Math 4800
25 October 2023

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Example: $f(x,y) = x^3 - y^3$

What is $\frac{\partial}{\partial z} f(x,y)$?

1st way: Since $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$

$$\Rightarrow \frac{\partial}{\partial z}(x^3 - y^3) = \frac{1}{2}\left(3x^2 + 3y^2 i\right)$$

2nd way: $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$

$$x^3 - y^3 = \left(\frac{z+\bar{z}}{2}\right)^3 - \left(\frac{z-\bar{z}}{2i}\right)^3$$

$$\frac{\partial}{\partial z}(x^3-y^3) = \frac{\partial}{\partial z}\left(\left[\left(\frac{z+\bar{z}}{2}\right)^3 - \left(\frac{z-\bar{z}}{2i}\right)^3\right]\right)$$

$$= \frac{3}{2}\left(\frac{z+\bar{z}}{2}\right)^2 - \frac{3}{2i}\left(\frac{z-\bar{z}}{2i}\right)^2.$$

<u>Example:</u> $E_{aff} = \{ (x,y) \in \mathbb{C}^2 : y^2 = x^3 - 1 \}$

$E = \{ [X:Y:Z] \in \mathbb{P}^2 : Y^2 Z = X^3 - Z^3 \}$

$$\begin{aligned} \pi : \quad E_{aff} &\to \mathbb{C} \\ (x,y) &\mapsto x \end{aligned}$$

Extension to E:

$$\begin{aligned} \pi : \quad E &\longrightarrow \mathbb{P}^1 \\ [X:Y:Z] &\mapsto [X:Z] \end{aligned}$$

a) Solved before:

What is $\pi[0:1:0]$?

$$[X:Z] = ?$$

$$\lim_{\substack{[X:Y:Z] \to [0:1:0] \\ \in E}}$$

Since $[0:1:0] \in \underbrace{\{ [X:Y:Z] \in \mathbb{P}^2 : Y \neq 0 \}}_{= U_2}$

So we can look at $E \cap U_2$

On U_2: $x_2 = X/Y$, $z_2 = Z/Y$

$E_2 = E \cap U_2 = \{ (x_2, y_2) \in \mathbb{C} : z_2 = x_2^3 - z_2^3 \}$

So we want to compute:

$$\lim_{(x_2, z_2) \to (0,0)} \frac{x_2}{z_2} = ?$$

$\uparrow$ E_2 Has the form $\frac{0}{0}$, so use l'Hospital rule.

First, we choose a coordinate for E_2 around $(0,0)$. $f_2 = z_2 + z_2^3 - x_2^3$

$$\nabla f_2 = (-3x_2^2, 1 + 3z_2^2)$$

$$\nabla f_2 \Big|_{x_2=0, z_2=0} = (0, 1).$$

$\Rightarrow$ x_2 can be chosen as local coordinate.

So z_2 is a function in x_2.

Use implicit differentiation:

$$z_2 + z_2^3 - x_2^3 = 0$$

$$\Rightarrow \frac{d}{dx_2}\left(z_2 + z_2^3 - x_2^3\right) = 0$$

$$= \frac{dz_2}{dx_2} + 3\frac{dz_2}{dx_2} \cdot z_2^2 - 3x_2^2 = 0.$$

Therefore at $(x_2, z_2) = (0,0)$

$$\Rightarrow \frac{dz_2}{dx_2} = 0$$

By l'Hospital rule:

$$\lim_{E_2 \ni (x_2, z_2) \to (0,0)} \frac{x_2}{z_2} = \lim_{E_2 \ni (x_2, z_2) \to (0,0)} \frac{1}{\left.\frac{dz_2}{dx_2}\right|} = \frac{1}{0} = \infty.$$

$\Rightarrow \pi[0:1:0] = [1:0]$!

b) What is $E \setminus E_{aff}$?

