar{z} = x - iy \Rightarrow d\bar{z} = dx - i\,dy$$

$\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}$ is dual basis to dz & $d\bar{z}$

$\uparrow V \otimes \mathbb{C}$

V is generated by $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$

$\Rightarrow \exists$ complex numbers $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that:

$$\frac{\partial}{\partial z} = \alpha_1 \frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial y}$$

Can we find α_1 & α_2?

$$dz\left(\frac{\partial}{\partial z}\right) = 1, \quad d\bar{z}\left(\frac{\partial}{\partial z}\right) = 0$$

$$dz\left(\frac{\partial}{\partial z}\right) = dz\left(\alpha_1 \frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial y}\right)$$
$$= \alpha_1 \; dz\left(\frac{\partial}{\partial x}\right) + \alpha_2 \; dz\left(\frac{\partial}{\partial y}\right)$$

$$dz = dx + i\,dy$$
$$\Rightarrow dz\left(\frac{\partial}{\partial x}\right) = (dx + i\,dy)\left(\frac{\partial}{\partial x}\right)$$
$$= dx\left(\frac{\partial}{\partial x}\right) + i\,dy\left(\frac{\partial}{\partial x}\right)$$
$$= 1$$
$$dz\left(\frac{\partial}{\partial y}\right) = (dx + i\,dy)\left(\frac{\partial}{\partial y}\right)$$
$$= i$$
$$d\bar{z} = dx - i\,dy \Rightarrow d\bar{z}\left(\frac{\partial}{\partial x}\right) = 1$$
$$d\bar{z}\left(\frac{\partial}{\partial y}\right) = -i$$

$$1 = dz\left(\frac{\partial}{\partial z}\right) = \alpha_1 + i\alpha_2$$
$$0 = d\bar{z}\left(\frac{\partial}{\partial z}\right) = \alpha_1 - i\alpha_2$$
$$\Rightarrow \alpha_1 = i\alpha_2, \quad 2\alpha_1 = 1$$
$$\Rightarrow \alpha_1 = \frac{1}{2}, \quad \alpha_2 = \frac{1}{2i} = -\frac{i}{2}$$
$$\Rightarrow \frac{\partial}{\partial z} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y}$$

Similarly:
$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{i}{2}\frac{\partial}{\partial y}$$

<u>Riemann - Cauchy equation:</u>

f is holomorphic in a disc

$\Rightarrow$ f has a power series:

$$f(z,\bar{z}) = \underbrace{\sum_{n=0}^{\infty} a_n z^n}_{\text{has no } \bar{z}}$$

$$\Rightarrow \frac{\partial}{\partial \bar{z}} f(z,\bar{z}) = 0$$

$$\begin{cases} z = x+iy \\ \bar{z} = x - iy \end{cases} \Rightarrow x = \frac{z+\bar{z}}{2}, \quad y = \frac{z-\bar{z}}{2i}$$

$$f = u(x,y) + iv(x,y)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$0 = \frac{\partial}{\partial \bar{z}} f = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u+iv)$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x}u + i\frac{\partial}{\partial x}v + i\frac{\partial}{\partial y}u - \frac{\partial}{\partial y}v\right)$$

$$= \frac{1}{2}\left[\left(\frac{\partial}{\partial x}u - \frac{\partial}{\partial y}v\right) + i\left(\frac{\partial}{\partial x}v + \frac{\partial}{\partial y}u\right)\right]$$

$$\Rightarrow \begin{cases} \frac{\partial}{\partial x}u = \frac{\partial}{\partial y}v \\ \frac{\partial}{\partial y}u = -\frac{\partial}{\partial x}v \end{cases}$$

<u>How can we define over a general RS X?</u>

* Extrinsic method:
  For every $x_0 \in X$, $\exists$ a coordinate neighborhood $U \subseteq X$ so that U is homeomorphic to an open subset V of $\mathbb{C}$.
  $\rightsquigarrow$ So we can identify the tangent vectors & differential forms on V with that on U.
* Intrinsic method:
  Look at $\mathcal{O}_U$ $\leftarrow$ sheaf of holomorphic germs over U.

$$m_{x_0} = \{\varphi \in \mathcal{E}_{x_0} : \varphi(x_0) = 0\}$$

$$m_{x_0}^2 = \{\varphi \in \mathcal{E}_{x_0} : \varphi(x_0) = 0, \ \varphi'(x_0) = 0\}$$

$\mathcal{E}$: differentiable functions $f: U \to \mathbb{C}$

Then define:

$$\Omega^1_{x_0} = \frac{m_{x_0}}{m^2_{x_0}}$$

$\rightarrow$ 1-forms

Ω^1 is the sheaf of 1-forms on X,
$\Omega^1_{x_0}$ is the stalk.

$$d_{x_0}(f) = (f - f(x_0)) \pmod{m_{x_0}^2}$$

(f: a smooth function on a neighborhood of x_0)

Explanation: Let's look at

$V = \mathbb{D} \subseteq \mathbb{C}$. f: smooth function

By Taylor's expansion:

$f(x,y) = ax + by +$ higher order terms

$f(0,0) = 0$.

