26 September 2023

MAT 4800

Proper mapping:

A map $f: X \to Y$ is proper if $\forall\, C \subseteq Y$ compact, then $f^{-1}(C)$ is also compact.

Example: $f: \mathbb{C} \to \mathbb{C}$, $z \mapsto z^3$ is proper.

Proof: Let $C \subseteq \mathbb{C}$ be compact. Want to show $f^{-1}(C)$ is compact. Now C is compact $\Rightarrow C$ is closed $\Rightarrow f^{-1}(C)$ is closed.

By Bolzano-Weierstrass theorem, for $f^{-1}(C)$ compact, it suffices to show that $f^{-1}(C)$ is bounded.

C is compact $\Rightarrow \exists\, R > 0$ so that $\forall z \in C$: $|z| \leq R$.

If $w \in f^{-1}(C) \Rightarrow f(w) = w^3 \in C \Rightarrow |w^3| \leq R \Rightarrow |w| \leq R^{1/3}$.

Ex: $f: \mathbb{C} \to \mathbb{C}$
$z \mapsto e^z$

not proper.

(Preimage of a point is not a compact set)

Ex: $f: \mathbb{C}^* \to \mathbb{C}$
$z \mapsto z^3$

is ~~not proper~~!

Example $f^{-1}(\{z : |z| \le 1\}) = \{z : |z| \le 1, z \neq 0\}$

Thm 4.22: $p: Y \to X$ is proper & local homeomorphism $\Rightarrow$ p is a covering map.

Proof: $\forall\ x_0 \in X$, need to show there is $x_0 \in U_0 \subseteq X$ (open) so that

$$p^{-1}(U_0) = \coprod_{i \in I} V_i ,$$

$p|_{V_i} : V_i \to U_0$ is homeomorphic.

Note: I is finite.

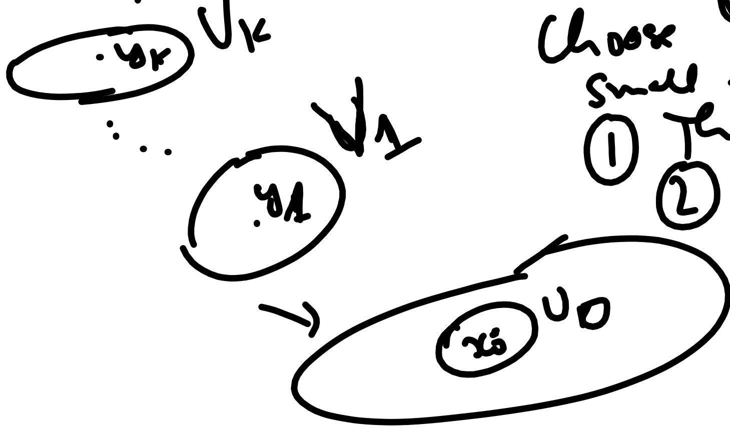
~~Let~~ First, note that

$p^{-1}(x_0)$ is a finite set.

$p^{-1}(x_0)$ is discrete, & also compact

$\Rightarrow$ it is finite.

$p^{-1}(x_0) = \{y_1, \dots, y_k\}$



Choose $V_1, \dots, V_k$ small enough so that:

① They are disjoint

② They map homeomorphically to the same $x_0 \in U_0 \subseteq X$.

Ex: $\mathbb{C}^* \to \mathbb{C}^*$

$z \to z^3$

is a covering map.

(Check proper: Preimage of compact set is compact.

Check local homeomorphism: Check derivative nonzero everywhere.)

Thm: (Proof uses local representation of holomorphic maps & definition of proper maps):

Thm 4.24: $f: X \to Y$ (RS)

proper holomorphic non-constant, then $\exists$ $n = \deg(f)$ so that $\forall$ point $y \in Y$:

$\# f^{-1}(y) = n.$
counted multiplicities

(Local representations: $z \mapsto z^k$ ← multiplicity)

(Show that the part which has degree n is non-empty, open & closed.)

Example: $f: \mathbb{C} \to \mathbb{C}$
$z \mapsto z^3$

is proper, non-constant, holomorphic.

So it has a degree n:

To compute n, we can choose a random z_0, & find how many points are in $f^{-1}(z_0)$. (Should avoid some bad points)

Degree $= 3$.

Ex: $f: \mathbb{C} \to \mathbb{C}$ polynomial of degree 5.

$\Rightarrow \# f^{-1}(y) = 5$.

§5. Universal covering & Covering transformations / Deck transformations

(Important for π_1).

X be connected, path-connected, ...

Def: A covering map ~~is~~
$$\pi : Y \to X$$
is called universal if $\pi_1(Y) = 0$.

Thm 5.3:

- If X is connected, then X has a universal covering $\pi : Y \to X$.
- Assume that $\pi : Y \to X$ is a universal covering, & $p : Z \to X$ is a covering map, then

$$\begin{tikzcd}
 & Z \arrow[d, "p"] \\
Y \arrow[ur, dotted, "\exists F\ ?"] \arrow[r, "\pi"'] & X
\end{tikzcd}$$

p = covering map

Y: simply-connected

(We proved in §4.) Moreover, F is also a covering map

So Y is the "biggest" covering map for X, that's why it is universal.

If both Y & Z are universal covering, then

$$\begin{tikzcd} & Z \arrow[d] \arrow[dl, dashed, bend left, "G"] \\ Y \arrow[r] \arrow[ur, dashed, "F"] & X \end{tikzcd}$$

F, G are homeomorphism. (If $F(y_0) = z_0$, $G(z_0) = y_0$, then F & G are inverses)

$\Downarrow$

Use uniqueness of lifting maps.