$E_{aff} = E \cap U_1$

$U_1 = \{[X:Y:Z] \in \mathbb{P}^2 : Z \neq 0\}$.

$$E \setminus E_{aff} = E \cap (\mathbb{P}^2 \setminus U_1) = E \cap \{[X:Y:Z] \in \mathbb{P}^2 : Z = 0\}.$$

$$= \{[X:Y:Z] \in \mathbb{P}^2 : Y^2 Z = X^3 - Z^3, Z = 0\}$$

$$= [0:1:0].$$

c) What are the critical points of the map $\pi : E \to \mathbb{P}^1$?

$$E = E_{aff} \cup \{[0:1:0]\}$$

so critical point of π are critical points of $\pi|_{E_{aff}}$ $\cup$ critical points of π in a neighborhood of $[0:1:0]$.

(actually, only need to check if $[0:1:0]$ is also a critical point)

* Critical points on $\pi|_{E_{aff}}$

$$E_{aff} = \{(x,y) \in \mathbb{C}^2 : y^2 = x^3 - 1\}$$

$\Downarrow$

$\pi(x,y) = x$.

Two cases to consider:

if x is a local coordinate of E_{aff} then $\pi'(x) = 1 \Rightarrow \pi$ has no critical points.

if x is not a local coordinate of E_{aff}, $\Rightarrow$ $x = h(y)$, h is holomorphic, & we need to check if $h'(y) = 0$ or not.

$$f_1 = y^2 - x^3 + 1$$

$$\nabla f_1 = (-3x^2 \ , \ 2y)$$

$\Rightarrow x$ is <u>not</u> a local coordinate $\Rightarrow y = 0$

$y = 0 \Rightarrow \ -x^3 + 1 = 0 \Rightarrow x = 1$ or $x = \zeta$ or $x = \zeta^2$, $\zeta^3 = 1$.

At these points: $\pi(x, y) = \ x = h(y)$

We want to compute $h'(y)$ & see if it is 0 or not.

Again, use implicit differentiation:

$$y^2 - x^3 + 1 = 0$$

. If y is a local coordinate of E_{aff}

$$\Rightarrow \frac{d}{dy}\left(y^2 - x^3 + 1\right) = 0$$

$$2y - 3x^2 \frac{dx}{dy} = 0$$

$y = 0$ & $x = 1$ or ζ or $\zeta^2 \Rightarrow x^2 \neq 0$ &

$2y = 0 \Rightarrow \ \frac{dx}{dy} = 0$.

So for $\pi(x,y) = x$ then at $(1,0), (\zeta,0), (\zeta^2,0)$ we have $\pi'(y) = 0 \Rightarrow$ these points are critical points of π.

Look near the point $[0:1:0]$ to determine if it is a critical point also.

In problem a, we know that we should work with $E_2 = \{ z_2 = x_2^3 - z_2^3 \}$ near $(x_2, z_2) = (0,0) = [0:1:0]$.

We also know that x_2 can be chosen as local coordinate there $\overset{\shortparallel}{} X/Y$.

& $\pi[0:1:0] = [1:0]$.

$$[1:0] \in \underbrace{\{[V:W] \in \mathbb{P}^1 : V \neq 0\}}_{= \Omega}$$

So we should work with Ω

$$\pi : E_2 \to \Omega$$

$$[X:Y:Z] \mapsto Z/X \;=\; [X:Z]$$

$$(x_2, z_2) \mapsto z_2/x_2$$

Want to compute $\pi'(x_2)\big|_{x_2=0}$.