We define $f(x,y) \cdot \frac{\partial}{\partial x} = a$

$f(x,y) \cdot \frac{\partial}{\partial y} = b$

So we are using a theorem: if V is finite dimensional vector space then $V^{**} = V$.

We know:

differential forms is dual to vectors

$\Rightarrow$ vectors is dual to differential forms

But vectors is also dual to functions of the form $ax + by$.

So functions of the form $ax + by$ can be identified with differential forms.

dx & dy is a basis for $\Omega^1_{x_0}$

So also dz & $d\bar{z}$ $\left(dx = \frac{dz + d\bar{z}}{2}, \; dy = \frac{dz - d\bar{z}}{2i} \right)$

$\Omega^{1,0}_{x_0}$ = subvector space of $\Omega^1_{x_0}$ generated by dz

$\Omega^{0,1}_{x_0}$ = - - - . - . generated by $d\bar{z}$.

Holomorphic 1-form: elements in $\Omega^{1,0}$ which has the form

$$f(z)\, dz,$$

where $f(z)$ holomorphic.

Meromorphic 1-forms: . . . .

$$f(z)\, dz,$$

where f is a meromorphic function.

Pull back of differential form:

Assume $f: X \to Y$ a smooth function.

Let ω be a 1 form on Y.

Locally: $\omega(y_1, \dots, y_n) = \sum_{i=1}^{n} g_i(y)\, dy_i$

Then: $f^*\omega$ is a 1-form on X, defined (locally) by:

$(f^*\omega)(x_1, \dots, x_n) =$

$$(f^* \omega)(x_1, \dots, x_n) = \sum_{i=1}^{n} g_i(f(x_1, \dots, x_n))\, d f_i(x_1, \dots, x_n)$$

where $f = (f_1, \dots, f_n)$.

<u>Example 1:</u>

$$f : \mathbb{R} \to \mathbb{R}^3$$
$$t \mapsto (\underbrace{t+1}_{=y_1}, \underbrace{t^2}_{=y_2}, \underbrace{e^t}_{=y_3})$$

$$\omega(y_1, y_2, y_3) = y_2\, dy_1 + y_1 y_3\, dy_2 + y_3\, dy_3$$

$$(f^*\omega)(t) =$$

$$\boxed{\begin{aligned}(y_1, y_2, y_3)(t) &= f(t) \\ &= (t+1, t^2, e^t)\end{aligned}}$$

$$= t^2\, d(t+1) + (t+1)e^t\, d(t^2) + e^t\, d(e^t)$$

$$= t^2\, dt + (t+1)e^t\, 2t\, dt + e^t \cdot e^t\, dt$$

$$= (t^2 + (t+1)e^t\, 2t + e^{2t})\, dt.$$

Example 2:

$$f: \mathbb{R}^2 \to \mathbb{R}^3$$

$$(x_1, x_2) \mapsto (\underbrace{x_1^2 - x_2^2}_{= y_1}, \underbrace{x_1^3 + e^{x_2}}_{= y_2}, \underbrace{e^{x_1 \cdot x_2}}_{= y_3})$$

$$\omega = y_2\, dy_1 + y_1 y_3\, dy_2 + y_3\, dy_3$$

Then $(f^*\omega)(x_1, x_2)$

$$= (x_1^3 + e^{x_2})\, d(x_1^2 - x_2^2)$$
$$+ (x_1^2 - x_2^2) e^{x_1 \cdot x_2}\, d(x_1^3 + e^{x_2})$$
$$+ e^{x_1 x_2}\, d(e^{x_1 x_2})$$

$$dh(x_1, x_2) = \frac{\partial h}{\partial x_1} dx_1 + \frac{\partial h}{\partial x_2} dx_2$$

$$h(x_1, x_2) = x_1^2 - x_2^2$$
$$\Rightarrow dh(x_1, x_2) = 2x_1\, dx_1 - 2x_2\, dx_2$$

Similarly (HW).

Theorem: Let X & Y be RS. $f: X \to Y$ holomorphic functn. If ω is a holomorphic 1-form on Y, then $f^*\omega$ is a holomorphic 1-form on X.

If ω is ... meromorphic, then $f^*(\omega)$ is meromorphic.

Proof:

locally $\omega(y) =_{\text{local}} h(y)\,dy$, where y is a coordinate of Y, & $h(y)$ is holomorphic in y. $y = f(x)$

$$(f^*\omega)(x) = h(f(x))\, d(f(x)) = \underbrace{h(f(x))\, f'(x)}_{\text{holomorphic}}\, dx$$

If ω is meromorphic $\Rightarrow$ h is meromorphic

$$\Rightarrow (f^*\omega)(x) = \underbrace{h(f(x))}_{\text{meromorphic}}\ \underbrace{f'(x)}_{\text{holomorphic}}\, dx.$$