So up to homeomorphism, $\exists$ 1 & only one covering map.

Proof: We will construct a simply connected space $\widetilde{X}$ & a covering map $p : \widetilde{X} \to X$.

Fixed a base point x_0.

For every x, let

$$\pi(x_0, x) = \text{curves joining } x_0 \to x \,/\, \text{homotopic equivalence}$$

$$\widetilde{X} = \{ (x, \alpha) : x \in X,\ \alpha \in \pi(x_0, x) \}$$

$$p : \widetilde{X} \to X$$
$$(x, \alpha) \mapsto x$$

- Show that $\tilde{X}$ is simply-connected.

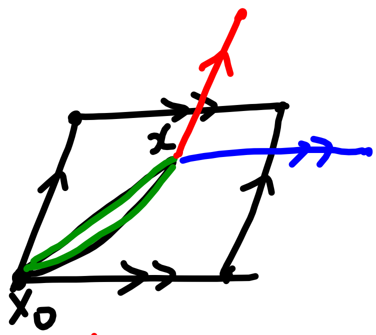
If $\varphi : [0,1] \to \tilde{X}$ is a closed curve.

Show that φ homotopic to a point. (Later)

- What is the topology on $\tilde{X}$.

(Proof : tomorrow)

Ex:



$E = \mathbb{C}/\Lambda$

$\tilde{X} = \tilde{E} = \mathbb{C}$

Calculation with $\mathbb{P}^2$

$E_{aff} = \{(x_1, y_1) \in \mathbb{C}^2 : y_1^2 = x_1^3 + 1\}$

$E = \{[X:Y:Z] \in \mathbb{P}^2 : Y^2 Z = X^3 + Z^3\}$

$\pi : E_{aff} \longrightarrow \mathbb{C}$

$(x, y) \mapsto x$

$\mathbb{P}^2 = U_1 \cup U_2 \cup U_3$

$U_1 = \{[X : Y : Z] : Z \neq 0\}$

$\simeq \mathbb{C}^2_{x_1, y_1}$

$x_1 = X/Z, \ y_1 = Y/Z$

$U_2 = \{[X : Y : Z] : Y \neq 0\}$

$\simeq \mathbb{C}^2_{x_2, z_2}$

$x_2 = X/Y, \ z_2 = Z/Y$

$U_3 = \{[X : Y : Z] : X \neq 0\}$

$\simeq \mathbb{C}^2_{y_3, z_3}$

$y_3 = Y/X, \ z_3 = Z/X$

$\cdot \ E_1 = E \cap U_1 = \{Y^2 Z = X^3 + Z^3\}$

$\cap \{Z \neq 0 : x_1 = X/Z, \ y_1 = Y/Z\}$

Equation for E_1:

Put $Z = 1 \Rightarrow x_1 = X, \ y_1 = Y$.

$E_1 = \{y_1^2 = x_1^3 + 1\} = E_{aff} \subseteq E$

We knew E_1 is smooth in $\mathbb{C}^2$.

- $E_2 = E \cap U_2 = \{Y^2 Z = X^3 + Z^3\}$
  $\cap \{Y \neq 0 : x_2 = X/Y, \; z_2 = Z/Y\}$.

Equation for E_2: put $Y = 1 \rightarrow$
$x_2 = X, \; z_2 = Z$

$$E_2 = \{ z_2 = x_2^3 + z_2^3 \}$$
$$= \{ f_2 = 0 \} \text{ where } f_2 = z_2 - x_2^3 - z_2^3.$$

$\nabla f_2 = (-3x_2^2, 1 - 3z_2^2)$
never 0 on $E_2 \Rightarrow E_2$ is smooth.

- $E_3 = E \cap U_3 = \{Y^2 Z = X^3 + Z^3\}$
  $\cap \{X \neq 0 : y_3 = Y/X, \; z_3 = Z/X\}$

$$E_3 = \{ y_3^2 z_3 = 1 + z_3^3 \}$$

Compute gradient, show that it is never 0. (HW)

- $\pi : E_1 \longrightarrow \mathbb{C}$
  $(x_1, y_1) \longmapsto x_1$

<u>Q</u>: What is the formula for π on E_2?

$$E_2 = \{ z_2 = x_2^3 + z_2^3 \}$$

Need to understand how to go from (x_1, y_1) to (x_2, z_2)

$$E = \{ Y^2 Z = X^3 + Z^3 \}$$

$$E_1 = \{ y_1^2 = x_1^3 + 1 \}$$

$$x_1 = X/Z, \quad y_1 = Y/Z$$

$$\begin{array}{ccc} E & & \\ \Downarrow & & \\ \left(\frac{X}{Z}, \frac{Y}{Z}\right) & \longrightarrow & \frac{X}{Z} \\ \left[\frac{X}{Z} : \frac{Y}{Z} : 1\right] & \mapsto & \left[\frac{X}{Z} : 1\right] \end{array}$$

$$\begin{array}{cccc} \pi : & E & \longrightarrow & \mathbb{P}^1 \\ & \Downarrow & & \\ & [X:Y:Z] & \mapsto & [X:Z] \end{array}$$

What is the image of $[0:1:0]$?

Use classification of singularities.

Hint: compute $\lim X/Z$

$E \ni [X:Y:Z] \to [0:1:0]$

$[0:1:0] \in U_2$ so we should look at E_2.