Use implicit function differentiation theorem & l'Hospital rule:

$$z_2 = x_2^3 - z_2^3 \quad (x_2 \text{ a local coordinate})$$

$$\Rightarrow \frac{dz_2}{dx_2} = 3x_2^2 - 3z_2^2 \cdot \frac{dz_2}{dx_2}$$

$$x_2 = z_2 = 0 \Rightarrow \frac{dz_2}{dx_2} = 0.$$

$$\pi(x_2, z_2) = \frac{z_2}{x_2}$$

$$\pi(0,0) = 0$$

$$\frac{d\pi}{dx_2} = \lim_{x_2 \to 0} \frac{\pi(x_2) - \pi(0)}{x_2}$$

$$= \lim_{x_2 \to 0} \frac{\pi(x_2)}{x_2} = \lim_{x_2 \to 0} \frac{z_2}{x_2^2} = 0.$$

l' Hospital : $\lim_{x_2 \to 0} \frac{z_2}{x_2^2} = \lim_{x_2 \to 0} \frac{z_2'}{2x_2} = \lim_{x_2 \to 0} z_2''$

WTS: $z_2''(0) = 0.$

Again implicit differentiation:

$$z_2 = x_2^3 - z_2^3$$

$$\Rightarrow z_2' = 3x_2^2 - 3z_2^2 \cdot z_2'$$

$$\Rightarrow z_2' = \frac{3x_2^2}{1 + 3z_2^2} \quad \text{near } (x_2, z_2) = (0,0)$$

$$\Rightarrow z_2'' = \cdots \quad \Rightarrow z_2'' = 0 \text{ when } (x_2, z_2) = (0,0).$$

So $\pi'(x_2)\big|_{x_2 = 0} = 0 \Rightarrow [0:1:0]$

is also a critical point.

So $\pi : E \to \mathbb{P}^1$ has in total 4 critical points.

d) What are the multiplicities of π at the critical points: $[1:0:1], [\zeta:0:1], [\zeta^2:0:1], [0:1:0]$, $\zeta^3 = 1$?

We know multiplicity at the critical point is ≥ 2. We show that here actually all the multiplicities are 2.

(Can use local representation of the map π around these points.)

Here we use Riemann-Hurwitz formula. $\pi: E \to \mathbb{P}^1$.

E elliptic curve $\Rightarrow g(E) = 1 \Rightarrow \chi(E) = 2 - 2g = 0$.

$\mathbb{P}^1 \Rightarrow g(\mathbb{P}^1) = 0 \Rightarrow \chi(\mathbb{P}^1) = 2 - 2g = 2$.

Degree of $\pi = 2$.

Riemann-Hurwitz formula:

$$\chi(E) = 2\chi(\mathbb{P}^1) - \sum_{p:\text{ critical point}} (e_p - 1)$$

$$0 = 4 - \sum_{p:\text{ critical point}} (e_p - 1)$$

$e_p - 1 \geq 1$, # critical points $= 4$

$\Rightarrow e_p - 1 = 1$ $\forall$ critical point $\Rightarrow e_p = 2$ $\forall p$ critical points.

π : is the Weierstrass map for $E \to \mathbb{P}^1$.

e) Let $\mathbb{P}^1 = U_1 \cup U_2$

$\mathbb{P}^1 = [X : Y]$

On $U_1 = \{Y \neq 0\}$ use coordinate $z = X/Y$

On $U_2 = \{X \neq 0\}$ use coordinate $w = Y/X$.

dz on $\mathbb{C} = U_1$ is a holomorphic 1-form with no zero.

Now we want to extend $\omega = dz$ to a 1-form on $\mathbb{P}^1$. How can we do it?

We want to be able to say what is dz on $U_1 \cap U_2$ in the other coordinate w. (Then we can extend to U_2.)

$$z = X/Y, \quad w = Y/X$$

$$\Rightarrow z = 1/w$$

$$\Rightarrow \omega = dz = d\left(\frac{1}{w}\right) = -\frac{1}{w^2}dw.$$

So our 1-form on $\mathbb{P}^1$ is:

$$\omega = \begin{cases} dz, & \text{on } U_1 \\ -\frac{1}{w^2}dw & \text{on } U_2. \end{cases}$$

So ω is not holomorphic on the whole $\mathbb{P}^1$ but only meromorphic.