$\square$

$X = Y = \mathbb{C}$

$f : \mathbb{C} \to \mathbb{C}$
$x \mapsto y = x^2$

$$f^*(dy) = d(x^2) = 2x\,dx.$$

Example 2: $f : \mathbb{C} \to \mathbb{C}$
$x \mapsto y = x^2$

$$\omega(y) = \frac{dy}{y}$$

$$\Rightarrow (f^*\omega)(x) = \frac{d(f(x))}{f(x)} = \frac{d(x^2)}{x^2} = \frac{2x\,dx}{x^2} = \frac{2\,dx}{x}.$$

Example 3: $f : \mathbb{D}^* \to \mathbb{D}^*$
$x \mapsto y = x^3$

$\omega(y) = y^2 \, dy$
$\Rightarrow f^*(\omega)(x) = (x^3)^2 \, d(x^3)$
$= 3x^8 \, dx.$

Example 4: $f : \mathbb{C} \to \mathbb{C}$ ($a \neq 0$ constant)
$x \mapsto y = ax$

$\omega_1 = \boxed{\frac{dy}{y}} \Rightarrow (f^*\omega_1)(x) = \boxed{\frac{dx}{x}}$

same form

$\omega_2 = dy \Rightarrow (f^*\omega_2)(x) = a \, dx.$

not the same, if $a \neq 1$

Differential form on X: is locally of the form $\sum_{i=1}^{n} a_i(x) \, dx_i$ ($\dim(X) = n$)

a_i smooth function in x.

Holomorphic 1-form on a RS X:

Locally of the form
$h(z) \, dz$,
z: local coordinate, $h(z)$ holomorphic in z.

Meromorphic 1-form:
Locally $h(z) \, dz$
z: local coordinate, $h(z)$ meromorphic in z.

Poles & zeros of meromorphic 1-form:

If ω is locally of the form $h(z)\,dz$, then we call zeros of $h(z)$ zeros of ω, & poles of $h(z)$ poles of ω.

Theorem: Poles & zeros of a meromorphic 1 form of a RS is independent of the choice of coordinates.

Proof:

Let $\omega(z) = h(z)\,dz$.

$z = \varphi(y)$ is a change of coordinates $\Rightarrow$ $\varphi'(y)$ is never zero.

$$
\begin{aligned}
(\varphi^*\omega)(y) &= h(z)\,dz \\
&= h(\varphi(y))\,d(\varphi(y)) \\
&= \underline{h(\varphi(y))\,\varphi'(y)}\,dy
\end{aligned}
$$

Since $\varphi'(y)$ is holomorphic & never zero, poles & zeros of $(\varphi^*\omega)(y)$ are the same as those for $h(\varphi(y))$ & so the same as ω. $\square$

Thm not true if φ' has a zero.

Example (again)

$$\varphi : \mathbb{C} \to \mathbb{C}$$
$$x \mapsto y = x^2$$

$\omega = dy$ has no root & pole

$(\varphi^* \omega)(x) = 2x\,dx$ has a root at $x = 0$.

And $\{x = 0\}$ is exactly the set where φ is not unbranched.

Define: Residue of a meromorphic form at a point x, in a local coordinate z where $z = 0$ at x, is α,

if $\omega = \left(\sum_{n=-N}^{\infty} a_n z^n \right) dz$

& $\alpha = a_{-1}$. Write $\mathrm{Res}[\omega, x] = \alpha$.

Theorem: Residue is independent of the choice of coordinate.

Proof: It is equivalent to show that if $\varphi(0) = 0$, $\varphi : \mathbb{C} \to \mathbb{C}$ is holomorphic such that $\varphi'(0) \neq 0$ & ω is a meromorphic 1-form on $\mathbb{C}$, then: $\mathrm{Res}[\omega, 0] = \mathrm{Res}[\varphi^* \omega, 0]$.

$\cancel{z} = \varphi(y)$

$$\omega(z) = \left(\sum_{n=-N}^{\infty} a_n z^n\right) dz$$

$\varphi(0) = 0$, $\varphi'(0) \neq 0$

so $\varphi(y) = y\,h(y)$, where $h(0) \neq 0$.

$$(\varphi^* \omega)(y) = \left(\sum_{n=-N}^{\infty} a_n (y\,h(y))^n\right) d(y h(y))$$

$$= \sum_{n=-N}^{\infty} a_n y^n h(y)^n h(y)\,dy + \sum_{n=-N}^{\infty} a_n y^n (h(y))^n h'(y)\,y\,dy$$

Special case: $h(y) = cy \Rightarrow$ done!

$z = \varphi(y) = y(1-y)$

$$\omega = \left(\frac{1}{z^2} + \frac{1}{z}\right) dz$$

$$(\varphi^*\omega)(y) = \left(\frac{1}{(y(1-y))^2} + \frac{1}{y(1-y)}\right] d(y(1-y))$$

$$= \left(\frac{1}{y^2}(1+y+y^2+\ldots)^2 + \frac{1}{y}(1+y+y^2+\ldots)\right] (1-2y)\,dy$$

$$\frac{1}{y^2}(1+y+y^2+\ldots)^2(1-2y)$$

$$= \frac{1}{y^2}(1+2y+y^2+\ldots)(1-2y) = \frac{1}{y^2} - 4 + \ldots$$