(HW/ later: on $\mathbb{P}^1$ there is no holomorphic 1-form except the form 0.)

f) What is $\pi^*(\omega)$?

$$\pi : E \to \mathbb{P}^1$$
$$[X:Y:Z] \mapsto [X:Z].$$

First, we work on $E_{aff.} = \{ y^2 = x^3 - 1 \}$

$\pi(x,y) = x$. $\pi : E_{aff} \to \mathbb{C}$

On $\mathbb{C}$, $\omega = dz$

& $\pi(x,y) = x$

$\Rightarrow \pi^* \omega = dx$ on E_{aff}.

x is always a holomorphic function on E_{aff} $\Rightarrow$ dx is a holomorphic 1-form on E_{aff}.

<u>Q</u>: Does $\underset{= \pi^*(\omega)}{dx}$ have zeros?

<u>Recall</u>: Let $\pi : \underset{\sim}{X} \to \underset{\sim}{Y}$ be a holomorphic map & Let ω be a holomorphic 1-form on Y, & ω has no zero.

RS

Then the zeros of $\pi^*(\omega)$ are exactly at the critical points of π, & the multiplicity at p is $e_p - 1$.

(Recall the proof:

Locally can take $\pi(z) = z^k$,

$p = 0$, $e_p = k$,

$\omega = dz$

$\Rightarrow \pi^*\omega = d(z^k) = kz^{k-1}dz$

So $\pi^*(\omega)$ has a zero of multiplicity $k-1$.)

g) $\pi : E \to \mathbb{P}^1$ is holomorphic, ω is a meromorphic 1-form on $\mathbb{P}^1 \Rightarrow$

$\pi^*(\omega)$ is a meromorphic 1-form on E.

<u>Q</u>: What is $\pi^*(\omega)$ near $E \setminus E_{aff} = [0:1:0]$?

$\Rightarrow$ 2 ways to do:

* Way 1: extend $\pi^*(\omega)|_{E_{aff}}$ to E.

$\pi^*(\omega) = dx$ in local coordinates (x,y), $x = X/Z$, $y = Y/Z$.

To extend to the point $[0:1:0]$, we need to use coordinates $x_2 = X/Y$, $z_2 = Z/Y$.

So $\pi^*(\omega) = dx = d(X/Z) = d(x_2/z_2)$.

So
$$\pi^*(\omega) = \begin{cases} dx & \text{on } E_{aff} \\ d(x_2/z_2) & \text{near } [0:1:0]. \end{cases}$$

HW: Does $\pi^*(\omega)$ have a zero or pole at $[0:1:0]$?

<u>2nd way to go:</u> (See a previous exercise)

Near $[0:1:0]$
$$\pi(x_2, z_2) = z_2/x_2 = w$$

& $\omega = -\dfrac{dw}{w^2}$ here.
$$\pi^*(\omega) = -\frac{d(z_2/x_2)}{(z_2/x_2)^2}$$

HW: Check that this is the same as the formula for $\pi^*(\omega)$ in the 1st method.

$\square$

<u>Residue theorem:</u>

Let ω be a meromorphic 1-form around $x_0 \in X = RS$.

Let $\mathbb{D} \subseteq X$ be a (small) open set containing x_0 so that ω has no poles in $\overline{\mathbb{D}} \setminus \{x_0\}$.

Let $C = \partial \mathbb{D}$ with positive orientation.

Then:

$$\oint_C \omega = 2\pi i \ \mathrm{Res}[\omega, x_0].$$

Proof: If $C =$ a unit circle then comes from:

$$\frac{1}{2\pi i} \oint_{|z|=1} z^n \, dz = \begin{cases} 1 & \text{if } n = -1 \\ 0 & \text{otherwise} \end{cases}$$

Use polar coordinates: $z = \rho e^{i\theta}$, $\theta \in [0, 2\pi]$

$\rho = 1 \Rightarrow z = e^{i\theta} \Rightarrow dz = i e^{i\theta} d\theta.$

In general, use Stokes theorem (next time) to reduce to this case. $\square$